Generating the bounded derived category and perfect ghosts

(joint with Steffen Oppermann)

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Outline

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Setup

Let k be a commutative noetherian ring and $\mathcal{T}$ be a Hom-finite skeletally small triangulated category over k. Examples to keep in mind: $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$, where:

- $\mathcal{A} = \mathrm{mod} R$ for a module finite k-algebra R; or
- $\mathcal{A} = \mathrm{coh} \mathbb{X}$ for projective scheme $\mathbb{X}$ over k.

Definition

A full subcategory $\mathcal{S}$ of $\mathcal{T}$ is thick if it is a triangulated subcategory closed under direct summands.

Fact

$\mathcal{S}$ is thick if and only if there is a triangulated functor $F \colon \mathcal{T} \to \mathcal{U}$ such that

$$\mathcal{S} = \{X \in \mathcal{T} \mid FX = 0\}.$$

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Generating thick subcategories

Let $M \in \mathcal{T}$. What does the smallest thick subcategory containing M look like?

Definition

- $\langle M \rangle = \langle M \rangle_1 =$ the closure of $\{M[i] \mid i \in \mathbb{Z}\}$ under products and summands in $\mathcal{T}$.
- $\langle M \rangle_{n+1} =$ summands of objects E appearing in a triangle

$$X \to E \to Y \to X[1] \quad \text{with } X \in \langle M \rangle \text{ and } Y \in \langle M \rangle_n.$$

- $\langle M \rangle_\infty = \bigcup_{n \geq 1} \langle M \rangle_n$, where $\langle M \rangle_1 \subseteq \langle M \rangle_2 \subseteq \langle M \rangle_3 \subseteq \cdots$

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$\langle M \rangle_\infty$ is the smallest thick subcategory of $\mathcal{T}$ containing M.

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Definition (Rouquier)

The dimension of $\mathcal{T}$, denoted $\dim \mathcal{T}$, is the minimum $n \geq 0$ with

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We put $\dim \mathcal{T} = \infty$ if no such n exists.

Examples

1. $\dim \mathcal{T} \leq \operatorname{gldim} R$ for $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$.
2. $\dim \mathcal{T} \leq \dim_k R - 1$ if k is a field and R is finite dimensional over k.

Remark

Computing $\dim \mathcal{T}$ is often not easy. How do we prove for instance that $\mathcal{T} \neq \langle M \rangle_{n+1}$ for given $n \geq 0$ and $M \in \mathcal{T}$?

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Ghost maps

Definition

A morphism $f\colon X \to Y$ is a (covariant) **M-ghost** if

$$\mathrm{Hom}_{\mathcal{T}}(f, M[i]) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

That is:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & & \downarrow{\scriptstyle \forall g} \\ & & M[i] \end{array}$$

Remark

Traditionally, contravariant ghosts are considered (i.e. morphisms such that $\mathrm{Hom}_{\mathcal{T}}(M[i], f) = 0$).
Example: $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(\mathrm{mod} R)$ and $M = R$, then

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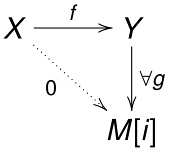

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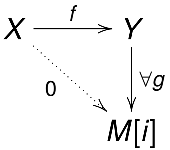

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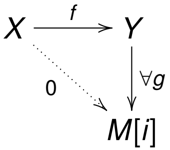

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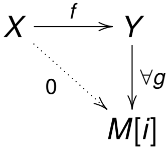

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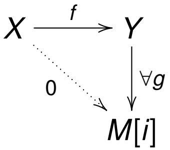

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Ghost Lemma

Lemma (several versions in the literature)

Suppose that $X \in \langle M \rangle_n$. Then every composition

$$X_n \xrightarrow{\text{ghost}} X_{n-1} \xrightarrow{\text{ghost}} \cdots \xrightarrow{\text{ghost}} X_1 \xrightarrow{\text{ghost}} X$$

of n consecutive M-ghosts ending at X vanishes.

Corollary

Suppose that $\mathcal{T} = \langle M \rangle_n$. Then every composition

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A full subcategory $\mathcal{C} \subseteq \mathcal{T}$ is preenveloping if each X admits a morphism $f_X \colon X \to C_X$ with $C_X \in \mathcal{C}$ such that

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If $\mathcal{C} = \langle M \rangle$ and we consider the triangle

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Then h is an M-ghost.

