The smashing spectrum of a valuation domain

Seminar on non-commutative motives and telescope-type problems

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Motivation

The goal

- Them main goal is to study the localizations of $\mathrm{D}(R)$, where R is a non-discrete valuation domain. The talk is based on

  S. Bazzoni, J. Šťovíček, Smashing localizations of rings of weak global dimension at most one, Adv. Math. 305 (2017)

- The setting is simple enough homologically in order to understand the situation completely in some cases (key fact: $\mathrm{w.gl.dim} R = 1$).
- On the other hand, the setting is complicated enough in that the telescope conjecture fails.

Keller's example

Example (Keller, 1994)

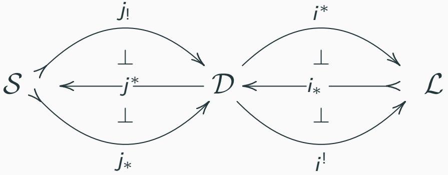
- Let $Q = \overline{\mathbb{C}(\!(x)\!)} = \{\sum_{i=N}^{\infty} c_i x^{\frac{i}{d}} \mid N \in \mathbb{Z}, d > 0, c_i \in \mathbb{C}\}$ — the field of Puiseux series (Newton 1670's, Puiseux 1850's).
- Put $A = \{\sum_{i=0}^{\infty} c_i x^{\frac{i}{d}} \mid d > 0, c_i \in \mathbb{C}\}$. This is a valuation domain and $\mathrm{Spec}(A) = \{0, \mathfrak{m} = \mathfrak{m}^2\}$.
- There are 5 smashing localizations given by ring epimorphisms, just the first three are compactly generated:

$$A \to 0,\ A \to Q,\ A \to A,\ A \to \mathbb{C},\ A \to Q \times \mathbb{C}.$$

Rings of weak global dimension at most one

Smashing localizations as recollements (after [Krause, 2000])

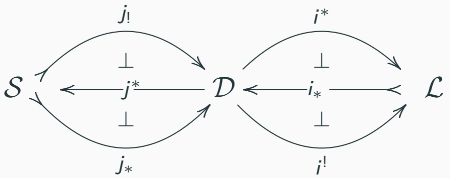
- If $\mathcal{D}$ is a compactly generated triangulated category, we study recollements



- A smashing subcategory of $\mathcal{D}$ is one of the form $\operatorname{im}(j_!)$. In this case i^* induces $\mathcal{D}/\mathcal{S} \simeq \mathcal{L}$.

Compactly generated localizations

Theorem (Neeman, 1992)
If $\mathcal{D}$ is compactly generated triangulated and $\mathcal{S}$ is a localizing subcategory generated by a set of objects, then it is a smashing subcategory. In particular, there is a recollement



Definition
$\mathcal{D}$ satisfies the telescope conjecture if all recollements (up to equivalence) arise in this way.

Homological epimorphisms (after [Nicolás-Saorín, 2009])

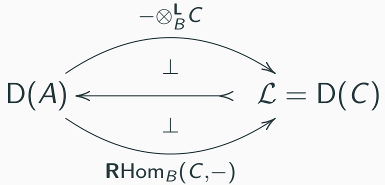
- Suppose that $\mathcal{D} = \mathrm{D}(A)$, where

$$A = (\cdots \to A^{-1} \xrightarrow{d} A^{0} \xrightarrow{d} A^{1} \xrightarrow{d} A^{2} \xrightarrow{d} \cdots)$$

  is a dg algebra over a commutative ring k.
- Then every recollement of $\mathrm{D}(A)$ is given by a homological epimorphism $A \to C$ in the homotopy category of dg k-algebras:

$$A \xleftarrow{\sim} B \longrightarrow C, \qquad\qquad C \otimes_B^{\mathbf{L}} C \xrightarrow{\sim} C.$$

- Then $\mathcal{S} = \{X \in \mathrm{D}(A) \mid X \otimes_B^{\mathbf{L}} C = 0\}$ and



The simplification for weak global dimension one

Theorem (Bazzoni, Š.)
If A is a k-algebra with $\mathrm{w.gl.dim} A \leq 1$, *then each homological epimorphism in the homotopy category of dg k-algebras is represented by a homological epimorphism of ordinary rings, $f: A \to C$. Then:*

- $I = \ker(f)$ *is an idempotent two-sided ideal and both $A \to A/I$ and $A/I \to C$ are homological epimorphisms of rings,*
- $\mathrm{w.gl.dim} A/I \leq 1$ *and* $\mathrm{w.gl.dim}\, C \leq 1$.

Moreover, there is a bijective between

1. *homological ring epimorphisms from R (up to equivalence) and*
2. *recollements of $\mathrm{D}(R)$ (up to equivalence).*

Valuation domains

Basic facts

Definition
A valuation domain is a commutative domain A whose ideals are totally ordered.

Theorem

1. *[Silver, 1967] If A is a commutative ring and $A \to C$ is a ring epimorphism, then C is also commutative.*
2. *[Glaz, 1989] A commutative ring satisfies $\mathrm{w.gl.dim} A \leq 1$ if and only if $R_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \mathrm{Spec}(A)$.*

Constructing examples of valuations domain

- Let A be a valuation domain and Q its quotient field. Then the collection of all non-zero finitely generated (=cyclic) submodules of Q forms a totally ordered abelian group $(\Gamma, \cdot, A, \leq^{\mathrm{op}})$, so-called value group of A.

