The smashing spectrum of a valuation domain
Seminar on non-commutative motives and telescope-type problems
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November 18, 2020
Motivation
The main goal is to study the localizations of $D(R)$, where $R$ is a non-discrete valuation domain. The talk is based on S. Bazzoni, J. Šťovíček, Smashing localizations of rings of weak global dimension at most one, Adv. Math. 305 (2017).

- The setting is simple enough homologically in order to understand the situation completely in some cases (key fact: $\text{w.gl.dim} R = 1$).
- On the other hand, the setting is complicated enough in that the telescope conjecture fails.
Example (Keller, 1994)

- Let $Q = \overline{\mathbb{C}((x))} = \{\sum_{i=N}^{\infty} c_i x^{\frac{i}{d}} \mid N \in \mathbb{Z}, d > 0, c_i \in \mathbb{C}\}$ — the field of Puiseux series (Newton 1670’s, Puiseux 1850’s).

- Put $A = \{\sum_{i=0}^{\infty} c_i x^{\frac{i}{d}} \mid d > 0, c_i \in \mathbb{C}\}$. This is a valuation domain and $\text{Spec}(A) = \{0, m = m^2\}$.

- There are 5 smashing localizations given by ring epimorphisms, just the first three are compactly generated:

  \[ A \rightarrow 0, \ A \rightarrow Q, \ A \rightarrow A, \ A \rightarrow \mathbb{C}, \ A \rightarrow Q \times \mathbb{C}. \]
Rings of weak global dimension at most one
If $\mathcal{D}$ is a compactly generated triangulated category, we study recollements

\[
\begin{array}{ccc}
S & \overset{j!}{\longrightarrow} & \mathcal{D} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
j_* & \overset{j^*}{\longrightarrow} & \mathcal{D} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
j_* & \overset{i^!}{\longrightarrow} & \mathcal{L}
\end{array}
\]

- A **smashing subcategory** of $\mathcal{D}$ is one of the form $\text{im}(j!)$. In this case $i^*$ induces $\mathcal{D}/S \simeq \mathcal{L}$. 


Theorem (Neeman, 1992) 
If $\mathcal{D}$ is compactly generated triangulated and $S$ is a localizing subcategory generated by a set of objects, then it is a smashing subcategory. In particular, there is a recollement

![Diagram]

Definition
$\mathcal{D}$ satisfies the telescope conjecture if all recollements (up to equivalence) arise in this way.
Suppose that \( D = D(A) \), where
\[
A = (\cdots \to A^{-1} \xrightarrow{d} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots)
\]
is a dg algebra over a commutative ring \( k \).

Then every recollement of \( D(A) \) is given by a homological epimorphism \( A \to C \) in the homotopy category of dg \( k \)-algebras:
\[
A \xleftarrow{\sim} B \xrightarrow{} C, \quad C \otimes_B^L C \xrightarrow{\sim} C.
\]

Then \( S = \{ X \in D(A) \mid X \otimes_B^L C = 0 \} \) and
The simplification for weak global dimension one

Theorem (Bazzoni, Š.)

If $A$ is a $k$-algebra with $\text{w.gl.dim} A \leq 1$, then each homological epimorphism in the homotopy category of dg $k$-algebras is represented by a homological epimorphism of ordinary rings, $f : A \to C$. Then:

- $I = \ker(f)$ is an idempotent two-sided ideal and both $A \to A/I$ and $A/I \to C$ are homological epimorphisms of rings,
- $\text{w.gl.dim} A/I \leq 1$ and $\text{w.gl.dim} C \leq 1$.

Moreover, there is a bijective between

1. homological ring epimorphisms from $R$ (up to equivalence) and
2. recollements of $\text{D}(R)$ (up to equivalence).
Valuation domains
Definition
A valuation domain is a commutative domain $A$ whose ideals are totally ordered.

Theorem
1. [Silver, 1967] If $A$ is a commutative ring and $A \rightarrow C$ is a ring epimorphism, then $C$ is also commutative.
2. [Glaz, 1989] A commutative ring satisfies $w.gl.dim A \leq 1$ if and only if $R_p$ is a valuation domain for each $p \in \text{Spec}(A)$.
Let $A$ be a valuation domain and $Q$ its quotient field. Then the collection of all non-zero finitely generated (=cyclic) submodules of $Q$ forms a totally ordered abelian group $(\Gamma, \cdot, A, \leq^{\text{op}})$, so-called value group of $A$.

