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Artin algebras and infinitely generated tilting modules

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CONTENTS

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ARTIN ALGEBRAS AND INFINITELY GENERATED TILTING MODULES

Introduction

This thesis is based on the following three papers that have been either published or accepted for publishing (as of August 2007):

(1) J. Šťovíček, *All n-cotilting modules are pure-injective*, Proc. Amer. Math. Soc. **134** (2006), 1891–1897.
(2) J. Šťovíček and J. Trlifaj, *All tilting modules are of countable type*, Bull. Lond. Math. Soc. **39** (2007), 121–132.
(3) S. Bazzoni and J. Šťovíček, *All tilting modules are of finite type*, to appear in Proc. Amer. Math. Soc.

The original goal for this work, three years ago, was to study artin algebras and the relatively new phenomenon of infinitely generated tilting modules over them. Most of the results, nevertheless, turned out to work for modules over much more general rings than artin algebras, often even for modules over any associative ring with unit.

In this survey accompanying the three papers mentioned above, I would like to present some basic facts and motivation, and put the results into a wider context. The scope of representation theory and tilting theory and developments in the last two or three decades are, however, well beyond the scope of this text and would be suitable rather for a monograph. Therefore, I have been forced to leave out many details and also topics that I consider interesting for one or another reason. Beside tens or hundreds of papers on representation theory and tilting theory, there are fortunately also some very useful books which should be considered as default references for this work. By those I mean in particular the textbook [13] for representation theory of artin algebras, and the recent monograph [44] on approximation and realization theory and the collection [4] of introductory and survey papers on tilting theory and related topics.

Proofs are mostly omitted or just sketched in this survey with exception of the last section which contains results that have not been published yet. However, a number of references are given both in this text and in the papers (1)–(3). The text is organized as follows: In the first two sections, basics of representation theory for artin algebras and tilting theory are discussed. In Sections 3 and 4, the results from the papers (1)–(3) are presented. Finally, some additional unpublished results on cotilting approximations are stated and proved in Section 5.

1. Artin algebras

1.1. **Basic properties.** We start by recalling basic properties of a special class of rings—artin algebras. The concept of an artin algebra is a generalization of the one of a finite dimensional algebra over a field and includes for example all finite rings. Although a large part of the tilting theory presented later in this text works in a more general setting, artin algebras are important for several reasons. They have a well developed representation theory and provide abundance of examples of tilting modules. Even though they must satisfy very restrictive properties when viewed as rings, their representation theory is very rich and complex. This is illustrated by the fact that many classical problems can be equivalently restated (and solved) in this language (for example finding canonical forms of pairs of bilinear forms [74, 52], or characterizing quadruples of subspaces of a finite dimensional vector space [43, 55]). Last but not least, the concept of tilting modules originally comes from this setting.

Definition 1.1. An associative ring R with unit is called an *artin algebra* if:

(1) The centre $Z(R)$ of R is an artinian subring; and
(2) R is a finitely generated module over $Z(R)$.

We give just a brief list of basic properties of artin algebras. For more details and most proofs, we refer to the textbook [13].

Any artin algebra R is left and right artinian. Moreover, if X and Y are finitely generated right R-modules, then $\mathrm{Hom}_R(X,Y)$ is a finitely generated $Z(R)$-module in a canonical way. This in particular implies that the endomorphism ring $\mathrm{End}_R(X)$ of a finitely generated right R-module X is an artin algebra, and we get an important consequence: The category mod-R of all finitely generated right R-modules is a Krull-Schmidt category. That is, every module in mod-R decomposes as a finite direct sum of indecomposables, and this decomposition is unique up to isomorphism and permutation of the summands. Hence, to understand the structure of all finitely generated right R-modules, it suffices, in principle, to undestand the structure of the indecomposable ones. Of course, corresponding properties can be obtained for the category R-mod of finitely generated left modules.

The left-right symmetry goes much further for artin algebras. Namely, let J be the minimal injective cogenerator over $Z(R)$ and denote by D the functor $\mathrm{Hom}_{Z(R)}(-,J)$. Then D induces a duality between mod-R and R-mod in the sense that $(\text{mod-}R)^{op}$ is equivalent to R-mod via D. If R is a finite dimensional algebra over a field k, this functor reduces just to the vector space duality $\mathrm{Hom}_k(-,k)$.

1.2. **Algebras over algebraically closed and perfect fields.** An important example of finite dimensional algebras are *path algebras*. Let

k be a field and let Q be a finite quiver where Q_0 is the set of vertices and Q_1 is the set of arrows. Let kQ be the vector space spanned by all paths in Q. Here, we include also the so called *trivial paths* e_i, one for each vertex $i \in Q_0$, which formally both start and end at i and have length 0. Then kQ with multiplication defined on paths by composition when possible and as 0 otherwise has naturally a structure of an associative k-algebra with the unit $1 = \sum_{i \in Q_0} e_i$. Although kQ is not finite dimensional unless Q is acyclic, we can obtain a finite dimensional factor. Let I be an *admissible ideal* of kQ, that is, there is an integer $t \geq 2$ such that $(kQ_1)^2 \supseteq I \supseteq (kQ_1)^t$ where kQ_1 is the ideal of kQ generated by the arrows. Then kQ/I is a finite dimensional algebra. On the other hand, every finite dimensional algebra over an algebraically closed field is obtained this way up to Morita equivalence by a fundamental observation due to Gabriel:

Theorem 1.2. *Let k be an algebraically closed field and R be a finite dimensional k-algebra. Then R is Morita equivalent to kQ/I for a finite quiver Q and an admissible ideal I.*

Regarding homological properties of path algebras, it is well-known that kQ/I is hereditary if and only if $I = 0$. Hence, we get the following characterization:

Corollary 1.3. *Let k be an algebraically closed field and R be a finite dimensional hereditary k-algebra. Then R is Morita equivalent to kQ for some finite acyclic quiver Q.*

It is often useful to view left kQ/I-modules as representations of Q with relations from I. A *representation* is a tuple $(V_i \mid i \in Q_0)$ of vector spaces together with a collection $(f_\alpha \mid \alpha \in Q_1)$ of k-linear maps between corresponding spaces such that the relations from I are satisfied. Given a left R-module M, we can get such a representation by taking the vector spaces $(e_i M \mid i \in Q_0)$ together with the multiplication maps $(\alpha \cdot - \mid \alpha \in Q_1)$. It is not difficult to see that given a representation, we can reverse this process. Actually, the category of left kQ/I-modules and the category of representations of Q with relations I are equivalent. Similarly, we can view right kQ/I-modules as representations of the opposite quiver Q^{op} with relations I^{op}.

When the base field k is not algebraically closed, the situation is more complex. Instead of representations of quivers, one can study a generalization—representations of k-species as introduced by Gabriel [41]. A *k-species* is a finite collection of division rings, $(D_i \mid i \in I)$, which are finite dimensional over k, together with a collection of D_i-D_j-bimodules $({}_iM_j \mid i, j \in I)$, all centralized by k and finite dimensional over k. A *representation* $(V_i, {}_i\varphi_j)$ of a k-species assigns to each $i \in I$ a left D_i-space V_i, and to each $i, j \in I$ a left D_i-homomorphism ${}_i\varphi_j : {}_iM_j \otimes_{D_j} V_j \to V_i$. These representations have been studied for example in [30, 31, 29], where in [30] it has been proved that they correspond to modules over the tensor algebra of the species.

The situation is especially nice when k is a perfect field (a finite field, for example), where a complete analogue of Theorem 1.2 and Corollary 1.3 with species instead of quivers was obtained in [20, Theorem 1.3.12].

1.3. **Representation type.** A crucial property that describes behaviour of modules over an artin algebra R is so called *representation type* of R. All algebras are fundamentally divided into two classes—representation finite and representation inifinite algebras:

- R is *representation finite* if there are, up to isomorphism, only finitely many indecomposable finitely generated right R-modules. If this is the case, then every (even infinitely generated) module is isomorphic to a direct sum of finitely generated indecomposables. Furthermore, if R is a finite dimensional algebra over an algebraically closed field, then the category of finitely generated indecomposable modules can be fully described combinatorially by means of the Auslader-Reiten quiver of R.
- Otherwise, R is called *representation infinite*. In this case, the situation is much more complicated. For example, there are always infinitely generated indecomposable R-modules. This follows from the Auslander-Reiten theory and compactness of the Ziegler spectrum—for more details and references see [27, §2.3 and §2.5].

Hereditary path algebras kQ of finite representation type have been originally characterized by Gabriel [42] as precisely those where the underlying graph of Q is a finite disjoint union of Dynkin diagrams. This resembles very much the characterization of finite dimensional semisimple Lie algebras and has motivated Bernstein, Gelfand and Ponomarev [21] to reprove Gabriel's theorem using reflection functors, which, though with other terminology, is probably the first application of tilting theory.

When dealing with representation infinite algebras oven an algebraically closed field k, two subclasses are usually considered:

- A finite dimensional algebra R is called *tame* if for all dimensions $d \geq 1$, its indecomposable right modules of dimension d all belong to finitely many one-parametric families. More precisely, for each d there are finitely many $k[x]$-R-bimodules $M_1, \dots, M_{n_d}$ which are free of rank d when viewed as left $k[x]$-modules, and such that every indecomposable right R-module of dimension d is isomorphic to $k[x]/(x-\lambda) \otimes_{k[x]} M_i$ for some $1 \leq i \leq n_d$ and $\lambda \in k$.
- A finite dimensional algebra R is called *wild* if there is a $k\langle x, y\rangle$-R-bimodule M which is free of finite rank when viewed as a

left $k\langle x,y\rangle$-module, and such that the functor $F = - \otimes_{k\langle x,y\rangle} M : \text{mod-}k\langle x,y\rangle \to \text{mod-}R$ preserves indecomposability (i.e. if M is indecomposable, so is FM) and reflects isomorphism classes (i.e. if $FM \cong FN$, then $M \cong N$). Here we denote by mod-$k\langle x,y\rangle$ the category of all finite dimensional right modules over the algebra $k\langle x,y\rangle$ of all polynomials in two non-commuting indeterminates.

If R is wild, S is another finite dimensional k-algebra and $\dim_k S = n$, then there is a composition of two exact fully faithful embeddings of categories

$$\text{mod-}S \to \text{mod-}k\langle x_1, \dots, x_n\rangle \to \text{mod-}k\langle x,y\rangle,$$

the existence of the second proved for example in [22]. In view of this, we can identify mod-S with some exact (possibly non-full) subcategory of mod-R. This is one of the reasons why full classification of finitely generated modules over wild algebras is considered hopeless—the category mod-R is at least as complex as the category of finitely generated right modules over *any* other finite dimensional k-algebra.

The fundamental result concerning infinite representation type is Drozd's theorem proving dichotomy between Tame and Wild:

Theorem 1.4 ([32, 28])**.** *Let k be an algebraically closed field and R be a finite dimensional k-algebra. Then R is either tame or wild, but not both.*

1.4. **Auslander-Reiten formulas.** As we shall see, all tilting and many cotilting classes of right modules over an artin algebra R are determined as $\operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{S},-)$ and $\operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{S})$, respectively, where $\mathcal{S}$ is a set of finitely generated right R-modules. Therefore, the functors $\operatorname{Ext}^1_R(X,-)$ and $\operatorname{Ext}^1_R(-,X)$ with X finitely generated play a very important role in this text.

In this context, there are important formulas, so called *Auslander-Reiten* formulas, relating Hom_R and Ext^1_R over any artin algebra R. We need to introduce some notation before:

Definition 1.5. Let R be a ring and X,Y be right R-modules. We denote by $\overline{\operatorname{Hom}}_R(X,Y)$ the quotient group of $\operatorname{Hom}_R(X,Y)$ by the subgroup of homomorphisms from X to Y which factor through an injective module. Similarly, denote by $\underline{\operatorname{Hom}}_R(X,Y)$ the quotient of $\operatorname{Hom}_R(X,Y)$ by the homomorphisms which factor through a projective module.

Let R be and artin algebra, $X \in \text{mod-}R$, and $P_1 \to P_0 \to X \to 0$ be a minimal projective presentation of X. Then the *Auslander-Reiten translation* of X, denoted τX, is defined as the kernel of

$$D\operatorname{Hom}_R(P_1,R) \to D\operatorname{Hom}_R(P_0,R).$$

Similarly, let $0 \to X \to I_0 \to I_1$ be a minimal injective presentation of X. The *inverse Auslander-Reiten translation*, $\tau^- X$, is then defined as the cokernel of $\mathrm{Hom}_R(DI_0, R) \to \mathrm{Hom}_R(DI_1, R)$.

It is a well known fact that in mod-R the translations τ and τ^- define mutually inverse bijections between indecomposable non-projective and indecomposable non-injective modules, and on morphisms they induce functorial isomorphisms $\underline{\mathrm{Hom}}_R(X, \tau^- Y) \cong \overline{\mathrm{Hom}}_R(\tau X, Y)$. Now we can state the Auslander-Reiten formulas:

Theorem 1.6 ([12, 50])**.** *Let R be an artin algebra and let $X, Y \in$ Mod-R, X finitely generated. Then there are following $Z(R)$-isomorphisms functorial in both X and Y:*

1. $D \,\mathrm{Ext}^1_R(X, Y) \cong \overline{\mathrm{Hom}}_R(Y, \tau X)$*;*
2. $\mathrm{Ext}^1_R(Y, X) \cong D\underline{\mathrm{Hom}}_R(\tau^- X, Y)$*.*

We conclude this section with a corollary for the case when X has projective or injective dimension at most one, that is the case corresponding to 1-tilting or 1-cotilting modules, respectively. It follows by [13, IV.1.16] that in this case we have the Hom-groups themselves in the formulas instead of the factors of the Hom-groups. This is very good news for computation since it is usually much easier to deal with Hom-groups than Ext-groups.

Corollary 1.7. *Let R be an artin algebra and let $X, Y \in$ Mod-R, X finitely generated. Then the following hold:*

1. *If* $\mathrm{pd}\, X \leq 1$*, then* $D \,\mathrm{Ext}^1_R(X, Y) \cong \mathrm{Hom}_R(Y, \tau X)$*.*
2. *If* $\mathrm{id}\, X \leq 1$*, then* $\mathrm{Ext}^1_R(Y, X) \cong D \,\mathrm{Hom}_R(\tau^- X, Y)$*.*

2. Tilting and cotilting modules

2.1. **Historical notes.** The concept of tilting modules has originally come from the representation theory of finite dimensional algebras. The first tilting functors ever used, if we do not count Morita duality, were probably BGP reflection functors [43, 21]. They were used by Bernstein, Gelfand and Ponomarev to reprove Gabriel's characterization of representation finite hereditary path algebras. Dlab and Ringel [31, 29] extended in 1974 the use of reflection functors to arbitrary representation finite hereditary artin algebras. The concept of reflection functors was then generalized by Auslander, Platzek and Reiten [10] in 1979. Finally, the term *tilting module* was defined by Brenner and Butler [23] in 1980 and simplified to the form that is usual today by Happel and Ringel [46].

Let us recall the definition of a tilting module from [46]. We will call this kind of tilting modules, in view of the generalizations studied further in this text, classical tilting modules. We remind that when M is a module, add$M$ stands for the class of all direct summands of finite direct sums of M.

Definition 2.1. Let R be an artin algebra. A right R-module T is called a *classical tilting module* if it satisfies:

(1) $\operatorname{pd} T \leq 1$;
(2) $\operatorname{Ext}^1_R(T,T) = 0$; and
(3) there is an exact sequence $0 \to R_R \to T_0 \to T_1 \to 0$ such that $T_0, T_1 \in \operatorname{add}T$.

The motivation for the name "tilting" coined by Brenner and Butler comes from looking at the action of tilting functors on Grothendieck groups. We refer for missing definitions to [13]. Suppose R is hereditary with n non-isomorphic simple modules, T is a tilting R-module and $S = \operatorname{End}_R(T)$. Then the functor $\operatorname{Hom}_R(T,-) : \text{mod-}R \to \text{mod-}S$ induces an isomorphism between the Grothendieck groups of R and S, both isomorphic to $\mathbb{Z}^n$. If one tries to identify the groups under this isomorphism, their positive cones will overlap but differ—the positive cone of S is tilted compared to that of R

The main objective of the tilting theory in this context has always been to construct category equivalences and to have a way of comparing properties of module categories over different algebras. If T is a tilting module over any artin algebra R and $S = \operatorname{End}_R(T)$, the fundamental results by Brenner and Butler and Happel and Ringel say that there are two equivalences between large subcategories of mod-R and mod-S:

$$\begin{array}{rcl} \operatorname{Hom}_R(T,-) : & \operatorname{Ker}\operatorname{Ext}^1_R(T,-) \leftrightarrows \operatorname{Ker}\operatorname{Tor}^S_1(-,T) & : - \otimes_S T \\ \operatorname{Ext}^1_R(T,-) : & \operatorname{Ker}\operatorname{Hom}_R(T,-) \leftrightarrows \operatorname{Ker}(- \otimes_S T) & : \operatorname{Tor}^S_1(-,T) \end{array}$$

These results have been extended in many ways. Miyashita [54] has considered tilting modules of arbitrary finite projective dimensions.

Colby and Fuller [26] have proved the Brenner-Butler theorem for arbitrary rings. Happel [45] has discovered the induced equivalence of derived bounded categories and Rickard [60, 61] has generalized this result and developed Morita theory for derived categories. Since then, these results have been used in many areas of mathematics including finite and algebraic group theory, algebraic geometry, and algebraic topology. Much more information and many more details and references can be found in the recently issued collection [4]. Historical account on the results mentioned above is especially in the introduction and in the contributions by Brüstle, Keller, and Ringel.

2.2. **Infinitely generated tilting and cotilting modules.** The main topic of this thesis is a generalization of tilting modules following different lines—namely infinitely generated tilting modules. For a more comprehensive survey on this topic including many related results and applications we refer to [72, 67]. One can generalize Miyashita tilting modules as follows:

Definition 2.2. Let R be an associative ring with unit. A right R-module T is called a *tilting module* if it satisfies:

(1) $\operatorname{pd} T < \infty$;
(2) $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and each set I; and
(3) there is an exact sequence

$$0 \to R_R \to T_0 \to T_1 \to \cdots \to T_k \to 0$$

such that $T_0, T_1, \ldots, T_k \in \operatorname{Add}T$. Here, $\operatorname{Add}T$ stands for the class of all direct summands of arbitrary (infinite) direct sums of copies of T.

If we define a tilting module in such a way, we cannot expect to have category equivalences in general. In fact, existence of a category equivalence in the sense above forces the tilting module to be finitely generated [25, §2], [71]. But there is a remedy for this loss—a very general approximation theory related to cotorsion pairs, which is discussed in the following section. The crucial fact is that those approximations always exist, but they need not be finitely generated; thus they are "invisible" for the classical tilting theory. At this point, also infinitely generated cotilting modules come into play:

Definition 2.3. Let R be an associative ring with unit. A right R-module C is called a *cotilting module* if it satisfies:

(1) $\operatorname{id} C < \infty$;
(2) $\operatorname{Ext}^i_R(C^I, C) = 0$ for each $i \geq 1$ and each set I; and
(3) there is an exact sequence

$$0 \to C_k \to \cdots \to C_1 \to C_0 \to W \to 0$$

such that W is an injective cogenerator for the category Mod-R and $C_0, C_1, \ldots, C_k \in \operatorname{Prod}C$. Here, $\operatorname{Prod}C$ denotes the class of all direct summands of arbitrary (infinite) products of copies of C.

The definition of cotilting modules is syntactically dual compared to that of tilting modules, but cotilting modules are not anymore mere duals of tilting modules as in the classical setting. This is not the case even when we consider some more general duality functor, such as $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, which does exist for any ring. In fact, a dual of an infinitely generated tilting module is always a cotilting module of cofinite type (see Section 4), but there are cotilting modules that are not of cofinite type over any valuation domain with a non-trivial idempotent prime ideal [15].

2.3. **Approximations and cotorsion pairs.** As it seem to be a hopeless task to classify all modules over a general ring (and even over a general artin algebra as explained above!), there have been several successful attempts to study approximations of general modules by modules from a suitable class. Such classes can be for example the classes of all projective, injective, or flat modules, but also many other classes defined in a different way.

The term approximation refers to one of the following notions studied among others in [75, 37, 44]:

Definition 2.4. Let R be a ring, M a right R-module, and $\mathcal{C}$ a class of right R-modules. Then:

(1) A homomorphism $f : M \to C$ with $C \in \mathcal{C}$ is called a $\mathcal{C}$-*preenvelope* provided that any other homomorphism $f' : M \to C'$ with $C' \in \mathcal{C}$ factors through f.
(2) A $\mathcal{C}$-preenvelope $f : M \to C$ is called *special* if f is injective and
$$\mathrm{Ext}^1_R(\mathrm{Coker}\, f, \mathcal{C}) = 0.$$
(3) A $\mathcal{C}$-preenvelope $f : M \to C$ is called a $\mathcal{C}$-*envelope* if the following minimality property is satisfied: If $g \in \mathrm{End}_R(C)$ such that the following diagram commutes
$$\begin{array}{ccc} M & \xrightarrow{f} & C \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & C \end{array}$$
then g is an automorphism.
(4) A homomorphism $f : C \to M$ with $C \in \mathcal{C}$ is called a $\mathcal{C}$-*precover* provided that any other homomorphism $f' : C' \to M$ with $C' \in \mathcal{C}$ factors through f.
(5) A $\mathcal{C}$-precover $f : C \to M$ is called *special* if f is surjective and
$$\mathrm{Ext}^1_R(\mathcal{C}, \mathrm{Ker}\, f) = 0.$$

(6) A $\mathcal{C}$-precover $f : C \to M$ is a *$\mathcal{C}$-cover* if the following minimality property is satisfied: If $g \in \mathrm{End}_R(C)$ such that $fg = f$, then g is an automorphism.