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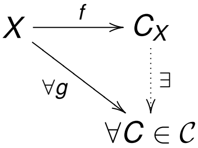
A full subcategory $\mathcal{C} \subseteq \mathcal{T}$ is preenveloping if each X admits a morphism $f_X \colon X \to C_X$ with $C_X \in \mathcal{C}$ such that



Observation

If $\mathcal{C} = \langle M \rangle$ and we consider the triangle

$$Z \xrightarrow{h} X \xrightarrow{f} C_X \to Z[1].$$

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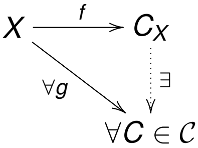

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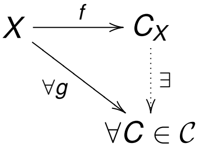

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Converse of the Ghost Lemma—v1

Lemma (Beligiannis?)

Suppose that $\langle M \rangle$ is preenveloping in $\mathcal{T}$. Then $X \in \langle M \rangle_n$ if and only if every composition

$$X_n \xrightarrow{\text{ghost}} X_{n-1} \xrightarrow{\text{ghost}} \cdots \xrightarrow{\text{ghost}} X_1 \xrightarrow{\text{ghost}} X$$

of n consecutive M-ghosts ending at X vanishes.

Corollary

If $\langle M \rangle$ is preenveloping, then $\mathcal{T} = \langle M \rangle_n$ if and only if every

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For our $\mathcal{T}$ one cannot expect that $\langle M \rangle$ will be preenveloping!

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Nevertheless...

Theorem (Š.-Oppermann)

Let R be a module finite algebra over a commutative noetherian ring and let $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(\mathrm{mod} R)$. TFAE for $M \in \mathcal{T}$ and $n \geq 1$:

1. $X \in \langle M \rangle_n$.
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$$X_n \xrightarrow{\text{ghost}} X_{n-1} \xrightarrow{\text{ghost}} \cdots \xrightarrow{\text{ghost}} X_1 \xrightarrow{\text{ghost}} X_0 \longrightarrow X,$$

where $X_0, X_1, \ldots, X_n$ are perfect complexes, vanishes.

Remarks

1. Perfect complex in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod} R)$ is a complex isomorphic to a bounded complex of finitely generated projective modules.
2. $\mathrm{gldim}\, R < \infty \iff$ every complex in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod} R)$ is perfect.
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The idea behind

- Consider a bigger triangulated category $\mathcal{U} = \mathbf{D}^{-}(\mathrm{mod} R)$. Objects are bounded above complexes

$$M\colon \qquad \cdots \longrightarrow M^{N-2} \xrightarrow{\partial} M^{N-1} \xrightarrow{\partial} M^{N} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

- We have some infinite products in $\mathcal{U} = \mathbf{D}^{-}(\mathrm{mod} R)$. Namely those of the form

$$\prod_{i\in\mathbb{Z}} M[i]^{n_i}$$

where n_i is finite for each i and $n_i = 0$ for $i \ll 0$.

- Consequence: Suppose that $M \in \mathcal{T} = \mathbf{D}^{\mathrm{b}}(\mathrm{mod} R)$. Then $\langle M \rangle$, taken in $\mathcal{U} = \mathbf{D}^{-}(\mathrm{mod} R)$ (i.e. it may contain infinite products of the form above), is preenveloping in $\mathcal{U}$. So the converse of the Ghost Lemma holds for M and $\mathcal{U}$.

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- We know: Given $M, X \in \mathcal{T} = \mathbf{D}^{\mathrm{b}}(\mathrm{mod}R)$ and $n \geq 1$, we have $X \in \langle M \rangle_n$ in $\mathcal{U}$ if and only if every composition

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  of M-ghost in $\mathcal{U}$ vanishes.
- Objects in $\mathcal{T}^{\mathrm{op}}$ are up to isomorphism the compact ones in $\mathcal{U}^{\mathrm{op}}$. This allows us to get rid of infinite products in $\mathcal{U}$, so that

$$X \in \langle M \rangle_n \text{ in } \mathcal{U} \iff X \in \langle M \rangle_n \text{ in } \mathcal{T}.$$

- On the other hand, if we have a composition of M-ghosts as above, we can w.l.o.g. assume that the complexes $X_0, \ldots, X_n$ have projective components. If we have a non-vanishing composition of M-ghosts, we get one also by suitably truncating $X_0, \ldots X_n$ to bounded complexes.

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- On the other hand, if we have a composition of M-ghosts as above, we can w.l.o.g. assume that the complexes $X_0, \ldots, X_n$ have projective components. If we have a non-vanishing composition of M-ghosts, we get one also by suitably truncating $X_0, \ldots X_n$ to bounded complexes.