Theorem (Krull, 1932)
Let k be a field and Γ a totally ordered group. There there exists a valuation domain A

- *whose residue field is isomorphic to k and*
- *whose value group is isomorphic to Γ.*

The Zariski spectrum of a valuation domain

Theorem
If A is a valuation domain, then $(\mathrm{Spec}(A), \leq)$ is a totally ordered set which is

1. *order complete (i.e. has all suprema and infima) and*
2. *nowhere dense (i.e. given primes $\mathfrak{p} \lneq \mathfrak{q}$, then there exist primes $\mathfrak{p} \leq \mathfrak{p}_1 \lneq \mathfrak{q}_1 \leq \mathfrak{q}$ with no other primes between $\mathfrak{p}_1 \lneq \mathfrak{q}_1$).*

Moreover, each total ordered set $(P, \leq)$ satisfying these two properties arises as $(\mathrm{Spec}(A), \leq)$ for a valuation domain A.

The compactly generated localizations of a valuation domain

- If A is a valuation domain, it is coherent as a ring, and $\mathrm{mod}(A) = \mathrm{add}(A/(x) \mid x \in A)$ is a hereditary Krull-Schmidt abelian category.
- Moreover, the category of compacts of $\mathrm{D}(A)$ is equivalent to $\mathrm{D}^b(\mathrm{mod}(A))$.

Theorem (Bazzoni-Š.)
Let A be a valuation domain. The homological epimorphisms $A \to C$ corresponding to compactly generated localizations are just the flat ones and they are all of the form $A \to A_{\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Spec}(A)$.

Idempotent ideals

- Suppose that A is a valuation domain and $\mathfrak{i} = \mathfrak{i}^2 \leq A$ is an idempotent ideal. Then $\mathfrak{i}$ is prime.
- On the other hand, the union of a strictly ascending chain $\cdots < \mathfrak{p}_\alpha < \cdots < \mathfrak{p}_\beta < \cdots$ of primes is an idempotent ideal.
- In particular, $(\mathrm{iSpec}(A), \leq)$ is a non-empty $(0 = 0^2!)$ subset of $(\mathrm{Spec}(A), \leq)$ which is closed under suprema.
- It follows that $(\mathrm{iSpec}(A), \leq)$ is order-complete, but I do not know whether it must be nowhere dense (it is in small examples).

Example
Suppose $\mathrm{Spec}(A) = \{0 = \mathfrak{p}_0 < \mathfrak{p}_1 < \cdots < \mathfrak{p}_n = \mathfrak{m}\}$. Then $\mathrm{iSpec}(R)$ can be any subset of $\mathrm{Spec}(A)$ containing $\mathfrak{p}_0$.

The failure of the telescope conjecture for valuation domains

- Suppose $0 \neq \mathfrak{i} = \mathfrak{i}^2 < A$ is a non-trivial idempotent ideal of a valuation domain A.
- Then $A \to A/\mathfrak{i}$ is a homological epimorphism of rings and the only compact object in

$$\mathcal{S}_{A/\mathfrak{i}} = \{X \in \mathsf{D}(A) \mid X \otimes^{\mathbf{L}}_A A/\mathfrak{i} = 0\}$$

is the zero object ([Keller, 1994], essentially the Nakayama lemma).

The classification of smashing localizations

Lemma
If $A \to C$ is a homological epimorphism from a valuation domain A and $\mathfrak{n} \in \mathrm{Spec}(C)$, then $A \to C \to C_{\mathfrak{n}}$ is equivalent to $A \to A_{\mathfrak{p}}/\mathfrak{i}$ from some $\mathfrak{i} \in \mathrm{iSpec}(A)$, $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathfrak{i} \leq \mathfrak{p}$.

Theorem (Bazzoni-Š.)
Let A be a valuation domain. Then there is a bijection between

- *homological epimorphisms $A \to C$ (up to equivalence) and*
- *collections $\mathcal{I} = \{[\mathfrak{i}_j = \mathfrak{i}_j^2, \mathfrak{p}_j] \mid j \in J\}$ of mutually disjoint formal intervals in $\mathrm{Spec}(A)$ which satisfy two properties:*
    1. *"no gaps condition",*
    2. *$\mathcal{I}$ is nowhere dense with the order inherited from $\mathrm{Spec}(A)$.*

The smashing spectrum

Spectral spaces

Definition
A topological space X is spectral if it is

1. quasi-compact,
2. has an open basis consisting of quasi-compact sets, and
3. it is sober (i.e. each irreducible closed set has a generic point).

Theorem (Hochster, 1969)
A space is spectral if and only if it is homeomorphic to $\mathrm{Spec}(A)$ *for a commutative ring* A*.*

The smashing spectrum of a valuation domain

Theorem (?)
Let A be a valuation domain such that $\mathrm{iSpec}(A)$ is nowhere dense. Then the homological epimorphisms starting from A (up to equivalence) are classified by the open sets of a spectral topological space $\mathrm{Smash}(A)$, the smashing spectrum of A.

There is a surjective continuous map $\mathrm{Smash}(A) \to \mathrm{Spec}(A)$, where we consider $\mathrm{Spec}(A)$ with the Thomason topology (Hochster dual to the Zariski topology).