Independently, essentially the same concepts have been introduced by Auslander, Reiten, and Smalø for categories of finitely generated modules over artin algebras [14, 11]. Instead of preenvelopes and precovers, they use the terms *left* and *right approximations*, respectively.

Note that the approximations (1), (3), (4) and (6) above are weaker versions of $\mathcal{C}$-reflections and $\mathcal{C}$-coreflections. Recall that a *$\mathcal{C}$-reflection*, or *$\mathcal{C}$-coreflection*, is the unit of the left, or the counit of the right adjoint to the inclusion functor $\mathcal{C} \subseteq \text{Mod-}R$, respectively, if such an adjoint exists. If for instance $f : M \to C$ is a $\mathcal{C}$-preenvelope, we only require that the induced maps $\mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ be epimorphisms for each $C' \in \mathcal{C}$ instead of isomorphisms. This in particular means that we can by no means expect that $\mathcal{C}$-preenvelopes and $\mathcal{C}$-precovers will be unique. $\mathcal{C}$-envelopes and $\mathcal{C}$-covers are, on the other hand, by definition unique up to isomorphism if they exist. However, there is still a big difference from reflections and coreflections—envelopes and covers might not be functorial [1].

Of special interest are classes that provide for approximations of every module:

Definition 2.5. Let R be a ring and $\mathcal{C}$ a class of right R-modules. Then $\mathcal{C}$ is called *preenveloping* (*special preenveloping, enveloping, precovering, special precovering, covering*) if every $M \in \text{Mod-}R$ has a $\mathcal{C}$-preenvelope (special $\mathcal{C}$-preenvelope, $\mathcal{C}$-envelope, $\mathcal{C}$-precover, special $\mathcal{C}$-precover, $\mathcal{C}$-cover, respectively).

Abundance of such classes originate from cotorsion pairs, a notion introduced by Salce [64], originally in the setting of abelian groups:

Definition 2.6. Let R be a ring. A pair $(\mathcal{A}, \mathcal{B})$ of subclasses in Mod-R is called a *cotorsion pair* if $\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, \mathcal{B})$ and $\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{A}, -)$.

If $\mathcal{S}$ is a subclass of Mod-R, then the cotorsion pair *generated by* $\mathcal{S}$ is the unique cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{S}, -)$. Dually, the cotorsion pair *cogenerated by* $\mathcal{S}$ is the unique cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, \mathcal{S})$.

As to the latter definition, the reader shall observe that in the papers on which this thesis is based ([19, 68, 69]), the notions of a *generated* and a *cogenerated* cotorsion pair are swapped. This is due to the recently issued monograph [44] and the corresponding change in the community agreement on using these terms.

Before stating the crucial results, we first put the concepts of special and minimal approximations into a relation:

Lemma 2.7 ([73])**.** *Let R be a ring, M be a right module, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then:*

(1) *If M has a $\mathcal{B}$-envelope f, then f is special. In particular, if $\mathcal{B}$ is enveloping then $\mathcal{B}$ is special preenveloping.*
(2) *If M has an $\mathcal{A}$-cover f, then f is special. In particular, if $\mathcal{A}$ is covering then $\mathcal{A}$ is special precovering.*

Now, we can state the fundamental approximation existence results obtained by Eklof and Trlifaj. The results are proved also in [44].

Theorem 2.8. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair.*

- [36] *If the cotorsion pair is generated by a set of modules, then $\mathcal{A}$ is special precovering and $\mathcal{B}$ is special preenveloping.*
- [35] *If the cotorsion pair is cogenerated by a class of pure-injective modules, then $\mathcal{A}$ is covering and $\mathcal{B}$ is enveloping.*

Note that the additional assumptions on the cotorsion pair in Theorem 2.8 are necessary. Eklof and Shelah proved in [34] that it is independent of the axioms of ZFC whether the class of all Whithead groups $\mathcal{W} = \operatorname{Ker} \operatorname{Ext}^1_{\mathbb{Z}}(-, \mathbb{Z})$ is precovering or not. For more details we refer to [33].

We also refer to [44] for a much more comprehensive treatment of approximations connected to cotorsion pairs and for a number of applications of the theory.

2.4. **Tilting and cotilting classes.** As we have seen above, cotorsion pairs give many examples of classes with interesting approximation properties. The link to tilting theory is the notion of tilting and cotilting cotorsion pairs:

Definition 2.9. Let R be a ring and T be a tilting module over R. The *tilting class* corresponding to T is defined as $\mathcal{T} = \bigcap_{i \geq 1} \operatorname{Ker} \operatorname{Ext}^i_R(T, -)$. If we, moreover, denote $\mathcal{S} = \operatorname{Ker} \operatorname{Ext}^1_R(-, \mathcal{T})$, then the cotosion pair $(\mathcal{S}, \mathcal{T})$ is called the *tilting cotorsion pair* corresponding to T.

Dually, if C is a cotilting R-module, then the *cotilting class* corresponding to C is defined as $\mathcal{C} = \bigcap_{i \geq 1} \operatorname{Ker} \operatorname{Ext}^i_R(-, C)$. If we denote $\mathcal{D} = \operatorname{Ker} \operatorname{Ext}^1_R(\mathcal{C}, -)$, then the cotorsion pair $(\mathcal{C}, \mathcal{D})$ is called the *cotilting cotorsion pair* corresponding to C.

Since tilting and cotilting classes are always parts of a cotorsion pair, we can apply Theorem 2.8 to prove existence of tilting and cotilting approximations for completely general tilting and cotilting modules, respectively. For tilting modules this is easy since the tilting cotorsion pair corresponding to a tilting module T is generated by syzygies of T. For cotilting modules, the new results in this thesis come into play. If C is a cotilting module, then C is pure-injective by [68, Theorem 13], and so are all cosyzygies of C by the well known result of Auslander's that a cosyzygy of a pure-injective module is pure-injective again. Therefore, Theorem 2.8 applies. A weaker version of the following theorem, with cotilting special precovers instead of covers, has been obtained before in [8, 3].

Theorem 2.10. *Let R be a ring. Then the following hold true:*

1. *Every tilting class in Mod-R is special preenveloping.*
2. *Every cotilting class in Mod-R is covering.*

Tilting and cotilting classes enjoy also other important properties, which will be discussed in Section 3. At this point we only introduce a characterization of tilting and cotilting cotorsion pairs in order to be able to identify several examples in the next subsection.

Theorem 2.11. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair.*

1. *$(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair if and only if $\operatorname{Ext}^2_R(\mathcal{A}, \mathcal{B}) = 0$, all modules in $\mathcal{A}$ have finite projective dimension, and $\mathcal{B}$ is closed under arbitrary direct sums.*
2. *$(\mathcal{A}, \mathcal{B})$ is a cotilting cotorsion pair if and only if $\operatorname{Ext}^2_R(\mathcal{A}, \mathcal{B}) = 0$, all modules in $\mathcal{B}$ have finite injective dimension, and $\mathcal{A}$ is closed under arbitrary products.*

Proof. (1). If $\operatorname{Ext}^2_R(\mathcal{A}, \mathcal{B}) = 0$, it is not difficult to see that $\mathcal{A}$ is closed under taking syzygies. This is well known to be equivalent to the fact that $\operatorname{Ext}^i_R(\mathcal{A}, \mathcal{B}) = 0$ for each $i \geq 1$. Since $\mathcal{A}$ is closed under taking direct sums, there is a non-negative integer n such that $\operatorname{pd} M \leq n$ for each $M \in \mathcal{A}$. Hence [69, Theorem 2] in this volume applies.

(2). Similarly as in (1), we infer that $\operatorname{Ext}^i_R(\mathcal{A}, \mathcal{B}) = 0$ for each $i \geq 1$ and there is some $n \geq 0$ such that $\operatorname{id} M \leq n$ for each $M \in \mathcal{B}$. It is then enough to apply [66, Theorem 2.4] to finish the proof. □

2.5. **Examples of tilting and cotilting modules.** To illustrate the concept of infinitely generated tilting modules, we list some examples over various rings. For more details, we refer either to the references given for each example of the the survey [72].

Example 2.12. *Fuchs tilting modules* [39, 40]. Let R be a commutative domain and S a multiplicative subset of R. Let $\delta_S = F/G$ where F is the free module with the basis formed by all finite sequences $(s_0, s_1, \ldots, s_{n-1})$ such that $n \geq 0$ and $s_0, s_1, \ldots, s_{n-1} \in S$. For $n = 0$, we have the empty sequence $e = ()$. The submodule G is defined to be generated by the elements of the form $(s_0, s_1, \ldots, s_n)s_n = (s_0, s_1, \ldots, s_{n-1})$ where $n \geq 0$ and $s_0, s_1, \ldots, s_{n-1} \in S$.

The module $\delta = \delta_{R\setminus\{0\}}$ has been introduced by Fuchs. Facchini [38] has proved that δ is tilting of projective dimension ≤ 1. The general case δ_S comes from [40]. The modules δ_S are tilting of projective dimension ≤ 1 for any multiplicative set S and they are called *Fuchs tilting modules*. The corresponding 1-tilting class is

$$\operatorname{Ker} \operatorname{Ext}^1_R(\delta_S, -) = \{M \in \text{Mod-}R \mid Ms = M \text{ for all } s \in S\},$$

the class of all S-divisible modules.

Example 2.13. *Tilting modules over Dedekind domains* [44, §6.2]. For Dedekind domains there is an easier construction than Fuchs tilting modules. Let R be a Dedekind domain, Q its quotient field, and P a set of maximal ideals. If we consider the exact sequence

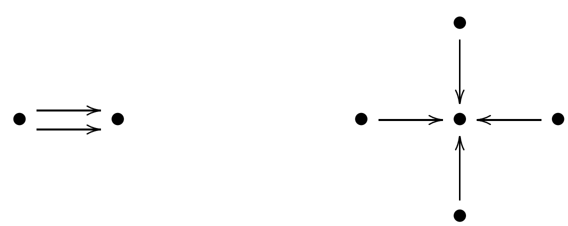
$$0 \to R \to Q \xrightarrow{\pi} \bigoplus_{p \in \mathrm{mSpec}(R)} E(R/p) \to 0$$

and take $Q_P = \pi^{-1}\big(\bigoplus_{p\in P} E(R/p)\big)$, then $T_P = Q_P \oplus \bigoplus_{p\in P} E(R/p)$ is a tilting module of projective dimension ≤ 1 and the corresponding tilting class is

$$\mathrm{Ker}\,\mathrm{Ext}^1_R(T_P, -) = \{M \in \text{Mod-}R \mid \mathrm{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\}.$$

Example 2.14. *Ringel tilting modules* [62, 63]. In his classical work [62], Ringel has discovered many analogies between modules over Dedekind domains and tame hereditary algebras. The analogies extend to the setting of infinitely generated tilting modules.

Let k be a field, Q be a quiver without oriented cycles whose underlying graph is an Euclidean diagram, and let $R = kQ$. If k is an algebraically closed field, this is equivalent to say that R is a finite dimensional basic connected tame hereditary algebra over k. We refer to [13] for missing terminology. This setting includes the Kronecker problem and the 4-subspace problem mentioned in Section 1 when Q is the quiver a) or b) depicted in the following figure, respectively:



a) Kronecker quiver b) 4-subspace quiver

Let G be the generic module. Then $S = \mathrm{End}_R(G)$ is a skew-field and $\dim_S G = n < \infty$. Denote by $\mathcal{P}$ the set of all tubes in the Auslander-Reiten quiver of R. If $\alpha \in \mathcal{P}$ is a homogenous tube, we denote by P_α the corresponding Prüfer module. If $\alpha \in \mathcal{P}$ is not homogenous, denote by P_α the direct sum of all Prüfer modules corresponding to the rays in α. Then there is an exact sequence

$$0 \to R_R \to G^n \xrightarrow{\pi} \bigoplus_{\alpha\in\mathcal{P}} P_\alpha^{(n_\alpha)} \to 0$$

where $n_\alpha > 0$ for all $\alpha \in \mathcal{P}$. The sum $T = G \oplus \bigoplus_{\alpha\in\mathcal{P}} P_\alpha$ is a tilting module of projective dimension ≤ 1, and the corresponding tilting class is the class of all *Ringel divisible modules*, that is, of all modules D such that $\mathrm{Ext}^1_R(M, D) = 0$ for each finitely generated regular module M.

More generally, let $\mathcal{Q} \subseteq \mathcal{P}$ and denote $G_{\mathcal{Q}} = \pi^{-1}\big(\bigoplus_{\alpha\in\mathcal{Q}} P_\alpha\big)$. Then the module $T_{\mathcal{Q}} = G_{\mathcal{Q}} \oplus \bigoplus_{\alpha\in\mathcal{Q}} P_\alpha$ is tilting of projective dimension ≤ 1, and we get different tilting classes for different subsets $\mathcal{Q}$.

Example 2.15. *Lucas tilting modules* [49, 53]. Let k be a field, Q be a connected wild quiver without oriented cycles, and let $R = kQ$. Then R is a basic connected wild hereditary algebra. For missing terminology we refer to [47, 13].

The class $\mathcal{D}$ of all Ringel divisible modules over R is again tilting by Theorem 2.11. But the tilting module is rather different. It can be chosen countable dimensional and obtained by a construction of Lucas.

More constructions of tilting modules are available in literature. Some of them will be listed in the next subsection devoted to applications of infinitely generated tilting theory. Another family of tilting modules over hereditary artin algebras has been constructed in the forthcoming paper [48]. Infinitely generated cotilting modules over tame hereditary algebras have been analysed in [24]. Finally, a systematic general way of obtaining tilting and cotilting classes (and in principle tilting and cotilting modules, even though they may be qiute difficult to compute explicitly) is explained the the part of Section 3 on finite type.

2.6. **Applications.** Here, we include a brief list of some recent applications of infinitely generated tilting modules, corresponding tilting classes, and the related approximation theory. There is no guarantee of completeness of the list. Closer description of the applications is beyond the scope of this volume, so the reader should follow references for more details.

1. Iwanaga-Gorenstein rings. Bass' Finitistic Dimension Conjectures (equality of the little and the big finitistic dimensions and thier finiteness) were confirmed for the class of Iwanaga-Gorenstein rings by Angeleri Hügel, Herbera and Trlifaj [7]. The result has been achieved by proving that the class of Gorenstein injective modules in the sense of [37] is a tilting class over these rings.

2. General Matlis localizations. The classical result by Matlis on divisible modules over commutative integral domains R such that $\operatorname{pd} Q = 1$ (Q is the quotient field) has been generalized in [6]. The generalization deals with modules over commutative rings with zero-divisors and over non-commutative rings, and their localizations with respect to left Ore sets of regular elements. If R is a ring and S is a left Ore set of regular elements, then $\operatorname{pd} S^{-1}R \leq 1$ if and only if the right R-module $S^{-1}R \oplus S^{-1}R/R$ is tilting of projective dimension ≤ 1. If R is commutative, these conditions are further equivalent to $S^{-1}R/R$ being a direct sum of countably presented modules. Shorter proof of the last equivalence has been given by the author and Jan Trlifaj in [70].

3. Baer modules over tame hereditary algebras. The structure of Baer modules over tame hereditary algebras have been analyzed in [5]. Let R be a tame hereditary artin algebra. In view of analogies between Dedekind domains and tame hereditary algebras described by Ringel [62], one can define the class $\mathcal{T}$ of *Ringel torsion modules* as the class generated by the finitely generated regular R-modules, and the class of *Baer modules* as $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{T})$.

Unlike in the commutative case, where the Baer modules have recently been proved to be projective [2], the situation in the tame hereditary case is considerably more complicated. Let $(\mathcal{L}, \mathcal{P})$ be the torsion pair such that $\mathcal{L}$ is defined by having no non-zero homomorphisms to finitely generated preprojective modules. Then $\mathcal{L}$ is a tilting class and, therefore, has an associated tilting module L, called a *Lukas tilting module*. The modules in $\mathcal{P}$ are called (infinitely generated) preprojective modules or, alternatively, $\mathcal{P}^\infty$-torsion-free modules. Then the modules in $\mathcal{P}$ are shown to bijectively correpond to equivalence classes of Baer modules under a certain equivalence relation defined using L. Since the modules in $\mathcal{P}$ are considered too complex to be classified, there is just a little hope that one can fully classify Baer modules.

3. Properties of tilting and cotilting classes

In this section, we describe various important properties of tilting and cotilting classes. New results from this volume will be substantially used.

3.1. Finite and cofinite type.

Even though we allow tilting modules themselves to be infinitely generated, we will show that tilting classes are parametrized by resolving classes of strongly finitely presented modules. If R is a ring, we call a module *strongly finitely presented* if it possesses a projective resolution consisting of finitely generated projective modules. We will denote the category of strongly finitely presented right (or left) modules by mod-R (or R-mod, respectively). Note that for artin algebras the notions of finitely generated and strongly finitely presented modules coincide, so our notation is consistent with the one used before.

We will start with defining classes of finite type. Note that definitions of this concept may vary in literature.

Definition 3.1. A class $\mathcal{C} \subseteq \text{Mod-}R$ is said to be of *finite type* if there is a set $\mathcal{S}$ of strongly finitely presented modules such that $\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{S}, -)$.

Now we can state one of the main results in this volume. For tilting modules of projective dimension ≤ 1, it has been obtained by Bazzoni, Herbera, Eklof, and Trlifaj [17, 18]. The general version comes from the papers [69, 19] included in this thesis and has been proved by Bazzoni, Trlifaj, and the author using induction on projective dimension of tilting modules. A more staightforward argument for this result is also given in the recent preprint [65].

Theorem 3.2 ([19, Theorem 4.2])**.** *Let R be a ring (associative, with unit) and $\mathcal{T}$ be a tilting class of right R-modules. Then $\mathcal{T}$ is of finite type.*

We can give an analogous definition of classes of cofinite type:

Definition 3.3. A class $\mathcal{C} \subseteq \text{Mod-}R$ is said to be of *cofinite type* if there is a set $\mathcal{S}$ of strongly finitely presented left R-modules such that $\mathcal{C} = \operatorname{Ker}\operatorname{Tor}^R_1(-, \mathcal{S})$.

There is, unfortunately, no general analogue of Theorem 3.2. Examples of cotilting modules that are not of cofinite type have been given [15]. Nevertheless, the following useful partial result has been proved by Trlifaj:

Theorem 3.4 ([72, Theorem 4.14])**.** *Let R be a right noetherian ring such that the classes of projective and flat left R-modules coincide* *(this holds when R is (i) left perfect or (ii) left hereditary or (iii) 1-Gorenstein, for example). Then every cotilting class in Mod-R corresponding to a cotilting module of injective dimension ≤ 1 is of cofinite type.*

Let us recall the notion of a resolving class in mod-R:

Definition 3.5. A subclass $\mathcal{S}$ of mod-R is called *resolving* if it contains R_R and is closed under extensions and kernels of epimorphisms.

The following theorem provides us with examples of tilting and cotilting classes in abundance:

Theorem 3.6. *Let R be a ring. Then there is a bijective correspondence among:*

(1) *tilting classes in Mod-R,*
(2) *resolving subclasses of mod-R with bounded projective dimension,*
(3) *cotilting classes of cofinite type in R-Mod.*

Proof. Given a tilting cotorsion pair $(\mathcal{U}, \mathcal{T})$ corresponding to a tilting module T, it is not difficult to see that $\mathcal{U} \cap \text{mod-}R$ is resolving and projective dimension of any module in $\mathcal{U} \cap \text{mod-}R$ is at most $\operatorname{pd} T$. When $\mathcal{S}$ is resolving with bounded projective dimension, then $\operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{S}, -)$ is a tilting class by Theorem 2.11. These correspondences are mutually inverse by Theorem 3.2 and [9, Theorem 2.3].

Similarly, the correspondence between (2) and (3) works as follows: If $\mathcal{S} \subseteq \text{mod-}R$ is resolving with bounded projective dimension, then $\mathcal{C} = \operatorname{Ker}\operatorname{Tor}^R_1(\mathcal{S}, -)$ is cotilting of cofinite type. If $\mathcal{C}$ is a cotilting class of cofinite type, then $\mathcal{S} = \operatorname{Ker}\operatorname{Tor}^R_1(-, \mathcal{C}) \cap \text{mod-}R$ is resolving in mod-R and projective dimension of modules in $\mathcal{S}$ is bounded by injective dimension of any cotilting module corresponding to $\mathcal{C}$. We refer to [7, Theorem 2.2] for more details. □

The problem to find a corresponding tilting module for a given tilting class is, however, not a trivial task. In principle, the construction is as follows (see [3]): Suppose we are given a resolving class $\mathcal{S} \subseteq \text{mod-}R$ such that each module in $\mathcal{S}$ has projective dimension at most n, and denote $\mathcal{T} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{S}, -)$ the corresponding tilting class. Then we can compute iterated special $\mathcal{T}$-preenvelopes—start with a preenvelope $f_0 : R \to T_0$ of R and, at each subsequent step, take a preenvelope $f_i : \operatorname{Coker} f_{i-1} \to T_i$. It follows that $\operatorname{Coker} f_{n-1} \in \mathcal{T}$, so we can take $T_n = \operatorname{Coker} f_{n-1}$. Now, $T = \bigoplus_{i=0}^n T_i$ is a tilting module and $\mathcal{T}$ is the tilting class corresponding to T. The main computational problem is, however, to find the preenvelopes. One can follow the proofs of [36, Theorem 2] or [44, Theorem 3.2.1], but it may be very difficult to write down the result explicitly.