Outline

Theorem (Š.-Oppermann)

Let R be a module finite algebra over a commutative noetherian ring and let $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,R)$. If $\mathcal{T}' \subseteq \mathcal{T}$ is a thick subcategory such that $\dim \mathcal{T}' < \infty$ and $R \in \mathcal{T}'$ then

$$\mathcal{T}' = \mathcal{T}.$$

Remark

1. The assumption "$R \in \mathcal{T}'$" is necessary.
2. Provided that $\dim \mathcal{T} < \infty$, we have "if and only if" in the theorem. This often happens in algebraic geometry and representation theory.
3. One can compute explicit examples of this phenomenon.
4. Again, there is a similar theorem for $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X})$.

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Sketch of the proof

- Suppose that $\mathcal{T}' = \langle M \rangle_n$ for some $M \in \mathcal{T}$ and $n \geq 1$.
- Since $R \in \mathcal{T}'$, all perfect complexes belong to $\mathcal{T}'$.
- Thus, every composition

$$X_n \xrightarrow{\text{ghost}} X_{n-1} \xrightarrow{\text{ghost}} \cdots \xrightarrow{\text{ghost}} X_1 \xrightarrow{\text{ghost}} X_0,$$

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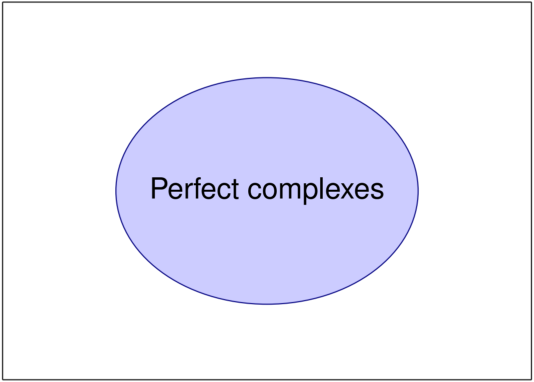

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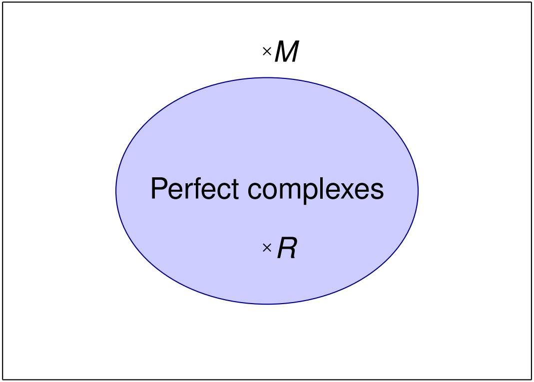

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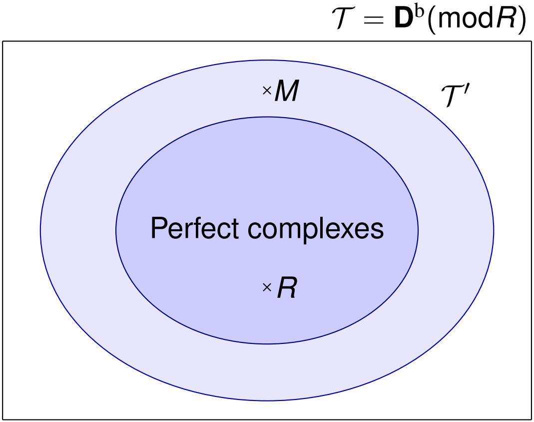

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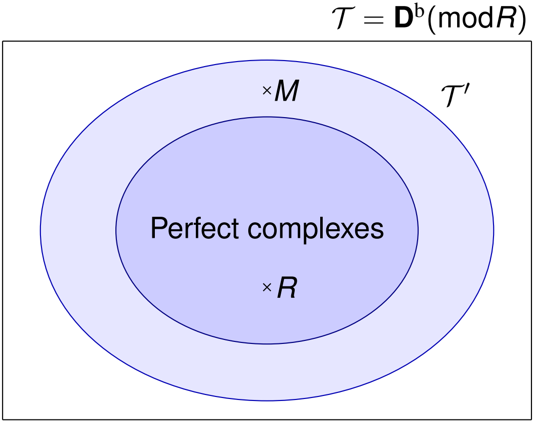

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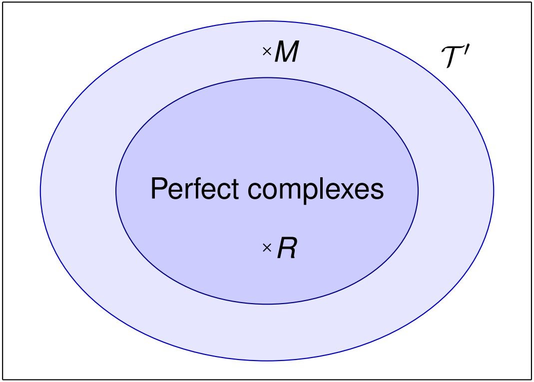

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