**Theorem (Krull, 1932)**

Let $k$ be a field and $\Gamma$ a totally ordered group. There exists a valuation domain $A$

- whose residue field is isomorphic to $k$
- whose value group is isomorphic to $\Gamma$. 
Theorem
If $A$ is a valuation domain, then $(\text{Spec}(A), \leq)$ is a totally ordered set which is

1. order complete (i.e. has all suprema and infima) and
2. nowhere dense (i.e. given primes $p \nleq q$, then there exist primes $p \leq p_1 \nleq q_1 \leq q$ with no other primes between $p_1 \nleq q_1$).

Moreover, each total ordered set $(P, \leq)$ satisfying these two properties arises as $(\text{Spec}(A), \leq)$ for a valuation domain $A$. 
The compactly generated localizations of a valuation domain

- If $A$ is a valuation domain, it is coherent as a ring, and $\text{mod}(A) = \text{add}(A/(x) \mid x \in A)$ is a hereditary Krull-Schmidt abelian category.
- Moreover, the category of compacts of $D(A)$ is equivalent to $D^b(\text{mod}(A))$.

**Theorem (Bazzoni-Š.)**

Let $A$ be a valuation domain. The homological epimorphisms $A \to C$ corresponding to compactly generated localizations are just the flat ones and they are all of the form $A \to A_p$, $p \in \text{Spec}(A)$.
Idempotent ideals

- Suppose that $A$ is a valuation domain and $i = i^2 \leq A$ is an idempotent ideal. Then $i$ is prime.
- On the other hand, the union of a strictly ascending chain $\cdots < p_\alpha < \cdots < p_\beta < \cdots$ of primes is an idempotent ideal.
- In particular, $(i\text{Spec}(A), \leq)$ is a non-empty ($0 = 0^2!$) subset of $(\text{Spec}(A), \leq)$ which is closed under suprema.
- It follows that $(i\text{Spec}(A), \leq)$ is order-complete, but I do not know whether it must be nowhere dense (it is in small examples).

**Example**

Suppose $\text{Spec}(A) = \{0 = p_0 < p_1 < \cdots < p_n = m\}$. Then $i\text{Spec}(R)$ can be any subset of $\text{Spec}(A)$ containing $p_0$. 

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Suppose $0 \neq i = i^2 < A$ is a non-trivial idempotent ideal of a valuation domain $A$. Then $A \to A/i$ is a homological epimorphism of rings and the only compact object in

$$S_{A/i} = \{ X \in D(A) \mid X \otimes^L_A A/i = 0 \}$$

is the zero object ([Keller, 1994], essentially the Nakayama lemma).
The classification of smashing localizations

Lemma
If $A \to C$ is a homological epimorphism from a valuation domain $A$ and $n \in \text{Spec}(C)$, then $A \to C \to C_n$ is equivalent to $A \to A_p/i$ from some $i \in \text{iSpec}(A)$, $p \in \text{Spec}(A)$ such that $i \leq p$.

Theorem (Bazzoni-Š.)
Let $A$ be a valuation domain. Then there is a bijection between

- homological epimorphisms $A \to C$ (up to equivalence) and
- collections $\mathcal{I} = \{[i_j = i_j^2, p_j] | j \in J\}$ of mutually disjoint formal intervals in $\text{Spec}(A)$ which satisfy two properties:
  1. “no gaps condition”,
  2. $\mathcal{I}$ is nowhere dense with the order inherited from $\text{Spec}(A)$.
The smashing spectrum
Definition
A topological space $X$ is **spectral** if it is

1. quasi-compact,
2. has an open basis consisting of quasi-compact sets, and
3. it is sober (i.e. each irreducible closed set has a generic point).

Theorem (Hochster, 1969)
A space is spectral if and only if it is homeomorphic to $\text{Spec}(A)$ for a commutative ring $A$. 
The smashing spectrum of a valuation domain

**Theorem (?)**

Let $A$ be a valuation domain such that $i\text{Spec}(A)$ is nowhere dense. Then the homological epimorphisms starting from $A$ (up to equivalence) are classified by the open sets of a spectral topological space $\text{Smash}(A)$, the *smashing spectrum* of $A$.

There is a surjective continuous map $\text{Smash}(A) \to \text{Spec}(A)$, where we consider $\text{Spec}(A)$ with the Thomason topology (Hochster dual to the Zariski topology).