3.2. **General module dualities.** Before giving more properties properties of tilting or cotilting classes, we need the notion of a duality between left and right modules over a fixed ring.

Definition 3.7. Let S be a commutative ring and R an S-algebra (that is, there is a ring homomorphism $S \to R$ whose image lies in the centre of R). Then the functor $D = \mathrm{Hom}_S(-, J)$ is called a *module duality functor* if:

(1) J is an injective cogenerator for Mod-S,
(2) for any S-module M, the cardinalities of M and DM are either equal or both infinite.

Two prominent examples of module dualities are:

(1) An arbitrary ring R has a unique structure of a $\mathbb{Z}$-algebra. If we take $J = \mathbb{Q}/\mathbb{Z}$, then $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is a module duality fuctor in the sense above. The second property of the definition follows from the fact that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_n$.
(2) If R is an algebra over a field k, then we can take $J = k$ and the vector space duality $D = \mathrm{Hom}_k(-, k)$ is a module duality in the sense above.

Both of the dualities above play an essential role. The vector space duality $\mathrm{Hom}_k(-, k)$ is an important tool in representation theory and is suitable for computations. But its range is limited to algebras over a field. The other duality, $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, is available for any ring, but it may be much less convenient for computations. For most purposes, it does not really matter which particular duality we use in what follows as long as it is in accordance with Definition 3.7. When we deal with finite dimensional algebras, the second one will be assumed if not stated otherwise. The important aspect here, usually not mentioned in literature, is part (2) of the definition which ensures that the double dual of a module is elementarily equivalent to the original module (for $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ this has been proved in [57, Proposition 1.6]). Hence, our duality is well-behaved with respect to model theory.

Let us summarize a few important properties of module duality functors. The following proposition of course holds true if we change left modules for right modules and vice versa in its statement.

Proposition 3.8. *Let R be a ring and D be a module duality functor for R. Then*

(1) *D is a contravariant functor Mod-$R \to R$-Mod and R-Mod $\to$ Mod-R.*
(2) *$M = 0$ if and only if $DM = 0$ for each $M \in R$-Mod.*
(3) *For each $M \in$ Mod-R, $N \in R$-Mod, and $i \geq 0$, there is a natural isomorphism*

$$\mathrm{Ext}^i_R(M, DN) \cong D\,\mathrm{Tor}^R_i(M, N) \cong \mathrm{Ext}^i_R(N, DM).$$

(4) *For each $M, N \in$ Mod-R such that M is finitely presented, there is a natural isomorphism*

$$\operatorname{Hom}_R(M, D^2N) \cong D^2 \operatorname{Hom}_R(M, N)$$

(5) $0 \to K \to L \to M \to 0$ *is an exact sequence of right R-modules if and only if* $0 \to DM \to DL \to DK \to 0$ *is an exact sequence of left R-modules.*

(6) $0 \to K \to L \to M \to 0$ *is a pure exact sequence of right R-modules if and only if* $0 \to DM \to DL \to DK \to 0$ *is a split exact sequence of left R-modules.*

(7) *DM is a pure-injective left R-module for any $M \in$ Mod-R.*

(8) *The natural evaluation map $\eta_M : M \to D^2M$ defined as*

$$\eta_M(m)(f) = f(m)$$

is an elementary embedding for any $M \in$ Mod-R.

Proof. (1) is well known, and (2) and (5) follow directly from the fact that J in Definition 3.7 is an injective cogenerator. For (3) we refer to [37]; for $i = 0$ it follows by the well-known adjoint formula between Hom_R and the tensor product. For (6) and (7) we refer to [44, §1.2].

(4). [37, Theorem 3.2.11] yields a functorial isomorphism

$$D \operatorname{Hom}_R(M, N) \cong M \otimes_R DN.$$

Now, we obtain the wanted functorial isomorphism by applying the Hom-tensor formula:

$$\operatorname{Hom}_R(M, D^2N) \cong D(M \otimes_R DN) \cong D^2 \operatorname{Hom}_R(M, N)$$

(8). We adapt accordingly the proof of [57, 1.7]. It is well known that η_M is a pure embedding [44, 1.2.15], so it is enough to prove that M and D^2M are elementarily equivalent [56, Corollary 2.26]. By refining the arguments in [27, §2.1] and using the Baur-Monk Theorem in its full strength [56, Corollary 2.18], we get the following criterion (see also [56, §12.1]): Two modules $Z, W \in$ Mod-R are elementarily equivalent if and only if for each functor F of the form $\operatorname{Coker} \operatorname{Hom}_R(f, -)$ where $f : X \to Y$ is a homomorphism between finitely presented right R-modules, the cardinalities of $F(Z)$ and $F(W)$ are either the same or both infinite. Now, we have the following isomorphisms by (4) and (5):

$$\operatorname{Coker} \operatorname{Hom}_R(f, D^2M) \cong \operatorname{Coker} D^2 \operatorname{Hom}_R(f, M) \cong D^2 \operatorname{Coker} \operatorname{Hom}_R(f, M).$$

Hence, M and D^2M are elementarily equivalent by the criterion above and (2) of Definition 3.7. □

3.3. **Definability.** In this section, we will discuss some nice closure properties of tilting and cotilting classes which have a close connection to model theory of modules. Let us recall that *primitive positive formulas* (shortly *pp-formulas*) are the first-order language formulas of the theory of right R-modules which are of the form $(\exists \bar{y})(\bar{x}A = \bar{y}B)$ for some matrices A, B over R. For proof of the following theorem, we refer to [56], [27, §2.3], and [76, Section 1]:

Theorem 3.9. *Let R be a ring and $\mathcal{C}$ be a class of right R-modules. Then the following are equivalent:*

(1) *$\mathcal{C}$ is closed under taking arbitrary products, direct limits, and pure submodules;*
(2) *$\mathcal{C}$ is defined by vanishing of some set of functors of the form* $\operatorname{Coker}\operatorname{Hom}_R(f, -)$ *where $f : X \to Y$ are homomorphisms between finitely presented right R-modules;*
(3) *$\mathcal{C}$ is axiomatizable in the first-order language of right R-modules and closed under direct sums and summands;*
(4) *$\mathcal{C}$ is defined in the first order language of R-modules by satisfying some implications $\varphi(\bar{x}) \to \psi(\bar{x})$ where $\varphi(\bar{x})$ and $\psi(\bar{x})$ are pp-formulas.*

Definition 3.10. Let R be a ring. A class $\mathcal{C}$ of right R-modules is called *definable* if it satisfies one of the four equivalent conditions of Theorem 3.9.

Theorem 3.9(3) implies that definable classes are closed under taking elementarily equivalent modules. As an immediate consequence of Proposition 3.8(8), we get the following:

Corollary 3.11 ([51, Lemma 4.4])**.** *Let R be a ring and D a module duality functor. Then for any definable class $\mathcal{C}$ of right R-modules and any $M \in$ Mod-R, $M \in \mathcal{C}$ if and only if $D^2M \in \mathcal{C}$.*

Now, another of the main results in this thesis follows. It is based on results from [19, 68].

Theorem 3.12. *Let R be a ring and $\mathcal{C}$ be a class of right R-modules. If $\mathcal{C}$ is tilting or cotilting, then $\mathcal{C}$ is definable.*

Proof. If $\mathcal{C}$ is tilting, the definability is an easy consequence of the fact that $\mathcal{C}$ is of finite type. If $\mathcal{C}$ is cotilting, we infer, using [68, Theorem 13] as in the proof of Theorem 2.10, that $\mathcal{C}$ is the left-hand class of a cotorsion pair generated by a set of pure-injective modules. Hence, $\mathcal{C}$ is closed under pure-epimorphic images. Since $\mathcal{C}$ is also closed under taking direct sums and kernels of epimorphisms, it follows that $\mathcal{C}$ is closed under taking direct limits and pure-submodules. Finally, $\mathcal{C}$ is closed under products by [16, Lemma 3.5]. $\square$

To conclude this section, we show that the correspondence between tilting and cofinite type cotilting classes from Theorem 3.6 can be realized directly on the classes without the intermediary step of Theorem 3.6(2).

Definition 3.13. Let R be a ring, D a module duality functor, and $\mathcal{C}$ a definable class of right (or left) R-modules. Then the *dual class of* $\mathcal{C}$, denoted by $\mathcal{C}^*$, is defined as the class of all pure submodules of modules of the form DM, $M \in \mathcal{C}$.

It is not difficult to check using Proposition 3.8 that $\mathcal{C}^*$ is definable. Now, we state the result:

Proposition 3.14. *Let R be a ring. Then $\mathcal{T} \mapsto \mathcal{T}^*$ and $\mathcal{C} \mapsto \mathcal{C}^*$ are mutually inverse bijective correspondences between tilting classes $\mathcal{T}$ in Mod-R and cotilting classes $\mathcal{C}$ of cofinite type in R-Mod. Moreover, these are the same correspondences as those in Theorem 3.6.*

Proof. Note first that $\mathcal{T}^{**} = \mathcal{T}$ and $\mathcal{C}^{**} = \mathcal{C}$ by Corollary 3.11 and Proposition 3.8. Now, when $\mathcal{S}$ is a resolving class with bounded projective dimension such that $\mathcal{T} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{S}, -)$ and $\mathcal{C} = \operatorname{Ker}\operatorname{Tor}^R_1(\mathcal{S}, -)$, we get by Proposition 3.8(2) and (3) that $M \in \mathcal{C}$ if and only if $DM \in \mathcal{T}$. It follows that $\mathcal{C}^* = \mathcal{T}$. Indeed, if $X \in \mathcal{C}^*$, then X is a pure submodule of DM for some $M \in \mathcal{C}$, so $X \in \mathcal{T}$. On the other hand, if $X \in \mathcal{T}$, so is D^2X. Hence $DX \in \mathcal{C}$, and X is in $\mathcal{C}^*$ because it is a pure submodule of D^2X. Finally, we get $\mathcal{C} = \mathcal{C}^{**} = \mathcal{T}^*$ by passing to the dual classes. $\square$

4. Structure of tilting and cotilting modules

4.1. **Pure-injectivity of cotilting modules.** As we have seen in the previous sections, one result has turned to be quite important at several places. Namely the fact that every cotilting module is pure-injective. The proof in full generality has been finished by the author in [68], but it is based on work of many people before. For more on historical account we refer to the introduction of [68] in this volume.

Theorem 4.1 ([68, Theorem 13])**.** *Let R be a ring and C be a cotilting right R-module. Then C is pure-injective.*

4.2. **Other structure results.** Since the main tool for studying tilting and cotilting modules is to study their corresponding tilting and cotilting classes, we often get structure results up to equivalence of tilting or cotilting modules:

Definition 4.2. Let R be a ring and T and T' be tilting modules. Then T and T' are called *equivalent* if they determine the same tilting classes. Similarly, two cotilting modules C and C' are said to be *equivalent* if they determine the same cotilting classes.

The following statement gives a convenient criterion for testing this equivalence:

Proposition 4.3 ([3, 16])**.** *The following hold true for an arbitrary ring R:*

(1) *Two tilting modules $T, T' \in$ Mod-R are equivalent if and only if* $\mathrm{Add}T = \mathrm{Add}T'$.
(2) *Two cotilting modules $C, C' \in$ Mod-R are equivalent if and only if* $\mathrm{Prod}C = \mathrm{Prod}C'$.

Let us recall that if M is a module and $\mathcal{S}$ a class modules, then M is called *$\mathcal{S}$-filtered* if there is a continuous well-ordered ascending chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M such that $M_0 = 0$, $M_\sigma = M$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to a module in $\mathcal{S}$ for each $\alpha < \sigma$.

We get the following structure theorem for general infinitely generated tilting modules:

Proposition 4.4 ([19, Remark 4.3])**.** *Let R be a ring, T be a tilting module, and $\mathcal{S}$ be the resolving subclass of mod-R corresponding to T via Theorem 3.6. Then there is an $\mathcal{S}$-filtered tilting module T' eqiuvalent to T.*

Cotilting modules, on the other hand, can be sometimes up to equivalence expressed as products of indecomposable modules. This is for instance always true for finitely generated cotilting modules over artin algebras. In the infinitely generated case, the topic is discussed in [72, §3] where also the following result is proved. Let us recall that a pure-injective module is called *superdecomposable* or *continuous* if it has no non-zero indecomposable direct summands.

Proposition 4.5 ([72, Theorem 3.5])**.** *Let R be a ring with no superdecomposable pure-injective right modules (this holds if R is (i) a Dedekind domain or (ii) a tame hereditary artin algebra, for example). Then for any cotilting module $C \in$ Mod-R, there is an equivalent cotilting module C' which is a product of indecomposable modules.*

Let us conclude this discussion by noting that in both Propositions 4.4 and 4.5, we really need to pass to an equivalent tilting or cotilting module. Without this, the statements would not be true.

4.3. **Duality on the level of modules.** We will once again briefly revisit the duality between tilting and cotilting classes from Theorem 3.6. This time, we focus on how it acts on tilting and cotilting modules or, more precisely, their equivalence classes. We say that a cotilting module is of *cofinite type* if its corresponding cotilting class is of cofinite type.

Proposition 4.6. *Let R be a ring and D a module duality functor for R. Then DT is cotilting whenever $T \in$ Mod-R is tilting, and $T \mapsto DT$ defines a bijective correspondence between equivalence classes of tilting right R-modules and equivalence classes of cotilting left R-modules of cofinite type. Moreover, this correspondence is the same as that in Theorem 3.6.*

Proof. This has been proved in [7, Proposition 2.3] for the duality functor $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, but the proof reads equally well for any module duality functor D. $\square$

5. Cotilting covers and reflections

In this section, we will take a closer look at covers and reflections induced by cotilting classes. We recall that a class of modules $\mathcal{C} \subseteq \text{Mod-}R$ is *reflective* if the full embedding $\mathcal{C} \subseteq \text{Mod-}R$ has a left adjoint. A *reflection* of a module $M \in \text{Mod-}R$ is then the corresponding component of the unit of this adjunction. In other words, a reflection of M is a homomorphism $f : M \to C$ such that $C \in \mathcal{C}$ and any other morphism $f' : M \to C'$ with $C' \in \mathcal{C}$ has a *unique* factorization through f. Therefore, reflections can be viewed as a functorial version of envelopes.

5.1. **Cotilting reflections.** We will give another characterization of the cotilting classes corresponding to cotilting modules of injective dimension ≤ 2. It will follow that every such class is reflective. In this way, we are provided with a large family of non-trivial reflective classes by Theorem 3.6. First, we recall some general facts.

Definition 5.1. Let $\mathcal{A}$ be a category with arbitrary products and $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory. Then $\mathcal{C}$ is said to be *locally initially small* in $\mathcal{A}$ (in the sense of [58]) if for every object $A \in \mathcal{A}$, there is a set $\mathcal{C}_A \subseteq \mathcal{C}$ such that every morphism $A \to C$ with $C \in \mathcal{C}$ factors through a product of objects of $\mathcal{C}_A$.

Then we have the following criteria for a class being preenveloping or reflective:

Proposition 5.2 ([58, Theorem 3.3])**.** *Let R be a ring and $\mathcal{C}$ be a class of right R-modules. Then the following assertions are equivalent:*

(1) *$\mathcal{C}$ is preenveloping,*
(2) *$\mathcal{C}$ is locally initially small in Mod-R and every product of modules from $\mathcal{C}$ is a direct summand in a module from $\mathcal{C}$.*

Proposition 5.3 ([59, Corollary 3.2])**.** *Let R be a ring and $\mathcal{C}$ be a class of right R-modules. Then the following assertions are equivalent:*

(1) *$\mathcal{C}$ is reflective,*
(2) *$\mathcal{C}$ is preenveloping and closed under equalizers,*
(3) *$\mathcal{C}$ is locally initially small in Mod-R and closed under equalizers and products.*

Note that $\mathcal{C}$ being closed under direct products and equalizers is exactly the same as $\mathcal{C}$ being closed under taking limits in the sense of theory of categories. This is due to the classical result that every limit can be constructed by means of products and equalizers. Notice also that a class $\mathcal{C}$ being closed under equalizers in a category of modules is equivalent to the condition that $\mathcal{C}$ is closed under taking kernels of any homomorphism between modules in $\mathcal{C}$. In the case of a left hand class of a cotorsion pair, the condition of being closed under equalizers can also be equivalently restated as follows:

Lemma 5.4. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod-R. Then the following are equivalent:*

1. *$\mathcal{A}$ is closed under equalizers,*
2. $\operatorname{Ext}^2(\mathcal{A}, \mathcal{B}) = 0$ *and every $B \in \mathcal{B}$ has injective dimension at most* 2.

Proof. (1) $\implies$ (2). By (1), $\mathcal{A}$ is clearly closed under kernels of epimorphisms, which by dimension shifting immediately implies $\operatorname{Ext}^2(\mathcal{A}, \mathcal{B}) = 0$. Next, let $B \in \mathcal{B}$, M be an arbitrary module, and

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

be a projective presentation of M. Then $\operatorname{Ker} f \in \mathcal{A}$ since $P_0, P_1 \in \mathcal{A}$. Thus, $\operatorname{Ext}^3_R(M, B) \cong \operatorname{Ext}^1_R(\operatorname{Ker} f, B) = 0$ and $\operatorname{id} B \leq 2$.

(2) $\implies$ (1). Let $f : X \to Y$ be a homomorphism with $X, Y \in \mathcal{A}$. Note first that $\operatorname{Ext}^2(\mathcal{A}, \mathcal{B}) = 0$ implies that $\mathcal{A}$ is closed under taking syzygies and, consequently, $\operatorname{Ext}^i(\mathcal{A}, \mathcal{B}) = 0$ for each $i \geq 1$. Then for any $B \in \mathcal{B}$, we have $\operatorname{Ext}^1_R(\operatorname{Ker} f, B) \cong \operatorname{Ext}^3_R(\operatorname{Coker} f, B) = 0$ using a dimension shifting argument. Hence $\operatorname{Ker} f \in \mathcal{A}$ and $\mathcal{A}$ is closed under equalizers. □

The following lemma is not difficult to prove but it has important consequences:

Lemma 5.5 ([58, Proposition 2.8])**.** *Let R be a ring and $\mathcal{C}$ be a class closed under pure submodules (i.e. any definable class). Then $\mathcal{C}$ is locally initially small in Mod-R.*

As a corollary, we get a handy criterion for a class to be preenveloping:

Corollary 5.6. *Let R be a ring and $\mathcal{C}$ be a definable class. Then $\mathcal{C}$ is preenveloping.*

Note, however, that definable classes are not enveloping in general. If R is a commutative domain, then the class of divisible R-modules is always definable. On the other hand, the regular module R does not have a divisible envelope unless R is a Matlis domain, that is, unless projective dimension of the quotient field Q of R is at most 1 [44, Corollary 6.3.18].

Now, we can characterize the cotorsion pairs which correspond to cotilting modules of injective dimension ≤ 2:

Theorem 5.7. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod-R. Then the following statements are equivalent:*

1. *$\mathcal{A}$ is the cotilting class corresponding to a cotilting module of injective dimension ≤ 2.*
2. *$\mathcal{A}$ is reflective.*
3. *$\mathcal{A}$ is closed under direct products and equalizers.*

(4) $\mathrm{Ext}^2_R(\mathcal{A},\mathcal{B})=0$, *all modules in* $\mathcal{B}$ *have injective dimension* ≤ 2, *and* $\mathcal{A}$ *is closed under arbitrary products.*

Proof. (1) $\iff$ (4) is a straightforward consequence of Theorem 2.11. (3) $\iff$ (4) is an immediate consequence of Lemma 5.4. (2) $\implies$ (3) follows immediately by Proposition 5.3. Finally, for $(1,3)$ $\implies$ (2) we first recall that any cotilting class is definable, hence locally initially small by Lemma 5.5. Thus, $\mathcal{A}$ is reflective by Proposition 5.3. □

As a consequence, we can give a direct characterization of the classes induced by cotilting modules of injective dimension ≤ 2, avoiding the notion of a cotorsion pair. It is a generalization of an analogous result for injective dimension one [8, Theorem 2.5], with "torsion-free" replaced by "reflective":

Theorem 5.8. *Let* R *be a ring and* $\mathcal{C}$ *be a class of right* R*-modules. Then the following statements are equivalent:*

(1) $\mathcal{C}$ *is the cotilting class corresponding to a cotilting module of injective dimension* ≤ 2,
(2) $\mathcal{C}$ *is a special covering reflective class,*
(3) $\mathcal{C}$ *is a special precovering class closed under direct products and equalizers.*

Proof. (1) $\implies$ (2). $\mathcal{C}$ is reflective by Theorem 5.7 and provides for special covers by Theorem 2.10 and Lemma 2.7.

(2) $\implies$ (3). This follows immediately from Proposition 5.3.

(3) $\implies$ (1). Note that since $\mathcal{C}$ is special precovering and closed under direct summands, it must contain all projective modules. Let W be an injective cogenerator, $f : U_0 \to W$ be a special $\mathcal{C}$-precover of W, and $g : U_1 \to \operatorname{Ker} f$ be a special $\mathcal{C}$-precover of $\operatorname{Ker} f$. In this way, we obtain an exact sequence

$$0 \to U_2 \to U_1 \to U_0 \to W \to 0$$

Then $U_2 = \operatorname{Ker} g \in \mathcal{C}$ since $\mathcal{C}$ is closed under equalizers. Moreover, it is easy to see that $U_0, U_1, U_2 \in \operatorname{Ker}\mathrm{Ext}^1_R(\mathcal{C},-)$ and, since $\mathcal{C}$ is closed under kernels of epimorphisms and contains all projectives, also $U_0, U_1, U_2 \in \operatorname{Ker}\mathrm{Ext}^i_R(\mathcal{C},-)$ for each $i \geq 2$.

Let us denote $U = U_0 \oplus U_1 \oplus U_2$. Then $U \in \mathcal{C} \cap \bigcap_{i\geq 1} \operatorname{Ker}\mathrm{Ext}^i_R(\mathcal{C},-)$ and, since $\mathcal{C}$ is closed under products, $\mathrm{Ext}^i_R(U^I, U) = 0$ for each $i \geq 1$ and every set I. We can deduce by a similar argument as in the proof of Lemma 5.4 that every module in $\operatorname{Ker}\mathrm{Ext}^1_R(\mathcal{C},-)$ has injective dimension at most 2. In particular, U is a cotilting module such that $\operatorname{id} U \leq 2$.

It remains to prove that $\mathcal{C}$ is the cotilting class corresponding to U. To this end, assume that $\mathcal{D}$ is the cotilting class for U. Then clearly $\mathcal{C} \subseteq \mathcal{D}$. On the other hand, $M \in \mathcal{D}$ if and only if there is an exact sequence $0 \to M \to U^\kappa \to U^\kappa$ for some cardinal κ, [16, Proposition 3.6]. But then $M \in \mathcal{C}$ since $U^\kappa \in \mathcal{C}$ and $\mathcal{C}$ is closed under equalizers. □

5.2. **Finitely presented modules in cotilting classes.** The aim of the last few paragraphs is to give a criterion under which all modules in a cotilting class $\mathcal{C}$ are obtained as direct limits of finitely presented modules in $\mathcal{C}$. This is well-known to be true for instance when R is a right noetherian ring and $\mathcal{C}$ is induced by a cotilting module of injective dimension ≤ 1 (and is, therefore, a torsion-free class). This fact has actually been an important part of the proof of Theorem 3.4 given in [72].

We will state the result in a more general setting for all preenveloping classes closed under direct limits. But, as we know by Theorem 3.12 and Corollary 5.6, any cotilting class has this property.

Theorem 5.9. *Let R be a ring and $\mathcal{C}$ be a preenveloping class of right R-modules which is closed under direct limits. Then the following are equivalent:*

(1) *Any module in $\mathcal{C}$ is a direct limit of finitely presented modules from $\mathcal{C}$.*
(2) *Any finitely presented module M has a $\mathcal{C}$-preenvelope $M \to C_M$ such that C_M is finitely presented.*

Proof. Note first that $\mathcal{C}$ being closed under direct limits implies that $\mathcal{C}$ is closed under direct summands. Hence $\mathcal{C}$ is closed under taking arbitrary products by Proposition 5.2. Using direct limits once again, we deduce that $\mathcal{C}$ is closed under arbitrary direct sums as well.

(1) $\Longrightarrow$ (2). Let M be finitely presented and $g : M \to D$ be a $\mathcal{C}$-preenvelope of M. Then, using (1) and Lenzing's characterization of direct limits of finitely presented modules [44, Lemma 1.2.9], g factors through a map $f : M \to C$ such that $C \in \mathcal{C}$ is finitely presented. It is easy to check that f is a $\mathcal{C}$-preenvelope.

(2) $\Longrightarrow$ (1). Let $U \in \mathcal{C}$. Then $U = \varinjlim U_i$ for some finitely presented modules U_i, $i \in I$. The direct limit induces a pure epimorphism $p : \bigoplus_i U_i \to U$. Let $U_i \to C_i$ be $\mathcal{C}$-preenvelopes of U_i such that C_i are all finitely presented. Then the corresponding coproduct map $h : \bigoplus_i U_i \to \bigoplus_i C_i$ is a $\mathcal{C}$-preenvelope of $\bigoplus_i U_i$, and there is, therefore, a pure epimorphism $q : \bigoplus C_i \to U$ such that $p = qh$. Thus, using Lenzing's characterization again, we infer that U is a direct limit of finitely presented modules from $\mathcal{C}$. ∎

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I.

ALL n-COTILTING MODULES ARE PURE-INJECTIVE

Abstract

We prove that all n-cotilting R-modules are pure-injective for any ring R and any $n \geq 0$. To achieve this, we prove that ${}^{\perp_1}U$ is a covering class whenever U is an R-module such that ${}^{\perp_1}U$ is closed under products and pure submodules.

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ALL n-COTILTING MODULES ARE PURE-INJECTIVE

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1. Introduction

Tilting theory has been developed as an important tool in the representation theory of algebras. In that context, tilting modules are usually assumed to be finite dimensional. However, some of the results have recently been extended to general modules over arbitrary associative unital rings, with interesting applications to finitistic dimension conjectures (see [2] and [15]). In contrast to the finite dimensional case, cotilting modules form a larger class in general than duals of tilting modules, [6].

So a natural question arises whether each cotilting module is at least pure-injective, that is, a direct summand in a dual module (where duals are considered in the sense of modules of characters for general rings, or vector space duals for algebras over a field). An affirmative answer has important consequences: for example, each cotilting class is then a covering class, [9].

The pure-injectivity of all 1-cotilting modules was first proved in the particular setting of abelian groups, and modules over Dedekind domains, as a consequence of their classification by Eklof, Göbel and Trlifaj in [10] and [9].

The crucial step towards a general solution was the proof of pure-injectivity of all 1-cotilting modules over any ring by Bazzoni, [4]. In [5], she was able to prove pure-injectivity of all n-cotilting modules, $n \geq 0$, modulo one of the following conjectures where (B) is weaker then (A):

- (A) If U is an R-module such that ${}^{\perp_1}U$ is closed under products and pure submodules, then ${}^{\perp_1}U$ is closed under direct limits.
- (B) If U is an R-module such that ${}^{\perp_1}U$ is closed under products and pure submodules, then ${}^{\perp_1}U$ is a special precovering class.

Recently, Conjecture (A) has been proved for countable rings and for divisible modules U over Prüfer domains by Bazzoni, Göbel and Strüngmann in [7]. A stronger version of Conjecture (B) was proved for

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any ring, but under the additional set theoretic assumption of Gödel's axiom of constructibility, by Šaroch and Trlifaj in [14].

In the present paper, we prove Conjecture (A) in ZFC, thus proving that all n-cotilting modules over any ring are pure-injective.

2. Preliminaries

Let R be a unital associative ring. All the modules will be left R-modules. For a class of modules $\mathcal{C}$ and $i \geq 1$, denote by ${}^{\perp_i}\mathcal{C}$ the class of all modules X such that $\mathrm{Ext}^i_R(X, C) = 0$ for all $C \in \mathcal{C}$. Dually, $\mathcal{C}^{\perp_i} = \{X \mid \mathrm{Ext}^i_R(C, X) = 0 \text{ for all } C \in \mathcal{C}\}$. We will write ${}^{\perp_i}U$ instead of ${}^{\perp_i}\{U\}$ for a single module U. Note that then ${}^{\perp_{i+1}}U = {}^{\perp_i}U'$ where U' is a cosyzygy of U.

A (non-strictly) increasing chain of sets $(S_\alpha \mid \alpha < \lambda)$ indexed by ordinals less than λ is called *smooth* if $S_\mu = \bigcup_{\alpha<\mu} S_\alpha$ for all limit ordinals $\mu < \lambda$. A smooth chain $(M_\alpha \mid \alpha < \lambda)$ of submodules of a module M is called a *filtration* of M if $M_0 = 0$ and $M = \bigcup_{\alpha<\lambda} M_\alpha$.

The following lemma is well-known (see eg. [8, Proposition XII.1.14]):

Lemma 1. *Let M, U be modules such that M has a filtration $(M_\alpha \mid \alpha < \lambda)$ with $M_{\alpha+1}/M_\alpha \in {}^{\perp_1}U$ for all $\alpha < \lambda$. Then $M \in {}^{\perp_1}U$.*

Let $\mathcal{C}$ be a class of modules. Then a homomorphism $f : C \to M$ is called a *special $\mathcal{C}$-precover* of M if f is epic and $\mathrm{Ker}\, f \in \mathcal{C}^{\perp_1}$. The class $\mathcal{C}$ is called *special precovering* if every module has a special $\mathcal{C}$-precover. The term comes from the fact that whenever $f : C \to M$ is a special precover and $g : C' \to M$ is any homomorphism such that $C' \in \mathcal{C}$, then g factorizes through f. Therefore, special precovers are indeed special instances of precovers as defined for example in [16]. A special $\mathcal{C}$-precover f is called a *$\mathcal{C}$-cover* if in addition $g : C \to C$ is an automorphism whenever $fg = f$. A *covering class* is defined in an obvious way.

A module U is called *n-cotilting*, where $n \geq 0$ is a natural number, if:

(1) $\mathrm{inj.dim}\, U \leq n$,
(2) $\mathrm{Ext}^i_R(U^\kappa, U) = 0$ for all $i \geq 1$ and all cardinals κ,
(3) There is an injective cogenerator W and an exact sequence $0 \to U_m \to \cdots \to U_1 \to U_0 \to W \to 0$ such that all U_j's are direct summands of some products of copies of U for all $0 \leq j \leq m$.

A class $\mathcal{A}$ is *n-cotilting* if $\mathcal{A} = \bigcap_{i\geq 1} {}^{\perp_i}U$ for some n-cotilting module U. In addition, we have adopted the following notation: Let M be a module. Then $PE(M)$ denotes the pure-injective hull of M.

Let $(M_\alpha \mid \alpha < \lambda)$ be a family of modules indexed by ordinal numbers less than λ. Then $\prod^b_{\alpha<\lambda} M_\alpha$ denotes the (pure) submodule of the direct product formed by the elements with a bounded support in λ. When $M_\alpha \cong M$ for all $\alpha < \lambda$, the corresponding "bounded power" is denoted by $M^{<\lambda}$.

Let M be a module, I a set, and let κ be a cardinal number. Then the submodule of M^I consisting of the elements with supports of cardinality $< \kappa$ is denoted $M^{[I;\kappa]}$.

3. Special embeddings into pure-injective modules

The aim of this section is to embed a module into a pure-injective module in such a way that we know more about the structure of the cokernel.

Lemma 2. *Let R be a ring and M a module. Then there is an increasing (non-smooth) chain of modules M_λ indexed by ordinal numbers, and homomorphisms $S_\lambda : \prod_{\alpha<\lambda} M_\alpha \to M_\lambda$, such that*

(a) *$M_0 = M$, $M_{\lambda+1} = M_\lambda$ for each ordinal λ,*
(b) *$M_\lambda / \bigcup_{\alpha<\lambda} M_\alpha \cong \prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha$ for each limit ordinal λ,*
(c) *The embeddings $M_\mu \subseteq M_\lambda$ are pure for each $\mu < \lambda$,*
(d) *The restrictions $S_\lambda \restriction M_\alpha : M_\alpha \to M_\lambda$ to any direct summand of the product $\prod_{\alpha<\lambda} M_\alpha$ are just the inclusions $M_\alpha \subseteq M_\lambda$ (that is, each S_λ extends the summation map $\bigoplus_{\alpha<\lambda} M_\alpha \to M_\lambda$).*
(e) *$S_\lambda \restriction \prod_{\alpha<\mu} M_\alpha = S_\mu$ for each $\mu < \lambda$.*

Proof. We will construct the modules M_λ by induction. By (a), $M_0 = M$ and $S_0 = 0$. If $\lambda = \mu + 1$, then $M_\lambda = M_\mu$ and $S_\lambda : (\prod_{\alpha<\mu} M_\alpha) \oplus M_\mu \to M_\lambda$ is just the coproduct homomorphism of S_μ and $id : M_\mu \to M_\lambda$.

Now, let λ be a limit ordinal. By induction hypothesis, $\tilde{S} = \bigcup_{\alpha<\lambda} S_\alpha$ is a well-defined homomorphism $\prod^b_{\alpha<\lambda} M_\alpha \to \bigcup_{\mu<\lambda} M_\mu$. Let us define M_λ and S_λ by the following push-out:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod^b_{\alpha<\lambda} M_\alpha & \xrightarrow{\subseteq} & \prod_{\alpha<\lambda} M_\alpha & \longrightarrow & \prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{S}} & & \downarrow{\scriptstyle S_\lambda} & & \| & & \\
0 & \longrightarrow & \bigcup_{\mu<\lambda} M_\mu & \xrightarrow{\subseteq} & M_\lambda & \longrightarrow & \prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha & \longrightarrow & 0
\end{array}$$

Then (b), (d), (e) are obvious. Moreover, $\bigcup_{\mu<\lambda} M_\mu$ is a pure submodule of M_λ, since the upper left horizontal map is a pure inclusion, thus (c) follows. □

Lemma 3. *Let R be a ring and M a module. Let λ be an ordinal,*

$$\sum_{j\in J} a_{ij} x_j = y_i, \quad y_i \in M, \quad i < \lambda \tag{1}$$

be a system of equations in any (finite or infinite) number of unknowns x_j, $j \in J$, and M_λ be the module corresponding to M and λ from the previous lemma. If (1) is finitely satisfied in M, then it is satisfied in M_λ.

Proof. Suppose that (1) is finitely satisfied. We will construct by induction partial solutions $x_j^\mu \in M_\mu$, $j \in J$, of the first μ equations such that

$$x_j^\mu = S_\mu\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu}\big) \tag{2}$$

We will set $x_j^0 = 0$ for each $j \in J$ by definition. If μ is non-zero finite, there is a solution of the first μ equations by the assumption and (2) is trivially satisfied. Since for any μ infinite: $\mathrm{card}(\mu) = \mathrm{card}(\mu+1)$, we can find a solution of the first $\mu+1$ equations just by renumbering the equations and using the induction hypothesis. Then

$$x_j^{\mu+1} = (x_j^{\mu+1} - x_j^\mu) + x_j^\mu = (x_j^{\mu+1} - x_j^\mu) + S_\mu\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu}\big) = \\ = S_{\mu+1}\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu+1}\big)$$

Now let μ be a limit ordinal. We will consider (2) as a definition of x_j^μ. Then for arbitrary $i < \mu$:

$$\sum_{j\in J} a_{ij} x_j^\mu = S_\mu\big(\sum_{j\in J} a_{ij}\,(x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu}\big) = \\ = S_{i+1}\big(\sum_{j\in J} a_{ij}\,(x_j^{\alpha+1} - x_j^\alpha)_{\alpha<i+1}\big) = \\ = \sum_{j\in J} a_{ij} S_{i+1}\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<i+1}\big) = \sum_{j\in J} a_{ij} x_j^{i+1} = y_i$$

Thus, x_j^μ is a solution of the first μ equations, and subsequently x_j^λ, $j \in J$, is a solution of the whole system. □

Corollary 4. *Let R be a ring, M a module and $\kappa = \max\{\aleph_0, \mathrm{card}(R)\}$. Let $(N_\alpha \mid \alpha \le \kappa^+)$ be a smooth chain of modules defined via: $N_0 = 0$, $N_1 = M$, $N_{\alpha+1}$ is the κ-th member of the chain from Lemma 2 when starting with the module N_α. Then N_{κ^+} is pure-injective.*

Proof. It is sufficient to prove that every system of linear equations

$$\sum_{j\in J} a_{ij} x_j = y_i, \quad y_i \in N_{\kappa^+}, \quad i < \kappa$$

in unknowns x_j, $j \in J$, which is finitely satisfied in N_{κ^+} is satisfied in N_{κ^+}, [8, V.1.2]. But all the y_i's are actually included in N_μ for some $\mu < \kappa^+$, thus the system is satisfied in $N_{\mu+1}$ by the preceding lemma. □

4. Cotilting modules

First, we need the following two set-theoretic lemmas that hold in ZFC. The first one was proven in [11] for the special case $\kappa = \aleph_0$. The second one is a straightforward generalization of [4, 2.3].

Lemma 5. *Let κ be an infinite regular cardinal. Then for every cardinal μ there is a cardinal $\lambda \geq \mu$ such that $\lambda^\kappa = 2^\lambda$ and $\lambda^\alpha = \lambda$ for each $\alpha < \kappa$.*

Proof. Let κ, μ be as above, and let λ be the union of the smooth chain $(\mu_i \mid i < \kappa)$ defined by $\mu_0 = \mu$ and $\mu_{i+1} = 2^{\mu_i}$. Then clearly λ is of cofinality κ and $\nu < \lambda$ implies $2^\nu < \lambda$. The power set $\mathcal{P}(\lambda)$ embeds in an obvious way in $\prod_{i<\kappa} \mathcal{P}(\mu_i)$, hence $2^\lambda \leq \lambda^\kappa$. If $\alpha < \kappa$, then the range of any map $\alpha \to \lambda$ is actually contained in some μ_i, thus $\lambda^\alpha = \operatorname{card}(\bigcup_{i<\kappa} \mu_i^\alpha) \leq \lambda$. □

Lemma 6. *Let λ, κ be cardinals such that $\lambda^\kappa = 2^\lambda$ and $\lambda^\alpha = \lambda$ for each $\alpha < \kappa$. Then there is a family $\mathcal{S}$ of subsets of λ of cardinality κ such that :*

(a) $\operatorname{card}(\mathcal{S}) = 2^\lambda$,
(b) $\operatorname{card}(X \cap Y) < \kappa$ *for each pair of distinct elements* $X, Y \in \mathcal{S}$.

Proof. Let D denote the disjoint union of the sets λ^α for all $\alpha < \kappa$. Then $\operatorname{card}(D) = \lambda$. Define a map $F : \lambda^\kappa \to \mathcal{P}(D)$ by $F(f) = \{(f \restriction \alpha) \mid \alpha < \kappa\}$. Then clearly $\operatorname{card}(F(f)) = \kappa$ and $\operatorname{card}(F(f) \cap F(g)) < \kappa$ for each distinct $f, g \in \lambda^\kappa$. The family $\mathcal{S}$ arises just by applying bijections between λ and D, and between λ^κ and 2^λ. □

The following lemma is a generalization of [4, 2.5] (which deals with the case of $\kappa = \aleph_0$):

Lemma 7. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then for any regular cardinal κ, $M \in {}^{\perp_1}U$ implies $M^\kappa / M^{<\kappa} \in {}^{\perp_1}U$.*

Proof. Let λ be a cardinal such that $\lambda^\kappa = 2^\lambda$ and $\lambda^\alpha = \lambda$ for each $\alpha < \kappa$. Consider a family $\mathcal{S}$ of subsets of λ as in Lemma 6. For each $X \in \mathcal{S}$, let $\eta_X : M^X \to M^\lambda / M^{[\lambda;\kappa]}$ be the composition of the canonical embedding $M^X \to M^\lambda$ with the canonical projection. Denote the module $M^\kappa / M^{<\kappa}$ by N. Then clearly $\operatorname{Im} \eta_X \cong N$ and $\operatorname{Ker} \eta_X = M^{[X;\kappa]}$. Moreover, it is easy to see that the sum $\sum_{X \in \mathcal{S}} \operatorname{Im} \eta_X$ is actually a direct sum.

Next, denote by V the preimage of $\sum_{X \in \mathcal{S}} \operatorname{Im} \eta_X$ in M^λ. We claim that V is a pure submodule of M^λ. In fact, $x \in V$ if and only if the support of x is a subset of some union of the form $G \cup X_1 \cup \cdots \cup X_n$, where $X_1, \ldots, X_n$ are finitely many elements of $\mathcal{S}$ and $\operatorname{card}(G) < \kappa$. Thus, any system of finitely many linear equations $\sum_{j \leq m} a_{ij} x_j = y_i$ with all the y_i's in V that can be solved in M^λ has a solution with supports of x_i's inside the union of the supports of y_i's, therefore it has a solution in V.

Now suppose that $M \in {}^{\perp_1}U$. Then $V \in {}^{\perp_1}U$ as well, and we have a short exact sequence of the following form:

$$0 \to M^{[\lambda;\kappa]} \to V \to N^{(\mathcal{S})} \to 0$$

and the corresponding induced exact sequence:

$$\mathrm{Hom}_R(M^{[\lambda;\kappa]}, U) \to \mathrm{Ext}^1_R(N^{(\mathcal{S})}, U) \to 0$$

We can always choose λ so that in addition $\lambda \geq \mathrm{card}(\mathrm{Hom}_R(M^\mu, U))$ for each $\mu < \kappa$ using Lemma 5. Let $\mathcal{L}$ denote the set of all the subsets of λ of cardinality $< \kappa$. Then any homomorphism $f : M^{[\lambda;\kappa]} \to U$ is uniquely determined by its restrictions to M^Z, Z running through all elements of $\mathcal{L}$. Therefore:

$$\mathrm{card}(\mathrm{Hom}_R(M^{[\lambda;\kappa]}, U)) \leq \prod_{Z \in \mathcal{L}} \mathrm{card}(\mathrm{Hom}_R(M^Z, U)) \leq \lambda^{\mathrm{card}(\mathcal{L})}$$

Moreover, $\mathrm{card}(\mathcal{L}) \leq \mathrm{card}(\bigcup_{\mu<\kappa} \lambda^\mu) = \lambda$. Hence

$$\mathrm{card}(\mathrm{Hom}_R(M^{[\lambda;\kappa]}, U)) \leq 2^\lambda.$$

On the other hand, if $\mathrm{Ext}^1_R(N, U) \neq 0$, then $\mathrm{card}(\mathrm{Ext}^1_R(N^{(\mathcal{S})}, U)) \geq 2^{\mathrm{card}(\mathcal{S})} = 2^{2^\lambda}$, a contradiction with the existence of an epimorphism. Thus $N \in {}^{\perp_1}U$. □

The following lemma generalizes [5, 3.7, part 2]. The proof is essentially the same as in [5].

Lemma 8. *Let $\mathcal{C}$ be a class of modules closed under pure submodules and products. Assume in addition that there is a limit ordinal λ such that $M \in \mathcal{C}$ implies $M^\lambda / M^{<\lambda} \in \mathcal{C}$. Then $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \in \mathcal{C}$ for any family $(M_\alpha \mid \alpha < \lambda)$ of modules of $\mathcal{C}$.*

Proof. Let us denote $W = \prod^b_{\alpha<\lambda} M_\alpha$ and let $\varepsilon_\alpha : M_\alpha \to W$ be the canonical embeddings. Since W is a pure submodule of $\prod_{\alpha<\lambda} M_\alpha$, we get $W \in \mathcal{C}$ and $W^\lambda / W^{<\lambda} \in \mathcal{C}$. Denote $f : \prod_{\alpha<\lambda} M_\alpha \to W^\lambda / W^{<\lambda}$ the composition of the product of the maps ε_α with the canonical projection. Then the kernel of f is exactly $\prod^b_{\alpha<\lambda} M_\alpha$ and the induced embedding $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \to W^\lambda / W^{<\lambda}$ is pure. Thus $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \in \mathcal{C}$. □

Now, we are able to extend Lemma 7 to all limit ordinals:

Lemma 9. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then for any limit ordinal λ, if $(M_\alpha \mid \alpha < \lambda)$ is a family of modules of ${}^{\perp_1}U$, then $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \in {}^{\perp_1}U$.*

Proof. In the view of the preceding lemma, it is sufficient to prove, by induction on λ, that $M \in {}^{\perp_1}U$ implies $M^\lambda / M^{<\lambda} \in {}^{\perp_1}U$. If λ is a regular cardinal, and this is in particular the case when $\lambda = \aleph_0$, then we use Lemma 7. If λ is not a regular cardinal, then there is a limit ordinal $\mu < \lambda$ and an increasing continuous map $f : \mu \to \lambda$ with an unbounded range and such that $f(0) = 0$. Let us denote $M_\alpha = M^{f(\alpha+1) \setminus f(\alpha)}$ for each $\alpha < \mu$. Then obviously $M^\lambda/M^{<\lambda} \cong \prod_{\alpha<\mu} M_\alpha / \prod^b_{\alpha<\mu} M_\alpha$, and the latter module is contained in ${}^{\perp_1}U$ by the induction hypothesis. □

Proposition 10. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then $M \in {}^{\perp_1}U$ implies $PE(M)/M \in {}^{\perp_1}U$.*

Proof. By Lemmas 1 and 9, $M_\lambda/M \in {}^{\perp_1}U$ whenever $M \in {}^{\perp_1}U$ for all M_λ in Lemma 2. Thus, using this and Corollary 4, M purely embeds into the pure injective module N_{κ^+} and $N_{\kappa^+}/M \in {}^{\perp_1}U$. Therefore, $PE(M)/M$ is isomorphic to a direct summand of N_{κ^+}/M, [12, Theorem 4.20]. Hence $PE(M)/M \in {}^{\perp_1}U$. □

Finally, we are ready to prove both the conjectures (A) and (B). The proof of Theorem 11 given here is inspired by the proof of Conjecture (A) in [7].

Theorem 11. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then ${}^{\perp_1}U$ is closed under pure epimorphic images.*

Proof. It suffices to prove that, whenever $i : Y \to X$ is a pure monomorphism such that $X \in {}^{\perp_1}U$, and $f : Y \to U$ is any homomorphism, then there is a homomorphism $g : X \to U$ such that $f = gi$. But in this case $Y \in {}^{\perp_1}U$ and $PE(Y)/Y \in {}^{\perp_1}U$ too (Proposition 10). Thus, there are homomorphisms $h : X \to PE(Y)$ and $k : PE(Y) \to U$ such that $j = hi$ and $f = kj$, where j is the embedding of Y into $PE(Y)$. The composition kh yields the desired map g. □

Corollary 12. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then ${}^{\perp_1}U$ is a covering class.*

Proof. This follows by [5, Proposition 5.4] and [9, Theorem 5]. □

The following is the main result of our paper:

Theorem 13. *Let R be an arbitrary ring, $n \geq 0$, and U be an n-cotilting module. Then U is pure-injective.*

Proof. This is immediate from Corollary 12 and [5, Theorem 5.5]. □

From [1, Theorem 4.1] and [3, Proposition 3.5], we get

Corollary 14. *Let U be an n-cotilting module over an arbitrary ring such that $n \geq 1$, and let U' be a cosyzygy of U. Then $\bigcap_{i\geq 1} {}^{\perp_i}U'$ is an $(n-1)$-cotilting class.*

Remark. It is possible to state Lemma 7 more generally with just a small change in the proof: If $\mathcal{U}$ is a class of modules such that ${}^{\perp_1}\mathcal{U}$ is closed under products and pure submodules, then $M \in {}^{\perp_1}\mathcal{U}$ implies $M^\kappa/M^{<\kappa} \in {}^{\perp_1}\mathcal{U}$ for any regular cardinal κ. The subsequent statements in this paper generalize in a similar way so we can consider a class of modules $\mathcal{U}$ instead of a single module U everywhere to Corollary 12. This fact was recently used by Šaroch and Trlifaj in [14] to improve the characterization of cotilting cotorsion pairs from [1], dropping out the assumption of the completeness of a cotorsion pair.

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II.

ALL TILTING MODULES ARE OF COUNTABLE TYPE

Abstract

Let R be a ring and T be an (infinitely generated) tilting module. Then T is of countable type, that is, there is a set, $\mathcal{C}$, of modules possessing a projective resolution consisting of countably generated projective modules such that the tilting class $T^\perp$ equals $\mathcal{C}^\perp$. Moreover, a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is tilting if and only if $\mathfrak{C}$ is hereditary, all modules in $\mathcal{A}$ have finite projective dimension, and $\mathcal{B}$ is closed under arbitrary direct sums.

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ALL TILTING MODULES ARE OF COUNTABLE TYPE

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Infinitely generated tilting modules occur naturally in various areas of contemporary module theory. For example, finiteness of the little finitistic dimension of a right noetherian ring R is equivalent to the existence of a particular tilting R-module T_f, [6]. Explicit computation of T_f then yields a proof of the equality of the little and the big finitistic dimensions for all (non-commutative) Iwanaga-Gorenstein rings, [5]. Similarly, if R is a commutative ring, S is some multiplicative set of regular elements in R, and Q denotes the localization of R in S, then the existence of a decomposition of Q/R into a direct sum of countably presented R-submodules is equivalent to $T_S = Q \oplus Q/R$ being a tilting module of projective dimension ≤ 1, [4].

Though the examples of tilting modules above are typically infinitely generated, there is an implicit finiteness condition connected to tilting. Namely, all known examples of tilting modules T are of *finite type*, that is, there is a set, $\mathcal{S}$, of modules possessing a projective resolution consisting of finitely generated projective modules such that $T^\perp = \mathcal{S}^\perp$ (see below for unexplained terminology). Then the tilting class $T^\perp$ is definable, and one can characterize modules in $T^\perp$ by formulas of the first-order language of module theory. For particular rings (Prüfer domains [8], et al.), it has been proved that all tilting modules are of finite type. The general case, however, remains open. [1]

The proofs in the particular cases rely on the following result from [8]: if T is a tilting module of projective dimension ≤ 1 then T is of *countable type*, that is, there is a set, $\mathcal{C}$, of modules possessing a projective resolution consisting of countably generated projective modules such that $T^\perp = \mathcal{C}^\perp$.

Our main result generalizes this to all tilting modules:

Theorem 1. *Let R be a ring and T a tilting module. Then T is of countable type.*

We also obtain a new characterization of tilting cotorsion pairs:

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[1]*Added in proof:* Applying the main results of the present paper (Theorem 1), and of a recent manuscript by Bazzoni and Herbera ("One dimensional tilting modules are of finite type"), Bazzoni and Šťovíček have recently proved that all tilting modules are of finite type.

Theorem 2. *Let R be a ring, $n < \omega$, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is an n-tilting cotorsion pair if and only if $\mathfrak{C}$ is hereditary, $\mathcal{A}$ consists of modules of projective dimension $\leq n$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

We prove Theorems 1 and 2 in ZFC, but we make essential use of the set-theoretic methods developed originally by Eklof, Fuchs, Hill, and Shelah in projective dimension 1 for structure theory of Whitehead and Baer modules, cf. [9], [10, Chap. XII]. In order to deal with modules of projective dimension > 1, we extend some of the key constructions avoiding at the same time the tools characteristic of the one dimensional case (the tight systems, for example). Thus, our methods seem to be of independent interest, and they are presented in separate theorems in the first two sections of the paper, and in Theorems 15 and 18. The proof of Theorem 1 appears in Section 3; Theorem 2 is proved in Section 4.

Preliminaries

For a ring R, we denote by Mod-R the category of all (unitary right R-) modules. Given a module M, $\mathrm{gen}(M)$ denotes the minimal cardinality of an R-generating subset of M, and $E(M)$ the injective hull of M.

For a ring R, $\dim(R)$ denotes the minimal infinite cardinal κ such that $\mathrm{gen}(I) \leq \kappa$ for all right ideals I of R. For example, $\dim(R) = \aleph_0$ if and only if R is right $\aleph_0$-noetherian.

Given a family of modules $(M_i \mid i \in I)$ and an infinite cardinal ν, we denote by $\prod_{i\in I}^{<\nu} M_i$ the submodule of the direct product $\prod_{i\in I} M_i$ formed by all elements with support of cardinality $< \nu$ (so $\prod_{i\in I}^{<\aleph_0} M_i = \bigoplus_{i\in I} M_i$, for example).

We denote by $\mathcal{E}$ the class of all modules of the form $\prod_{i\in I}^{<\nu} E_i$ where ν is a regular infinite cardinal and $(E_i \mid i \in I)$ a family of injective modules. The subclass of $\mathcal{E}$ consisting of arbitrary direct sums of injective modules (using only $\nu = \aleph_0$) is denoted by $\mathcal{E}_0$.

For $n < \omega$, $\mathcal{P}_n$ stands for the class of all modules of projective dimension $\leq n$. The class of n-th cosyzygies of all R-modules is denoted by Cos^n-R; that is, $M \in \mathrm{Cos}^n$-R if and only if there is an exact sequence $E_{n-1} \to \cdots \to E_0 \to M \to 0$ with injective modules E_j, $0 \leq j < n$.

For a class of modules $\mathcal{C}$ and an infinite cardinal κ, denote by $\mathcal{C}^{<\kappa}$ and $\mathcal{C}^{\leq\kappa}$ the subclass of all modules in $\mathcal{C}$ possessing a projective resolution consisting of $< \kappa$-generated and $\leq \kappa$-generated, respectively, projective modules. For example, if R is right $\aleph_0$-noetherian or $\mathcal{C} \subseteq \mathcal{P}_1$, then $\mathcal{C}^{\leq\aleph_0}$ consists of all countably presented modules in $\mathcal{C}$.

For a class of modules $\mathcal{C}$ and $1 \leq i < \omega$, we define

$$\mathcal{C}^{\perp_i} = \operatorname{Ker}\operatorname{Ext}^i_R(\mathcal{C}, -) = \{M \in \text{Mod-}R \mid \operatorname{Ext}^i_R(C, M) = 0 \ \ \forall C \in \mathcal{C}\}$$

and $\mathcal{C}^\perp = \bigcap_{1\leq i<\omega} \mathcal{C}^{\perp_i}$. Similarly, ${}^{\perp_i}\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^i_R(-, \mathcal{C})$ and ${}^\perp\mathcal{C} = \bigcap_{1\leq i<\omega}({}^{\perp_i}\mathcal{C})$. If $\mathcal{C} = \{M\}$ for a single module M, we write $M^\perp$ instead of $\{M\}^\perp$ etc. Note that $\mathcal{P}_n = {}^{\perp_1}(\text{Cos}^n\text{-}R) = {}^\perp(\text{Cos}^n\text{-}R)$ for each $n < \omega$.

A pair of classes of modules, $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. $\mathfrak{C}$ is *hereditary* provided that $\operatorname{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $2 \leq i < \omega$ (that is, provided that $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$). $\mathfrak{C}$ is *complete* provided that for each module M there is an exact sequence $0 \to B \to A \to M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

For example, given a class of modules $\mathcal{C}$, $({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1})$ is a cotorsion pair, called the cotorsion pair *cogenerated* by $\mathcal{C}$. Any cotorsion pair cogenerated by a set of modules is complete, cf. [11].

A module T is a *tilting module* provided that T has finite projective dimension, $\operatorname{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i < \omega$ and all cardinals κ, and there are $m < \omega$ and an exact sequence $0 \to R \to T_0 \to \dots \to T_m \to 0$ such that T_i is a direct summand in a (possibly infinite) direct sum of copies of T for each $i \leq m$. The class $\mathcal{T}_T = T^\perp$ is called the *tilting class* induced by T, and the (complete, hereditary) cotorsion pair $\mathfrak{C}_T = ({}^\perp(T^\perp), T^\perp)$ the *tilting cotorsion pair* induced by T.

If T has projective dimension $\leq n$, then T, $\mathcal{T}_T$, and $\mathfrak{C}_T$, are called the *n-tilting module*, *n-tilting class*, and *n-tilting cotorsion pair*, respectively. In this case, ${}^\perp(T^\perp) \subseteq \mathcal{P}_n$, and $\mathcal{T}_T$ is closed under arbitrary direct sums, cf. [16].

For a module M, an ascending chain, $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa)$ of submodules of M is called *continuous* provided that $M_0 = 0$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \kappa$. If moreover $M_\kappa = M$, then $\mathcal{M}$ is called a *filtration* of M. Similarly (componentwise), we define continuous chains and filtrations consisting of short exact sequences of modules. If $\mathcal{M}$ is a filtration of a module M such that κ is an infinite cardinal and $\operatorname{gen}(M_\alpha) < \kappa$ for all $\alpha < \kappa$, then $\mathcal{M}$ is called a *κ-filtration* of M.

Assume κ is a regular uncountable cardinal. A strictly ascending function $f : \kappa \to \kappa$ is called *continuous* provided that $f(0) = 0$, and $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$ for all limit ordinals $\alpha < \kappa$. If $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa)$ is a filtration of a module M, and $f : \kappa \to \kappa$ a continuous function, then $\mathcal{M}' = (M_{f(\alpha)} \mid \alpha \leq \kappa)$ (where we put $f(\kappa) = \kappa$) is again a filtration of M, called the *subfiltration* of $\mathcal{M}$ induced by f. Any two κ-filtrations of M coincide on a closed and unbounded subset of κ, cf. [10], thus they possess a common subfiltration.

Given a class of modules $\mathcal{C}$ and a module M, we say that M is $\mathcal{C}$-*filtered* provided that M possesses a filtration $(M_\alpha \mid \alpha \leq \lambda)$ such that $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \lambda$.

We will need the following well-known lemmas:

Lemma 3. *Let M be a module, and $\mathcal{C}$ a class of modules. Assume that M is ${}^{\perp_1}\mathcal{C}$-filtered. Then $M \in {}^{\perp_1}\mathcal{C}$.*

Proof. See e.g. [10, XII.1.14]. □

Lemma 4. *Let R be a ring, and κ be a cardinal such that $\kappa \geq \dim(R)$. Then any submodule of a $\leq \kappa$-generated module is $\leq \kappa$-generated.*

Proof. First, all submodules of cyclic modules are $\leq \dim(R)$-generated, since they are homomorphic images of right ideals. Further, any $\leq \kappa$-generated module M has a filtration $(M_\alpha \mid \alpha \leq \kappa)$ such that all factors $M_{\alpha+1}/M_\alpha$ are cyclic. If $K \subseteq M$, then $K \cap M_{\alpha+1}/K \cap M_\alpha$ embeds into $M_{\alpha+1}/M_\alpha$ for each $\alpha < \kappa$, and the assertion follows. □

1. Closure properties of cotorsion classes

Proposition 5. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair cogenerated by a class $\mathcal{C}$ of modules of finite projective dimension. Assume that $\mathcal{B}$ is closed under arbitrary direct sums and X is a $\mathcal{B}$-filtered module. Then $X \in \mathcal{B}$.*

Proof. Let $(X_\alpha \mid \alpha \leq \kappa)$ be a $\mathcal{B}$-filtration of X. By induction on κ, we will prove that $X \in \mathcal{B}$ and there is a continuous chain of short exact sequences

$$\delta_\alpha \quad : \quad 0 \to K_\alpha \to \bigoplus_{\lambda<\alpha} B_\lambda \to X_\alpha \to 0 \quad (\alpha < \kappa)$$

such that

(1) $K_0 = 0$ and $K_{\alpha+1}/K_\alpha \in \mathcal{B}$ for any $\alpha < \kappa$,
(2) $B_\lambda \in \mathcal{A} \cap \mathcal{B}$ for all $\lambda < \kappa$,
(3) The embedding of the middle term of δ_α into the middle term of δ_β is the canonical inclusion, for all $\alpha < \beta < \kappa$.

For $\kappa = 0$, clearly $0 \in \mathcal{B}$, and we just take the short exact sequence of zeros. Let $\kappa = \beta + 1$. Then $X_\kappa \in \mathcal{B}$ immediately by the inductive assumption and the fact that $\mathcal{B}$ is closed under extensions.

For the construction of δ_κ, we use an idea from [15, 2.3]: since $\mathfrak{C}$ is complete, there is a short exact sequence $0 \to B' \to B_\beta \to X_\kappa/X_\beta \to 0$ with $B' \in \mathcal{B}$ and $B_\beta \in \mathcal{A}$. We form a pull-back:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_\beta & = & X_\beta & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & P & \longrightarrow & X_\kappa & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B_\beta & \longrightarrow & X_\kappa/X_\beta & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $X_\kappa/X_\beta \in \mathcal{B}$, we have $B_\beta \in \mathcal{A} \cap \mathcal{B}$. Thus, the middle column of the diagram splits, and we can use the exact sequence δ_β to form the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_\beta & \longrightarrow & \bigoplus_{\lambda<\beta} B_\lambda & \longrightarrow & X_\beta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & K_\kappa & \longrightarrow & \bigoplus_{\lambda<\kappa} B_\lambda & \longrightarrow & X_\kappa & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B_\beta & \longrightarrow & X_\kappa/X_\beta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The diagram is commutative and has exact rows and columns by the 3×3 lemma, and the map i can be w. l. o. g. taken as the canonical inclusion. We define δ_κ as the middle row of this diagram.

Finally, assume κ is a limit ordinal. Then we define $\delta_\kappa = \varinjlim_{\alpha<\kappa} \delta_\alpha : 0 \to K_\kappa \to C_1 \to X_\kappa \to 0$. To complete the proof, we use an idea from [7, 3.2]: we replace X_κ with K_κ (and $(X_\beta \mid \beta \le \kappa)$ with $(K_\beta \mid \beta \le \kappa)$), and, step by step, construct a long exact sequence:

$$\cdots \to C_3 \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} X_\kappa \to 0, \qquad C_i \in \mathcal{B} \text{ for } i \ge 1$$

If $A \in \mathcal{C}$ has projective dimension n, then

$$\operatorname{Ext}^1_R(A, X_\kappa) \cong \operatorname{Ext}^{n+1}_R(A, \operatorname{Ker} f_n) = 0.$$

This proves that $X_\kappa \in \mathcal{B}$. $\square$

Theorem 6. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair cogenerated by a class of modules of finite projective dimension, and let $\mathcal{B}$ be closed under arbitrary direct sums. Then $\mathcal{B}$ is closed under arbitrary direct limits.*

Proof. First, we prove that $\mathcal{B}$ is closed under unions of arbitrary chains. Let $\mathcal{C} \subseteq \mathcal{B}$ be a chain of modules with respect to inclusion. We construct a $\mathcal{B}$-filtration of $X = \bigcup \mathcal{C}$ by transfinite induction: we set $X_0 = 0$, and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for α limit. If $\alpha = \beta + 1$ and $X_\beta \subsetneq X$, we consider $x \in X \setminus X_\beta$ and take X_α as an element of $\mathcal{C}$ containing x. Since $X_\alpha \nsubseteq X_\beta$ and X_β is a union of elements of $\mathcal{C}$, we have $X_\beta \subseteq X_\alpha$. Since the cotorsion pair $\mathfrak{C}$ is hereditary, we have $X_\alpha/X_\beta \in \mathcal{B}$, and Proposition 5 applies.

Now, the result follows from the well-known fact that closure under unions of well-ordered chains implies closure under arbitrary direct limits (see e.g. [1, 1.7]). □

Later on, we will need the following corollary:

Corollary 7. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be as in Theorem 6. Then $\prod_{i\in I}^{<\nu} M_i \in \mathcal{B}$ whenever $(M_i \mid i \in I)$ is a family of modules in $\mathcal{B}$ and ν is a regular infinite cardinal. In particular, $\mathcal{E} \subseteq \mathcal{B}$.*

Proof. The statement follows from the fact that $\prod_{i\in I}^{<\nu} M_i$ is a directed union of the products $\prod_{i\in J} M_i$ for subsets $J \subseteq I$ of cardinality $< \nu$. □

2. Filtrations of regular length

The following theorem is a partial converse of Lemma 3. It generalizes [10, Theorem XII.3.3] (which has the additional assumption of proj.dim$(M_{\alpha+1}/M_\alpha) \leq 1$ for all $\alpha < \kappa$). Our theorem essentially says that if $\mathcal{B}$ is closed under arbitrary direct sums, then each module $M \in^{\perp_1} \mathcal{B}$ with a κ-filtration in ${}^{\perp_1}\mathcal{B}$ is actually ${}^{\perp_1}\mathcal{B}$-filtered by a subfiltration.

Theorem 8. *Let R be a ring, κ a regular uncountable cardinal, and $\mathcal{B}$ a class of modules closed under arbitrary direct sums. Let $M \in {}^{\perp_1}\mathcal{B}$ be a module possessing a κ-filtration $(M_\alpha \mid \alpha \leq \kappa)$ such that $M_\alpha \in {}^{\perp_1}\mathcal{B}$ for all $\alpha < \kappa$. Then there is a continuous function $f : \kappa \to \kappa$ such that $M_{f(\beta)}/M_{f(\alpha)} \in {}^{\perp_1}\mathcal{B}$ for all $\alpha < \beta < \kappa$.*

Proof. Assume the claim is false. Then the set

$$E = \{\alpha < \kappa \mid \exists \beta : \alpha < \beta < \kappa \;\&\; M_\beta/M_\alpha \notin {}^{\perp_1}\mathcal{B}\}$$

has a non-empty intersection with each closed and unbounded subset of κ. Possibly passing to a subfiltration, we can w.l.o.g. assume that $E = \{\alpha < \kappa \mid \mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, \mathcal{B}) \neq 0\}$. Then for each $\alpha \in E$ there are a $B_\alpha \in \mathcal{B}$ and a homomorphism $\delta_\alpha : M_{\alpha+1}/M_\alpha \to E(B_\alpha)/B_\alpha$ that cannot be factorized through the projection $\tau_\alpha : E(B_\alpha) \to E(B_\alpha)/B_\alpha$. For $\alpha < \kappa$, $\alpha \notin E$, we put $B_\alpha = 0$ and $\delta_\alpha = 0$.

Let $I = \prod_{\alpha<\kappa} E(B_\alpha)$, $D = \bigoplus_{\alpha<\kappa} B_\alpha \,(\subseteq I)$, and $F = I/D$. For each subset $A \subseteq \kappa$, define $I_A = \{x \in I \mid x_\beta = 0 \text{ for all } \beta < \kappa, \beta \notin A\}$. In particular, $I_\kappa = I$, and $I_\alpha \cong \prod_{\beta<\alpha} E(B_\beta)$ is injective for each $\alpha \leq \kappa$.

For each $\alpha < \kappa$, we let $F_\alpha = (I_\alpha + D)/D \,(\subseteq F)$ and π_α be the epimorphism $I_\alpha \to F_\alpha$ defined by $\pi_\alpha(x) = x + D$. Then $\mathrm{Ker}(\pi_\alpha) \cong \bigoplus_{\beta<\alpha} B_\beta \,(\in \mathcal{B})$.

Let $U = \bigcup_{\alpha<\kappa} I_\alpha$. Then $D \subseteq U \subseteq I$, and we let $G = U/D \,(\subseteq F)$ and $\pi : U \to G$ be the projection modulo D.

For each $\alpha < \kappa$, define $E_\alpha = (I_{\{\alpha\}} + D)/D$. Then there is an isomorphism $\iota_\alpha : E(B_\alpha)/B_\alpha \cong E_\alpha$, and $F_{\alpha+1} = E_\alpha \oplus F_\alpha \,(\subseteq G)$. Moreover, taking $C_\alpha = (I_{(\alpha,\kappa)} + D)/D$, we have $F = F_{\alpha+1} \oplus C_\alpha$, so $G = E_\alpha \oplus F_\alpha \oplus (C_\alpha \cap G)$. Denote by ξ_α the projection onto the first component, E_α, in the latter decomposition of G. Then ξ_α maps $x + D \in G$ to $y + D \in E_\alpha$ where $y_\alpha = x_\alpha$ and $y_\beta = 0$ for all $\alpha \neq \beta < \kappa$.

In order to prove that $\mathrm{Ext}^1_R(M, \mathcal{B}) \neq 0$, it suffices to construct a homomorphism $\varphi : M \to G$ that cannot be factorized through π (because then the map $\mathrm{Hom}_R(M, \pi)$ is not surjective, so $\mathrm{Ext}^1_R(M, D) \neq 0$).

φ will be constructed by induction on $\alpha < \kappa$ as a union of a continuous chain of homomorphisms $(\varphi_\alpha \mid \alpha < \kappa)$ where $\varphi_\alpha : M_\alpha \to F_\alpha$ for all $\alpha < \kappa$, and $\varphi \restriction M_0 = 0$.

For $\alpha < \kappa$, we use the assumption of $\mathrm{Ext}^1_R(M_\alpha, \bigoplus_{\beta<\alpha} B_\alpha) = 0$ to find a homomorphism $\eta_\alpha : M_\alpha \to I_\alpha$ such that $\varphi_\alpha = \pi_\alpha \eta_\alpha$. The injectivity of the module I_α yields a homomorphism $\psi_\alpha : M_{\alpha+1} \to I_\alpha$ such that $\psi_\alpha \restriction M_\alpha = \eta_\alpha$.

Denote by ρ_α the projection $M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$. Define $\varphi_{\alpha+1} = \iota_\alpha \delta_\alpha \rho_\alpha + \pi_\alpha \psi_\alpha$. Then $\varphi_{\alpha+1} \restriction M_\alpha = \pi_\alpha \psi_\alpha \restriction M_\alpha = \pi_\alpha \eta_\alpha = \varphi_\alpha$.

Finally, assume there is $\phi : M \to U$ such that $\varphi = \pi\phi$. Since $U = \bigcup_{\alpha<\kappa} I_\alpha$, the set $C = \{\alpha < \kappa \mid \phi(M_\alpha) \subseteq I_\alpha\}$ is closed and unbounded in κ. So there exists $\alpha \in C \cap E$. Denote by σ the projection $I \to E(B_\alpha)$. Then ϕ induces a homomorphism $\bar{\phi} : M_{\alpha+1}/M_\alpha \to E(B_\alpha)$ defined by $\bar{\phi}\rho_\alpha(m) = \sigma(\phi(m))$ for all $m \in M_{\alpha+1}$.

By the definition of ξ_α, we have $\iota_\alpha \tau_\alpha \sigma(x) = \xi_\alpha \pi(x)$ for each $x \in U$, $\xi_\alpha \restriction F_\alpha = 0$, and $\xi_\alpha \restriction E_\alpha = \mathrm{id}$. So, for each $m \in M_{\alpha+1}$, we get

$$\tau_\alpha \bar{\phi} \rho_\alpha(m) = \iota_\alpha^{-1} \xi_\alpha \pi \phi(m) = \iota_\alpha^{-1} \xi_\alpha \varphi_{\alpha+1}(m) = \\ = \iota_\alpha^{-1} \xi_\alpha \iota_\alpha \delta_\alpha \rho_\alpha(m) = \delta_\alpha \rho_\alpha(m).$$

Since ρ_α is surjective, this proves that $\tau_\alpha \bar{\phi} = \delta_\alpha$, in contradiction with the definition of δ_α. □

3. Classes of countable type

We start with a technical lemma that will be applied later on to estimate the number of generators of submodules in our particular setting.

Lemma 9. *Let R be a ring, κ a regular infinite cardinal, and M a module. Let $(M_\alpha \mid \alpha \leq \kappa)$ be a strictly ascending filtration of M. Then there is a family of non-zero injective modules $(E_\alpha \mid \alpha < \kappa)$ and an embedding $e : M \to \prod_{\alpha<\kappa}^{<\kappa} E_\alpha$ such that, for each submodule $N \subseteq M$ with $N \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$ for all $\alpha < \kappa$, the union of supports of all elements of $e(N)$ equals κ.*

Proof. Let $i_\alpha : M_{\alpha+1}/M_\alpha \to E_\alpha$ be an injective envelope of $M_{\alpha+1}/M_\alpha$ for each $\alpha < \kappa$. We will construct a continuous chain of injective maps $e_\alpha : M_\alpha \to \prod_{\beta<\alpha} E_\beta$ as follows:

$e_0 = 0$; if e_α is already constructed, we can extend it to $f_\alpha : M_{\alpha+1} \to \prod_{\beta<\alpha} E_\beta$ since all E_β's are injective, and put $e_{\alpha+1} = f_\alpha + i_\alpha p_\alpha$ where $p_\alpha : M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$ is the projection.

Consider $e = \bigcup_{\alpha<\kappa} e_\alpha : M \to \prod_{\beta<\kappa}^{<\kappa} E_\beta$. If $N \subseteq M$ and $x \in N \cap (M_{\alpha+1} \setminus M_\alpha)$, then the α-th component of $e(x)$ $(= e_{\alpha+1}(x))$ is $i_\alpha p_\alpha(x)$ $(\neq 0)$, and the claim follows. $\square$

While Theorem 8 will take care of filtrations of regular length, our arguments for singular cardinals will be based on the following two lemmas essentially going back to [9], see also [10, Chap. XII].

Definition 10. Let M be a module, $\mathcal{Q}$ a set of modules, and κ a regular infinite cardinal. Then M is called *κ-$\mathcal{Q}$-free* provided there is a set $\mathcal{S}_\kappa$ consisting of $< \kappa$-generated $\mathcal{Q}$-filtered submodules of M such that:

(a) $0 \in \mathcal{S}_\kappa$,
(b) $\mathcal{S}_\kappa$ is closed under well-ordered chains of length $< \kappa$, and
(c) each subset of M of cardinality $< \kappa$ is contained in an element of $\mathcal{S}_\kappa$.

The set $\mathcal{S}_\kappa$ is said to witness the κ-$\mathcal{Q}$-freeness of M. If $\mathcal{S}_\kappa$ also satisfies

(d) M/N is $\mathcal{Q}$-filtered for each $N \in \mathcal{S}_\kappa$,

then we call M *κ-$\mathcal{Q}$-separable*, and the set $\mathcal{S}_\kappa$ is said to witness the κ-$\mathcal{Q}$-separability of M.

Clearly, every κ-$\mathcal{Q}$-separable module is $\mathcal{Q}$-filtered. The following lemma says that the converse is also true under rather weak assumptions:

Lemma 11. *Let R be a ring, μ an infinite cardinal and $\mathcal{Q}$ a set of $\leq \mu$-presented modules. Then M is κ-$\mathcal{Q}$-separable whenever M is $\mathcal{Q}$-filtered and κ is a regular cardinal $> \mu$. Moreover, it is possible to choose the witnessing sets so that $\mathcal{S}_\kappa \subseteq \mathcal{S}_{\kappa'}$ for all regular cardinals κ, κ' such that $\mu < \kappa < \kappa'$.*

Proof. The result is implicit in (the proof of) [10, XII.1.14], so we only sketch the main points. First, we recall the setting: a module N is called "free" iff N is $\mathcal{Q}$-filtered. A filtration $(N_\alpha \mid \alpha \leq \sigma)$ demonstrating that N is "free" is called the "free" chain for N.

If N is "free" then Y is a "basis" of N provided that Y is a set of subsets of N, and there is a "free" chain $\mathcal{N} = (N_\alpha \mid \alpha \leq \sigma)$ for N such that $Z \in Y$ iff for all $\alpha < \sigma$, $< Z > \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$ implies $M_{\alpha+1} \subseteq M_\alpha + < Z >$. (Note that Y is completely determined by the "free" chain $\mathcal{N}$.)

A submodule $B \subseteq N$ is a "free" factor of N provided that $B =< Z >$ for some member Z of a "basis" Y of N. If B is a "free" factor of N and Y', Y are "bases" of B and N, respectively, then (Y, Y') is called a designated pair provided there is a "free" chain $\mathcal{N} = (N_\alpha \mid \alpha \leq \sigma)$ for N such that $B = N_\beta$ for some $\beta < \sigma$, Y is a "basis" of N determined by $\mathcal{N}$, and $Y' = \{Z \in Y \mid Z \subseteq B\}$.

Now, by assumption, M is "free". We fix a "basis" X of M, and define $\mathcal{S}_\kappa$ to be the set of all "free" $< \kappa$-generated factors of M with respect to X. Then $\mathcal{S}_\kappa$ witnesses the κ-$\mathcal{Q}$-separability of M by [10, Lemma XII.1.14]. □

Alternatively, Lemma 11 can be proved as a consequence of a generalized version of Hill's lemma (see [12, Theorem 2.1] and [14, Lemma 1.4]) as follows. First, given a $\mathcal{Q}$-filtration, $(M_\alpha \mid \alpha \leq \sigma)$, of M, we fix $\leq \mu$-generated submodules A_α of $M_{\alpha+1}$ so that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$. A subset $S \subseteq \sigma$ is called 'closed' provided that $M_\alpha \cap A_\alpha \subseteq \sum_{\beta<\alpha, \beta \in S} A_\beta$ for each $\alpha \in S$. Now, it suffices to define $\mathcal{S}_\kappa$ as the set of all submodules of M of the form $\sum_{s \in S} A_\alpha$ where S is a 'closed' subset of σ of cardinality $< \kappa$.

The following is a particular case of the celebrated Shelah's Singular Compactness Theorem:

Lemma 12. [10, XII.1.14 and IV.3.7] *Let R be a ring, λ a singular cardinal, and $\aleph_0 \leq \mu < \lambda$. Let $\mathcal{Q}$ be a set of $\leq \mu$-presented modules. Let M be a module with* $\mathrm{gen}(M) = \lambda$. *Assume that M is κ-$\mathcal{Q}$-free for each regular uncountable cardinal $\mu < \kappa < \lambda$. Then M is $\mathcal{Q}$-filtered.*

In our particular setting, we will apply Theorem 8 under more general conditions. For this purpose, we need

Lemma 13. *Let R be a ring, $n < \omega$, and $\mathcal{B}$ a class of modules closed under arbitrary direct sums such that ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}_0$. Then ${}^\perp\mathcal{B} \cap \mathcal{P}_n = {}^\perp\mathcal{B}_n$ where $\mathcal{B}_n$ is the closure of $\mathcal{B} \cup$ Cos^n-R under arbitrary direct sums.*

Proof. Clearly, ${}^\perp\mathcal{B}_n \subseteq {}^\perp\mathcal{B} \cap \mathcal{P}_n$. Conversely, let $M \in {}^\perp\mathcal{B} \cap \mathcal{P}_n$ and $X \in \mathcal{B}_n$. Then X is of the form $B \oplus \bigoplus_{i \in I} C_i$ where $B \in \mathcal{B}$ and $(C_i \mid i \in I)$ is a family of modules from Cos^n-R. That is, we have exact sequences $E_{i,n-1} \to \cdots \to E_{i,0} \to C_i \to 0$ with $E_{i,j}$'s injective for all

$i \in I$ and $0 \leq j < n$. Since $M \in {}^\perp\mathcal{E}_0$, the exact sequence

$$\bigoplus_{i\in I} E_{i,n-1} \overset{f_{n-1}}{\to} \dots \overset{f_1}{\to} \bigoplus_{i\in I} E_{i,0} \overset{f_0}{\to} \bigoplus_{i\in I} C_i \to 0$$

implies $\operatorname{Ext}^k_R(M, \bigoplus_{i\in I} C_i) \cong \operatorname{Ext}^{k+n}_R(M, \operatorname{Ker} f_{n-1}) = 0$ for each $0 < k < \omega$. Thus $\operatorname{Ext}^k_R(M, X) = 0$, and we deduce $M \in {}^\perp\mathcal{B}_n$. □

The following lemma will serve as the induction step in the proof of Theorem 1.

For a class of modules $\mathcal{C}$, denote by $\mathfrak{A}_{\aleph_0}(\mathcal{C})$ the assertion: "All modules in $\mathcal{C}$ are $\mathcal{C}^{\leq\aleph_0}$-filtered".

Lemma 14. *Let R be a ring and $\mathcal{B}$ a class of modules closed under arbitrary direct sums such that ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}$. Then $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_n)$ implies $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_{n+1})$ for each $n < \omega$.*

Proof. Assume $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_n)$ holds. Let κ be a regular uncountable cardinal, $M \in {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}$ be a module, and $\lambda = \operatorname{gen}(M)$. W.l.o.g., there is a short exact sequence $0 \to K \hookrightarrow F \overset{\pi}{\to} M \to 0$ where $F = R^{(\lambda)}$, and K is a submodule of F.

Since $M \in {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}$, we have $K \in {}^\perp\mathcal{B} \cap \mathcal{P}_n$. Let $\mathcal{Q} = {}^\perp\mathcal{B} \cap \mathcal{P}_n^{\leq\aleph_0}$. By assumption and Lemma 11 (for $\mu = \aleph_0$), there are sets $\mathcal{S}_\kappa \subseteq \mathcal{S}_{\kappa^+}$ witnessing the κ-$\mathcal{Q}$-separability and κ^+-$\mathcal{Q}$-separability of K, respectively. Denote by $\mathcal{S}'_\kappa$ the set of all submodules $N \subseteq M$ such that there is a subset $A \subseteq \lambda$ of cardinality $< \kappa$ with $\pi(R^{(A)}) = N$ and $K \cap R^{(A)} \in \mathcal{S}_\kappa$.

Consider $L \in \mathcal{S}_\kappa$. Then L is $\mathcal{Q}$-filtered, so $L \in {}^\perp\mathcal{B}$ by Lemma 3. Moreover, L is $< \kappa$-generated and $\mathcal{P}_n^{\leq\aleph_0}$-filtered, so $L \in \mathcal{P}_n^{<\kappa}$ by the Horseshoe Lemma. This shows that $\mathcal{S}_\kappa \subseteq {}^\perp\mathcal{B} \cap \mathcal{P}_n^{<\kappa}$, and hence $\mathcal{S}'_\kappa \subseteq \mathcal{P}_{n+1}^{<\kappa}$.

We claim that $\mathcal{S}'_\kappa$ witnesses the κ-$\mathcal{Q}'_\kappa$-freeness of M where $\mathcal{Q}'_\kappa = {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}^{<\kappa}$. Clearly, $0 \in \mathcal{S}'_\kappa$, and $\mathcal{S}'_\kappa$ is closed under well-ordered unions of chains of length $< \kappa$. Moreover, we have the exact sequence $0 = \operatorname{Ext}^i_R(M, B) \to \operatorname{Ext}^i_R(N, B) \to \operatorname{Ext}^{i+1}_R(M/N, B) \cong \operatorname{Ext}^i_R(K/K \cap R^{(A)}, B) = 0$ for all $\pi(R^{(A)}) = N \in \mathcal{S}'_\kappa$, $B \in \mathcal{B}$ and $i \geq 1$. Thus $\mathcal{S}'_\kappa \subseteq Q'_\kappa$.

It remains to prove condition (c) of Definition 10. Let X be a subset of M of cardinality $< \kappa$. There is a subset $A_0 \subseteq \lambda$ of cardinality $< \kappa$ such that $X \subseteq \pi(R^{(A_0)})$. Let $L_0 = K \cap R^{(A_0)}$. We will prove that there is a module $K_0 \in \mathcal{S}_\kappa$ containing L_0.

If not, we can inductively construct a strictly ascending κ-filtration $(\tilde{K}_\alpha \mid \alpha \leq \kappa)$ such that $\tilde{K}_\alpha \in \mathcal{S}_\kappa$ and $L_0 \cap (\tilde{K}_{\alpha+1} \setminus \tilde{K}_\alpha) \neq \emptyset$ for all $\alpha < \kappa$. Indeed, take $\tilde{K}_0 = 0$, and for each $\alpha < \kappa$, $L_0 \nsubseteq \tilde{K}_\alpha$ by assumption, so we can find $\tilde{K}_{\alpha+1} \in \mathcal{S}_\kappa$ containing both $\tilde{K}_\alpha$ and an element $x \in L_0 \setminus \tilde{K}_\alpha$. Put $U = \tilde{K}_\kappa$ and consider the map $e : U \to \prod^{<\kappa}_{\alpha<\kappa} E_\alpha$ from Lemma 9. Then the union of supports of all elements of $e(U \cap L_0)$ equals κ. On the other hand, $U \in \mathcal{S}_{\kappa^+}$, so $K/U \in {}^\perp\mathcal{B}$. Since $F/K \cong M \in {}^\perp\mathcal{B}$ and ${}^\perp\mathcal{B} \subseteq {}^{\perp_1}\mathcal{E}$, we can extend e to K, then to F, to get a homomorphism $g : F \to \prod_{\alpha<\kappa}^{<\kappa} E_\alpha$ with $g \restriction U = e$. However, since $|A_0| < \kappa$, the union of supports of all elements of $g(R^{(A_0)})$ has cardinality $< \kappa$, a contradiction.

This proves that there exists $K_0 \in \mathcal{S}_\kappa$ such that $L_0 \subseteq K_0$. Take $A_1 \supseteq A_0$ such that $K_0 \subseteq R^{(A_1)}$ and $|A_1| < \kappa$. Put $L_1 = K \cap R^{(A_1)}$. Continuing in this way, we define a sequence $K_0 \subseteq K_1 \subseteq \dots$ of elements of $\mathcal{S}_\kappa$, and a sequence $A_0 \subseteq A_1 \subseteq \dots$ of subsets of λ of cardinality $< \kappa$ such that $K \cap R^{(A_i)} \subseteq K_i$ and $K_i \subseteq R^{(A_{i+1})}$ for all $i < \omega$. Then $K' = \bigcup_{i<\omega} K_i \in \mathcal{S}_\kappa$ and $K' = K \cap R^{(A')}$ where $A' = \bigcup_{i<\omega} A_i$. So $\pi(R^{(A')})$ is an element of $\mathcal{S}'_\kappa$ containing X, and $\mathcal{S}'_\kappa$ witnesses the κ-$\mathcal{Q}'_\kappa$-freeness of M. This completes the proof of the claim.

Now, we will prove $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_{n+1})$ by induction on $\lambda = \mathrm{gen}(M)$ for all $M \in {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}$. Define $\mathcal{R} = {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}^{\leq\aleph_0}$. If $\lambda \leq \aleph_0$, then we use Lemma 9 similarly as above to prove that the first syzygy, K, of M is countably generated. Since $K \in {}^\perp\mathcal{B} \cap \mathcal{P}_n$, by induction, we get that K has a projective resolution consisting of countably generated projective modules, so $M \in \mathcal{R}$.

If λ is regular, then we select from $\mathcal{S}'_\lambda$ a λ-filtration, $\mathcal{F}$, of M. Denote by $\mathcal{B}_{n+1}$ the closure of $\mathcal{B} \cup \mathrm{Cos}^{n+1}\text{-}R$ under arbitrary direct sums as in Lemma 13; we have ${}^\perp\mathcal{B} \cap \mathcal{P}_{n+1} = {}^\perp\mathcal{B}_{n+1}$. Since $0 = \mathrm{Ext}^i_R(N', B) \to \mathrm{Ext}^{i+1}_R(N/N', B) \to \mathrm{Ext}^{i+1}_R(N, B) = 0$ for all modules $N' \subseteq N \in \mathcal{F}$, $B \in \mathcal{B}_{n+1}$ and $i \geq 1$, we have $\mathrm{Ext}^i_R(N/N', \mathcal{B}_{n+1}) = 0$ for all $i \geq 2$. Then Theorem 8 yields a λ-subfiltration of $\mathcal{F}$ which is a ${}^\perp\mathcal{B}_{n+1}$-filtration of M. Using induction hypothesis, we refine this filtration to the desired $\mathcal{R}$-filtration of M.

If λ is singular then $\mathcal{S}'_\kappa$ witnesses κ-$\mathcal{R}$-freeness of M, for each regular uncountable cardinal $\kappa < \lambda$. So the existence of an $\mathcal{R}$-filtration of M follows by Lemma 12 for $\mu = \aleph_0$. □

A classical result of Kaplansky says that any projective module over any ring is a direct sum of countably generated modules. So $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_0)$ holds for any class of modules $\mathcal{B}$. Lemma 14 thus gives:

Theorem 15. *Let R be a ring and $\mathcal{B}$ a class of modules closed under arbitrary direct sums such that ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}$. Then for any $n < \omega$, all modules in ${}^\perp\mathcal{B} \cap \mathcal{P}_n$ are ${}^\perp\mathcal{B} \cap \mathcal{P}_n^{\leq\aleph_0}$-filtered.*

Now, it is easy to prove our main theorem:

Proof of Theorem 1. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be the tilting cotorsion pair induced by T, and n be the projective dimension of T. Then $\mathcal{A} = {}^\perp\mathcal{B} \cap \mathcal{P}_n$, so Corollary 7 applies and yields $\mathcal{E} \subseteq \mathcal{B}$. Let $\mathcal{C} = \mathcal{A}^{\leq\aleph_0}$. Then $\mathcal{C}^\perp = \mathcal{B} \,(= T^\perp)$ by Lemma 3 and Theorem 15, so T is of countable type. □

In general, there is a proper class of cotorsion pairs over a fixed ring R, cf. [13]. Since there is always a representative set of isomorphism classes of $\aleph_0$-presented modules, we get:

Corollary 16. *Let R be a ring. Then the cotorsion pairs induced by all tilting modules form a set.*

4. Tilting cotorsion pairs

If we omit the assumption of ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}$ in Lemma 14 we can still obtain a similar result, with $\aleph_0$ replaced by $\dim(R)$.

For a class of modules $\mathcal{C}$, denote by $\mathfrak{A}_\mu(\mathcal{C})$ the assertion: "All modules in $\mathcal{C}$ are $\mathcal{C}^{\leq\mu}$-filtered".

Lemma 17. *Let R be a ring and $\mathcal{B}$ a class of modules closed under arbitrary direct sums. Let $\mu = \dim(R)$. Then $\mathfrak{A}_\mu({}^\perp\mathcal{B} \cap \mathcal{P}_n)$ implies $\mathfrak{A}_\mu({}^\perp\mathcal{B} \cap \mathcal{P}_{n+1})$ for each $n < \omega$.*

Proof. The proof is the same as for Lemma 14 except that the induction on $\mathrm{gen}(M)$ starts at μ rather than $\aleph_0$, and when proving condition (c) of Definition 10 for $\mathcal{S}'_\kappa$, we find $K_i \in \mathcal{S}_\kappa$ containing L_i using Lemma 4 rather than Lemma 9. Finally, Lemma 13 is not needed when λ is a regular cardinal $\geq \dim(R)$, since each module $M \in \mathcal{P}_n$ with $\mathrm{gen}(M) = \lambda$ has a λ-filtration with successive factors in $\mathcal{P}_n^{<\lambda}$ by [2]. □

As in the case of Theorem 15, Lemma 17 implies:

Theorem 18. *Let R be a ring, $n < \omega$, and $\mathcal{B}$ be a class of modules closed under arbitrary direct sums. Then all modules in ${}^\perp\mathcal{B} \cap \mathcal{P}_n$ are ${}^\perp\mathcal{B} \cap \mathcal{P}_n^{\leq\dim(R)}$-filtered.*

Now, our second main result follows easily:

Proof of Theorem 2. It is well known that $\mathfrak{C}$ is n-tilting if and only if $\mathfrak{C}$ is complete, hereditary, $\mathcal{A} \subseteq \mathcal{P}_n$ and $\mathcal{B}$ is closed under arbitrary direct sums, see [3]. So it remains to prove the if-part. Indeed, it is enough to prove that the completness of $\mathfrak{C}$ is implied by the other three conditions. But then $\mathfrak{C}$ is cogenerated by a set by Lemma 3 and Theorem 18, so $\mathfrak{C}$ is complete by [11, Theorem 2]. □

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III.

ALL TILTING MODULES ARE OF FINITE TYPE

Abstract

We prove that any infinitely generated tilting module is of finite type, namely that its associated tilting class is the Ext-orthogonal of a set of modules possessing a projective resolution consisting of finitely generated projective modules.

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ALL TILTING MODULES ARE OF FINITE TYPE

SILVANA BAZZONI AND JAN ŠŤOVÍČEK

1. Introduction

In the early eighties Brenner and Butler [9], Happel and Ringel [18] generalized the classical Morita equivalence induced by a progenerator by introducing the notion of a tilting module over a finite dimensional artin algebras. In order to obtain equivalences between subcategories of the module category, tilting modules were assumed to be finitely generated and moreover of projective dimension at most one. Later, Miyashita [20] considered tilting modules of finite projective dimension and studied the equivalences induced by them. Colby and Fuller extended the setting to arbitrary rings, but in all the above mentioned papers the tilting modules were always assumed to be finitely generated. The notion of infinitely generated tilting modules over arbitrary rings was introduced by Colpi and Trlifaj in [11] for the one dimensional case and by Angeleri-Hügel and Coelho in [1] in the general case of finite projective dimension.

One of the advantages in dealing with infinitely generated tilting modules is evident in connection with the good approximation theory that they induce and which we are going to illustrate (cf. [3] and [1]). We recall that the tilting class $\mathcal{B}$ associated to a tilting module T over a ring R is the class of R-modules which are in the kernel of all the functors $\mathrm{Ext}^i_R(T,-)$. Such intersection of kernels is also called a right orthogonal of T. If $\mathcal{A}$ denotes the class of the R-modules which are in the kernel of all the functors $\mathrm{Ext}^i_R(-,B)$, for any $B \in \mathcal{B}$, then the pair $(\mathcal{A},\mathcal{B})$ of classes of R modules is a hereditary cotorsion pair which induces special precovers and preenvelopes (cf. [16] and [24]). Alternatively, in the terminology used in [4, 5] for classes of finitely generated modules, it induces right and left approximations.

There remained the problem to illustrate at what extent finitely and infinitely generated tilting modules are related. More precisely, the question was to decide whether the tilting classes are determined

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by finitely presented data in the sense that they are the right Ext-orthogonals of a set of finitely generated modules. There have been various efforts by many authors in order to solve this problem. A first answer was given by Eklof, Trlifaj and the first author in [7]. Using set theoretic methods, they show that one dimensional tilting modules are of countable type in the sense that their associated tilting classes are the right Ext-orthogonal of a set of countably generated modules. Moreover, in [7] it is shown that the problem to determine whether a one dimensional tilting module is of finite type is equivalent to prove that the associated tilting class is a definable class (see Section 2 for precise definitions). Herbera and the first author in [8] proved that every tilting module of projective dimension one is of finite type. The solution of the problem is obtained by proving that suitable countable inverse limits of groups of homomorphisms satisfy the Mittag-Leffler condition and that this condition is inherited by pure submodules. Almost at the same time Trlifaj and the second author in [22] were able to extend the set theoretic methods used in [7] to the case of tilting modules of any finite projective dimension, and to prove that they are of countable type.

In the present paper, we prove that tilting modules of any finite projective dimension are of finite type. First of all we note that the reduction to the countable case as proved in [22] is crucial. The next step is to prove that a tilting class is of finite type if and only if it is definable. In our situation this amounts to prove that, if $(\mathcal{A}, \mathcal{B})$ is the cotorsion pair induced by a tilting module, then every countably presented module in the class $\mathcal{A}$ is a direct limit of finitely presented modules in the class $\mathcal{A}$. This result will be achieved in Section 3 by using a particular notion of "freeness" for modules whose origin (for general algebraic structures) goes back to ideas introduced by Shelah to prove the singular compactness theorem. We say that a module is "free" if it admits a filtration of submodules with finitely presented factors. An adaptation of the proof of [14, XII.1.14] to our situation allow us to represent every countably presented module in the class $\mathcal{A}$ as a countable direct limit of finitely presented modules in the class $\mathcal{A}$. Our final result, Theorem 4.2, is obtained by induction on the projective dimension of a tilting module: we induct on the tilting cotorsion pairs induced by the syzygies of a tilting module and then we use arguments similar to the ones developed in [8] for the one dimensional case.

2. Preliminaries

In what follows, R is always an associative ring with unit and all modules are right R-modules. First, we fix the terminology and recall some definitions.

$\mathcal{P}_n$ will denote the class of all modules of projective dimension at most n.

For a class $\mathcal{C}$ of modules and an infinite cardinal κ, we define $\mathcal{C}^{<\kappa}$ to be the class of all modules in $\mathcal{C}$ possessing a projective resolution consisting of $<\kappa$ generated projective modules.

An ascending chain $(M_\alpha \mid \alpha < \kappa)$ of submodules of a module M indexed by a cardinal κ is called *continuous* if $M_\alpha = \cup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \kappa$. It is called a filtration of M if $M_0 = 0$ and $M = \cup_{\alpha<\kappa} M_\alpha$.

Given a class $\mathcal{C}$ of modules, we say that a module M is *$\mathcal{C}$-filtered* if it admits a filtration $(M_\alpha \mid \alpha < \kappa)$ such that $M_{\alpha+1}/M_\alpha$ is isomorphic to some module in $\mathcal{C}$ for every $\alpha < \kappa$. In this case we say that $(M_\alpha \mid \alpha < \kappa)$ is a *$\mathcal{C}$-filtration* of M.

A class of modules is called *definable* if it is closed under arbitrary direct products, direct limits, and pure submodules (cf. [12, §2.3]). Definable classes are axiomatizable by first order formulas and they are characterized by the subclass of pure-injective modules that they contain.

Let $\mathcal{C} \subseteq \operatorname{Mod} R$. Define $\mathcal{C}^{\perp_1} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{C}, -) = \{M \in \operatorname{Mod} R \mid \operatorname{Ext}^1_R(X, M) = 0 \text{ for all } X \in \mathcal{C}\}$, ${}^{\perp_1}\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{C})$, $\mathcal{C}^{\perp} = \bigcap_{i\geq 1} \operatorname{Ker}\operatorname{Ext}^i_R(\mathcal{C}, -)$, and ${}^{\perp}\mathcal{C} = \bigcap_{i\geq 1} \operatorname{Ker}\operatorname{Ext}^i_R(-, \mathcal{C})$. If the class $\mathcal{C}$ has only one element, say $\mathcal{C} = \{X\}$, we write just $X^{\perp_1}$ instead of $\{X\}^{\perp_1}$, and similarly in the other cases.

A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. Note that for every class $\mathcal{C}$, ${}^{\perp}\mathcal{C}$ is a *resolving* class, that is, it is closed under extensions, kernels of epimorphisms and contains the projective modules. In particular, it is syzygy-closed. Dually, $\mathcal{C}^{\perp}$ is *coresolving*: it is closed under extensions, cokernels of monomorphisms and contains the injective modules. In particular, it is cosyzygy-closed. A pair $(\mathcal{A}, \mathcal{B})$ is called a *hereditary cotorsion pair* if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. It is easy to see that $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{A}$ is resolving if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{B}$ is coresolving.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* provided that every R-module M admits a *special $\mathcal{B}$-preenvelope*, that is, if there exists an exact sequence of the form $0 \to M \to B \to A \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. For a class $\mathcal{C}$ of modules, the pair $\mathfrak{C} = ({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp})$ is a (hereditary) cotorsion pair; it is called the cotorsion pair *cogenerated* by $\mathcal{C}$. Every cotorsion pair cogenerated by a set of modules is complete, [15]. If all the modules in $\mathcal{C}$ have projective dimension $\leq n$, then ${}^{\perp}(\mathcal{C}^{\perp}) \subseteq \mathcal{P}_n$ as well.

If $\mathcal{C}$ is a set, then a complete description of the modules in ${}^{\perp_1}(\mathcal{C}^{\perp_1})$ is available. In fact, by results in [15] and by [23, Theorem 22], a module belongs to ${}^{\perp_1}(\mathcal{C}^{\perp_1})$ if and only if it is a direct summand of a $\mathcal{C}'$-filtered module where $\mathcal{C}' = \mathcal{C} \cup \{R\}$. Clearly, ${}^{\perp}(\mathcal{C}^{\perp}) = {}^{\perp_1}(\mathcal{C}^{\perp_1})$ provided that a first syzygy of M is contained in $\mathcal{C}$ whenever $M \in \mathcal{C}$.

A hereditary cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in $\operatorname{Mod} R$ is of *countable type*, and *finite type*, provided there is a class $\mathcal{S}$ of modules in $\mathcal{A}^{<\aleph_1}$, and $\mathcal{A}^{<\aleph_0}$, respectively, such that $\mathcal{S}$ cogenerates $\mathfrak{C}$, that is $\mathcal{B} = \mathcal{S}^{\perp}$.

We are interested in cotorsion pairs cogenerated by an n-tilting module. Recall that for $n < \omega$, a module T is *n-tilting* provided

(T1) $T \in \mathcal{P}_n$,
(T2) $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I, and
(T3) there exist $r \geq 0$ and a long exact sequence

$$0 \to R \to T_0 \to \cdots \to T_r \to 0$$

such that $T_i \in \operatorname{Add}(T)$ for each $0 \leq i \leq r$.

Here, $\operatorname{Add}(T)$ denotes the class of all direct summands of arbitrary direct sums of copies of T.

A class of modules $\mathcal{T}$ is *n-tilting* provided there is an n-tilting module T such that $\mathcal{T} = T^{\perp}$. In this case, the cotorsion pair cogenerated by T, namely $({}^{\perp}\mathcal{T}, \mathcal{T})$, is called an *$n$-tilting cotorsion pair*. Every module in ${}^{\perp}\mathcal{T}$ has then projective dimension $\leq n$ and the class $\mathcal{T}$ is closed under arbitrary direct sums. If $({}^{\perp}\mathcal{T}, \mathcal{T})$ is of countable type or of finite type, then $\mathcal{T}$ and T are called n-tilting of *countable type* or of *finite type*, respectively.

Notice that any tilting class of finite type is definable. More generally, for any cotorsion pair $(\mathcal{A}, \mathcal{B})$ of finite type the class $\mathcal{B}$ is definable. This follows from the well-known facts that if F posses a projective resolution of finitely generated projective modules, then $\operatorname{Ext}^n_R(F, -)$ commutes with direct limits for all $n < \omega$, and that $G^{\perp_1}$ is closed under pure submodules whenever G is finitely presented.

3. Countably generated modules

The aim of this section is to investigate conditions under which, for a given class $\mathcal{C}$, a module $M \in \mathcal{C}^{<\aleph_1}$ is a countable direct limit of objects in $\mathcal{C}^{<\aleph_0}$. The key idea is to look at conditions which imply that the first sygyzy module of M is $\mathcal{C}^{<\aleph_0}$-filtered.

A particular notion of "freeness" will be used. Its origin goes back to the Shelah's singular compactness theorem. The original version of the theorem appears in [21], a shorter and more algebraic proof is given in [19]. In these papers, an algebraic structure is called "free", if it satisfies certain rather general axioms wich are generalizations of properties valid for free structures and for their bases. For R-modules, this notion applies to more general situation than just free R-modules. In our setting for instance, M is considered as "free" provided that M is $\mathcal{Q}$-filtered by a family of $< \mu$-presented modules for some cardinal μ. Rather than stating the general axioms for the "freeness", we will concentrate on our case, where the key results concerning this notion have been collected and very well illustrated in [14, XII.1.14] or [13].

The following result is inspired by the proof of [14, XII.1.14]. The difference in our situation is that we need to take care of possibly finitely presented modules, and to explicitly state some properties contained only implicitly in the proofs in the original papers. For the sake of further simplification, we prove the proposition only for $\mu = \aleph_0$ or $\mu = \aleph_1$. It could be, however, extended to any regular cardinal by following the original proofs.

Proposition 3.1. *Let $\mu = \aleph_0$ or $\mu = \aleph_1$. Let M be a $\mathcal{Q}$-filtered module where $\mathcal{Q}$ is a family of $< \mu$-presented modules. Then there exists a subset $\mathcal{S}$ of $\mathcal{Q}$-filtered submodules of M satisfying the following properties:*

1. *$0 \in \mathcal{S}$.*
2. *$\mathcal{S}$ is closed under unions of arbitrary chains.*
3. *For every $N \in \mathcal{S}$, N and M/N are $\mathcal{Q}$-filtered.*
4. *For every subset $X \subseteq M$ of cardinality $< \mu$, there is a $< \mu$-presented module $N \in \mathcal{S}$ such that $X \subseteq N$.*

Proof. Let $(M_\alpha \mid \alpha < \lambda)$ be a $\mathcal{Q}$-filtration of M and let $\mathcal{S}$ be the set of all submodules N of M with the property:

$(*)$ for all $\alpha < \lambda$, $N \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$ implies $N + M_\alpha \supseteq M_{\alpha+1}$.

The properties (1) and (2) are then clear.

(3). We first show that every $N \in \mathcal{S}$ is $\mathcal{Q}$-filtered. Consider the family $(N \cap M_\alpha \mid \alpha < \lambda)$; then $(N \cap M_{\alpha+1})/(N \cap M_\alpha)$ is zero if $N \cap (M_{\alpha+1} \setminus M_\alpha) = \emptyset$ and isomorphic to $M_{\alpha+1}/M_\alpha$ otherwise. Thus, possibly by omitting some indicies, we get a $\mathcal{Q}$-filtration of N.

Next, we define by induction on ρ a $\mathcal{Q}$-filtration $(N_\rho/N \mid \rho < \tau)$ of M/N such that in addition $N_\rho \in \mathcal{S}$ for all ρ. Let $N_0 = N$ and assume N_ρ have been defined for all $\rho < \sigma$ ($\sigma > 0$). If σ is limit ordinal, let $N_\sigma = \bigcup_{\rho<\sigma} N_\rho$; then again $N_\sigma \in \mathcal{S}$ by (2). If $\sigma = \beta + 1$, choose a minimal cardinal δ such that $M_\delta \nsubseteq N_\beta$. If there is no such cardinal, then $N_\beta = M$, and we have already the filtration. Otherwise by continuity, $\delta = \gamma + 1$ for some $\gamma < \lambda$. We define $N_{\beta+1} = N_\beta + M_{\gamma+1}$. By $(*)$ and by the choice of $\gamma + 1$, it is easy to verify that $M_{\gamma+1}/M_\gamma \cong N_{\beta+1}/N_\beta$ via $x + M_\gamma \mapsto x + N_\beta$. So $N_{\beta+1}/N_\beta$ is isomorphic to an element of $\mathcal{Q}$. It remains to show that $N_{\beta+1}$ satisfies $(*)$. Let $N_{\beta+1} \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$, for some $\alpha < \lambda$. If $\alpha \leq \gamma$, then $N_{\beta+1} + M_\alpha \supseteq M_{\alpha+1}$, since $N_{\beta+1} \supseteq M_{\gamma+1} \supseteq M_{\alpha+1}$. If $\alpha > \gamma$, let $x = y + z \in N_{\beta+1} \cap (M_{\alpha+1} \setminus M_\alpha)$ with $y \in N_\beta$ and $z \in M_{\gamma+1}$. Then $y \in M_{\alpha+1}$ and $y \notin M_\alpha$; thus by induction $N_\beta + M_\alpha \supseteq M_{\alpha+1}$ and $N_{\beta+1}$ satisfies $(*)$.

(4). Let X be a subset of M of cardinality $< \mu$. Let $\rho(X)$ be the least ordinal number such that $X \subseteq M_\rho$. We prove by induction on ρ that there exists a $< \mu$-presented module $N \in \mathcal{S}$ such that $X \subseteq N \subseteq M_\rho$. If $\rho = 0$ there is nothing to prove. If ρ is a limit ordinal, then X is not a finite subset of M, so $\mu = \aleph_1$. Let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of X and let $\beta_n < \rho$ be the least ordinal such that $\{x_1, x_2, \ldots, x_n\} \subseteq M_{\beta_n}$, for every $n < \omega$. By induction, we can define an ascending chain $(N_n \mid n < \omega)$ of modules of $\mathcal{S}$ with $N_n < \mu$-presented, $N_n \subseteq M_{\beta_n}$ and $N_{n+1} \supseteq N_n \cup \{x_{n+1}\}$. Set $N = \bigcup_{n<\omega} N_n$; by (2) $N \in \mathcal{S}$. Clearly, $X \subseteq N$ and N is $< \mu$-presented contained in M_ρ.

If ρ is not a limit ordinal, let $\rho = \beta + 1$ for some $\beta < \lambda$. Let L be a $< \mu$-generated submodule of $M_{\beta+1}$ such that $L \supseteq X$ and $L + M_\beta = M_{\beta+1}$. Since $M_{\beta+1}/M_\beta$ is $< \mu$-presented, we infer that $L \cap M_\beta$ is $< \mu$-generated. Let Y be a generating system of $L \cap M_\beta$ of cardinality $< \mu$. By inductive hypothesis, there exists a $< \mu$-presented module $N_0 \in \mathcal{S}$ such that $Y \subseteq N_0 \subseteq M_\beta$. We claim that $N = L + N_0$ satisfies the wanted conditions. Clearly $X \subseteq N$. Moreover, $N/N_0 \cong L/(N_0 \cap L)$ and $L/(N_0 \cap L) \cong M_{\beta+1}/M_\beta$ via $x + (N_0 \cap L) \mapsto x + M_\beta$, since $N_0 \subseteq M_\beta$. Thus, N is $< \mu$-presented. It remains to show that $N \in \mathcal{S}$. Let $\gamma < \lambda$ be such that there is $x \in N \cap (M_{\gamma+1} \setminus M_\gamma)$. Then $\gamma \leq \beta$, since $N \subseteq M_{\beta+1}$. If $\gamma < \beta$, then $x \in N_0$, since $N_0 \subseteq M_\beta$. So $N + M_\gamma \supseteq N_0 + M_\gamma \supseteq M_{\gamma+1}$. If $\gamma = \beta$, then $N + M_\beta = L + M_\beta = M_{\beta+1}$. We conclude that N satisfies the condition $(*)$.

□

We will underline an immediate consequence of the condition (4) in Proposition 3.1 stating it as a corollary:

Corollary 3.2. *Let M be a countably generated $\mathcal{Q}$-filtered module where $\mathcal{Q}$ is a family of finitely presented modules. Then there is a filtration $(M_n \mid n < \omega)$ of M consisting of finitely presented submodules of M such that M_n and M/M_n are $\mathcal{Q}$-filtered for every $n < \omega$.*

The following technical lemma will be of use later:

Lemma 3.3. *Let $\mathcal{Q}$ be a family of finitely presented modules containing the regular module R. Let M be a countably presented module and let*

$$0 \to K \to F \to M \to 0$$

be a free presentation of M with F and K countably generated. Assume that K is a direct summand of a $\mathcal{Q}$-filtered module. Then, there exists an exact sequence:

$$0 \to H \to G \to M \to 0$$

where H and G are countably generated $\mathcal{Q}$-filtered modules.

Proof. Let K be a summand of a $\mathcal{Q}$-filtered module P. Since K is countably generated, Proposition 3.1 implies that K is contained in a countably generated $\mathcal{Q}$-filtered submodule of P; thus we may assume that P is countably generated. By Eilenberg's trick, $K \oplus P^{(\omega)} \cong P^{(\omega)}$. Consider the exact sequence

$$0 \to K \oplus P^{(\omega)} \to F \oplus P^{(\omega)} \to M \to 0$$

and let $H = K \oplus P^{(\omega)} \cong P^{(\omega)}$, $G = F \oplus P^{(\omega)}$. Then G and H are countably generated $\mathcal{Q}$-filtered modules. □

We apply the preceding results to the context of n-tilting cotorsion pairs.

Lemma 3.4. *Let $(\mathcal{A}, \mathcal{B})$ be an n-tilting cotorsion pair. If $A \in \mathcal{A}$ is a countably or finitely generated module, then $A \in \mathcal{A}^{<\aleph_1}$ or $A \in \mathcal{A}^{<\aleph_0}$, respectively.*

Proof. Since $\mathcal{A}$ is a resolving class, it is enough to show that every countably or finitely generated module in $\mathcal{A}$ is countably or finitely presented, respectively.

By [22, Theorems 1 and 15], $(\mathcal{A}, \mathcal{B})$ is of countable type and every module $A \in \mathcal{A}$ is $\mathcal{A}^{<\aleph_1}$-filtered. By Proposition 3.1 (4), a countably generated $A \in \mathcal{A}$ is countably presented.

Assume now that $A \in \mathcal{A}$ is finitely generated and let $0 \to K \to R^n \to A \to 0$ be a presentation of A. By the first part of the proof, K is countably generated. Write $K = \bigcup_{n \in \omega} K_n$, where K_n is a chain of finitely generated submodules of K. Consider the map $\phi\colon K \to \prod_{n<\omega} E_n$ defined by $\phi(x) = (x + K_n)_{n<\omega}$, where E_n is an injective envelope of K/K_n for every $n < \omega$. The image of ϕ is contained in $\bigoplus_{n<\omega} E_n$ which is an object of $\mathcal{B}$. Thus, ϕ extends to some $\psi\colon R^n \to \bigoplus_{n<\omega} E_n$. As a consequence, the image of ϕ is contained in $\bigoplus_{n \leq m} E_n$, for some $m < \omega$. Hence K is finitely generated. □

In order to use an inductive argument we show now that for an n-tilting module T, $n \geq 1$, the cotorsion pair cogenerated by a first syzygy of T is an $(n-1)$-tilting cotorsion pair.

Lemma 3.5. *Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by an n-tilting module T with $n \geq 1$. Let $(\mathcal{A}_1, \mathcal{B}_1)$ be the cotorsion pair cogenerated by a first syzygy module of T. Then:*

1. *$(\mathcal{A}_1, \mathcal{B}_1)$ is an $(n-1)$-tilting cotorsion pair,*
2. *$X \in \mathcal{B}_1$ if and only if (any) first cosyzygy of X belongs to $\mathcal{B}$,*
3. *$M \in \mathcal{A}$ implies that (any) first sysygy of M belongs to $\mathcal{A}_1$.*

Proof. Let $0 \to \Omega(T) \to F \to T \to 0$ be a presentation of T with F free and $\Omega(T)$ a first syzygy module of T. Then $\Omega(T)$ has projective dimension at most $(n-1)$ and, by [6, Lemma 3.4], $\Omega(T)^{\perp}$ is closed under direct sums. By the characterization of tilting classes (cfr. [1] or [22]), $\Omega(T)^{\perp}$ is an $(n-1)$-tilting class. Let $(\mathcal{A}_1, \mathcal{B}_1)$ be the associated cotorsion pair, namely $\mathcal{B}_1 = \Omega(T)^{\perp}$ and $\mathcal{A}_1 = {}^{\perp}\mathcal{B}_1 = {}^{\perp_1}\mathcal{B}_1$. For modules X and M consider exact sequences $0 \to X \to I \to \Omega^{-}(X) \to 0$ and $0 \to \Omega(M) \to F' \to M \to 0$, where I is an injective module, F' is a free module and $\Omega^{-}(X)$, $\Omega(M)$ are first cosyzygy and syzygy module of X and M, respectively. Then $\operatorname{Ext}^i_R(\Omega(M), X) \cong \operatorname{Ext}^{i+1}_R(M, X)$ and $\operatorname{Ext}^{i+1}_R(M, X) \cong \operatorname{Ext}^i_R(M, \Omega^{-}(X))$ for all $i \geq 1$. The last two statements follow immediately by these formulas. □

The following provides the key ingredient for proving our main result.

Lemma 3.6. *Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by an n-tilting module T. Assume that the cotorsion pair $(\mathcal{A}_1, \mathcal{B}_1)$ cogenerated by a first syzygy of T is of finite type. Then any countably generated module $A \in \mathcal{A}$ is isomorphic to a direct limit of a countable direct system of the form:*

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \to \ldots \to C_n \xrightarrow{f_n} C_{n+1} \to \ldots$$

where the modules C_n's are in $\mathcal{A}^{<\aleph_0}$.

Proof. Since the cotorsion pair $(\mathcal{A}_1, \mathcal{B}_1)$ is of finite type, it is cogenerated by (a representative subset of) $\mathcal{A}_1^{<\aleph_0}$. As recalled in Section 2, $\mathcal{A}_1$ coincides with the class of all direct summands of the $\mathcal{A}_1^{<\aleph_0}$-filtered modules.

Fix a countably generated module $A \in \mathcal{A}$ and let

$$0 \to K \to \bigoplus_{n<\omega} x_n R \to A \to 0$$

be a presentation of A. Then $K \in \mathcal{A}_1$, thus K is a summand in an $\mathcal{A}_1^{<\aleph_0}$-filtered module. By Lemma 3.4, A is countably presented; so by the hypotheses and Lemma 3.3, there exists an exact sequence $0 \to H \to G \to A \to 0$ where G and H are countably generated $\mathcal{A}_1^{<\aleph_0}$-filtered modules. By Corollary 3.2, we can write $H = \bigcup_{n<\omega} H_n$ and $G = \bigcup_{n<\omega} G_n$ where, for every $n < \omega$, H_n and G_n are finitely presented $\mathcal{A}_1^{<\aleph_0}$-filtered modules, and H/H_n, G/G_n are $\mathcal{A}_1^{<\aleph_0}$-filtered. W.l.o.g. we can assume that H is a submodule of G. Given $n < \omega$, there is an $m(n)$ such that $H_n \subseteq G_{m(n)}$; and we can choose the sequence $(m(n))_{n<\omega}$ to be strictly increasing.

We claim that $G_{m(n)}/H_n \in \mathcal{A}^{<\aleph_0}$. By Lemma 3.4, it is enough to show that $G_{m(n)}/H_n \in \mathcal{A}$. Let $B \in \mathcal{B}$; we have to show that $\operatorname{Ext}_R^1(G_{m(n)}/H_n, B) = 0$. But H/H_n is $\mathcal{A}_1^{<\aleph_0}$-filtered, thus in $\mathcal{A}_1$, and it is immediate to check that $\mathcal{A}_1 \subseteq \mathcal{A}$. Moreover, $G/H \cong A \in \mathcal{A}$. Hence, every homomorphism $f \colon H_n \to B$ can be extended to a homomorphism $g \colon G \to B$, and the restriction of g to $G_{m(n)}$ obviously induces an extension of f to $G_{m(n)}$. Thus $\operatorname{Ext}_R^1(G_{m(n)}/H_n, B) = 0$, since $G_{m(n)} \in \mathcal{A}_1 \subseteq \mathcal{A}$.

Set $C_n = G_{m(n)}/H_n$. Since $(m(n))_{n<\omega}$ is increasing and unbounded in ω, the inclusions $G_{m(n)} \subseteq G_{m(n+1)}$ induce maps $f_n \colon C_n \to C_{n+1}$, and A is a direct limit of the direct system $(C_n; f_n)_{n<\omega}$. □

4. Finite type

In the last step before stating the main result, we give a criterion similar to [2, Proposition 2.6] for a tilting cotorsion pair to be of finite type.

Proposition 4.1. *Let $(\mathcal{A}, \mathcal{B})$ be an n-tilting cotorsion pair. Assume that every countably generated module $A \in \mathcal{A}$ is isomorphic to a direct limit of some modules in $\mathcal{A}^{<\aleph_0}$. Then the following are equivalent:*

(1) *the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type;*
(2) *$\mathcal{B}$ is closed under pure submodules;*
(3) *$\mathcal{B}$ is a definable class.*

Proof. The implication (3) $\Rightarrow$ (2) is clear. For the converse, recall that $\mathcal{B}$ is a coresolving class closed under direct sums. So if $\mathcal{B}$ is closed under pure submodules then it is also closed under direct limits. $\mathcal{B}$ is always closed under products, thus (2) implies (3).

(1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) First of all recall that, by [22, Theorem 1], every n-tilting cotorsion pair is of countable type. Hence $\mathcal{B} = (\mathcal{A}^{<\aleph_1})^{\perp}$. Let $\mathcal{B}' = (\mathcal{A}^{<\aleph_0})^{\perp}$; then $\mathcal{B}'$ is a definable class containing $\mathcal{B}$ and it is well known (cfr. [12]) that two definable classes coincide if and only if they contain the same pure-injective modules. Let M be a pure-injective module in $\mathcal{B}'$ and let $A \in \mathcal{A}^{<\aleph_1}$. By hypothesis $A \cong \varinjlim C_n$, $C_n \in \mathcal{A}^{<\aleph_0}$. Then, from a well known result by Auslander, $\mathrm{Ext}^i_R(A, M) \cong \varprojlim \mathrm{Ext}^i_R(C_n, M) = 0$. Hence, $M \in \mathcal{B}$ and we conclude that $\mathcal{B} = \mathcal{B}'$. □

We are now in a position to prove our main result.

Theorem 4.2. *Let R be any ring and T be an n-tilting R-module, $n \geq 0$. Then T is of finite type.*

Proof. The proof is by induction on the projective dimension n of T.

If $n = 0$, the conclusion is obvious. Next, assume that all m-tilting modules are of finite type for every $m < n$. Let T be a tilting module of projective dimension n and let $(\mathcal{A}, \mathcal{B})$ be the n-tilting cotorsion pair cogenerated by T. By [22, Theorem 1], $(\mathcal{A}, \mathcal{B})$ is of countable type, hence $\mathcal{B} = (\mathcal{A}^{<\aleph_1})^{\perp}$.

Consider a free presentation of T:

$$0 \to \Omega(T) \to F \to T \to 0,$$

and let $(\mathcal{A}_1, \mathcal{B}_1)$ be the cotorsion pair cogenerated by $\Omega(T)$. By Lemma 3.5, $(\mathcal{A}_1, \mathcal{B}_1)$ is an $(n-1)$-tilting cotosion pair; so it is of finite type by inductive hypothesis. In particular, $\mathcal{B}_1 = (\mathcal{A}_1^{<\aleph_0})^{\perp}$.

Let $A \in \mathcal{A}$ be a countably generated module. We can then apply Lemma 3.6 to conclude that there is a countable direct system of the form

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \to \dots \to C_n \xrightarrow{f_n} C_{n+1} \to \dots$$

where C_n's are finitely presented modules in $\mathcal{A}$ and $A \cong \varinjlim C_n$. As in [8], we consider the representation of the countable direct limit given by the exact sequence:

$$(\dagger) \qquad 0 \to \bigoplus_{n<\omega} C_n \xrightarrow{\phi} \bigoplus_{n<\omega} C_n \to \varinjlim C_n \cong A \to 0$$

where, for every $n < \omega$, $\phi\varepsilon_n = \varepsilon_n - \varepsilon_{n+1} f_n$ and $\varepsilon_n \colon C_n \to \oplus_{n<\omega} C_n$ denotes the canonical map.

By Proposition 4.1, $(\mathcal{A}, \mathcal{B})$ is of finite type if and only if $\mathcal{B}$ is closed under pure submodules. Let Y be a pure submodule of a module $Z \in \mathcal{B}$. Since $(\mathcal{A}, \mathcal{B})$ is of countable type, Y is in $\mathcal{B}$ if and only if $\operatorname{Ext}^i_R(A, Y) = 0$, for every $A \in \mathcal{A}^{<\aleph_1}$ and $i \geq 1$. Moreover, $\mathcal{A}^{<\aleph_1}$ is clearly closed under countable syzygies, thus $Y \in \mathcal{B}$ if and only if $\operatorname{Ext}^1_R(A, Y) = 0$, for every $A \in \mathcal{A}^{<\aleph_1}$.

We continue now in a similar way as in the proof of [8, Theorem 2.5, 2.6]. We repeat the argument for the sake of completion.

Set $Z_n = Z$ for any $n < \omega$. Since $\mathcal{B}$ is closed under direct sums, $\bigoplus_{n<\omega} Z_n \in \mathcal{B}$, hence $\operatorname{Ext}^1_R(A, \bigoplus_{n<\omega} Z_n) = 0$. From the short exact sequence (†), we see that for every homomorphism $\gamma \colon \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} Z_n$ there exists $\psi \colon \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} Z_n$ such that $\psi\phi = \gamma$. By [8, Theorem 3.7] the inverse system of abelian groups

$$(\operatorname{Hom}_R(C_n, Z), \operatorname{Hom}_R(f_n, Z))_{n<\omega}$$

is Mittag-Leffler. Since Y is a pure submodule of Z and the modules C_n's are finitely presented, [8, Theorem 4.3] yields that the inverse system:

$$(\operatorname{Hom}_R(C_n, Y), \operatorname{Hom}_R(f_n, Y))_{n<\omega}$$

is Mittag-Leffler, too.

Applying the functors $\operatorname{Hom}_R(C_n, -)$ to the pure exact sequence

$$0 \to Y \to Z \to Z/Y \to 0$$

we obtain an inverse system of pure exact sequences of the form

$$0 \to \operatorname{Hom}_R(C_n, Y) \to \operatorname{Hom}_R(C_n, Z) \to \operatorname{Hom}_R(C_n, Z/Y) \to 0.$$

As $(\operatorname{Hom}_R(C_n, Y), \operatorname{Hom}_R(f_n, Y))_{n<\omega}$ is Mittag-Leffler, we can apply [17, Proposition 13.2.2] to conclude that there is an exact sequence

$$0 \to \varprojlim \operatorname{Hom}_R(C_n, Y) \to \varprojlim \operatorname{Hom}_R(C_n, Y) \to \varprojlim \operatorname{Hom}_R(C_n, Z/Y) \to 0,$$

which in turn gives the exact sequence

$$0 \to \operatorname{Hom}_R(A, Y) \to \operatorname{Hom}_R(A, Z) \to \operatorname{Hom}_R(A, Z/Y) \to 0.$$

Therefore, we also have the exact sequence

$$0 \to \operatorname{Ext}^1_R(A, Y) \to \operatorname{Ext}^1_R(A, Z) = 0$$

from which we deduce that $\operatorname{Ext}^1_R(A, Y) = 0$ as desired.

□

Remark 4.3. *Let T be an n-tilting module and $(\mathcal{A}, \mathcal{B})$ be the induced n-tilting cotorsion pair. Unlike in the case of countable type, we cannot prove in general that T is $\mathcal{A}^{<\aleph_0}$-filtered. Consider a projective module P without finitely generated direct summands. Then $R \oplus P$ is a projective generator, thus a tilting module, but it is not a direct sum of finitely generated projective modules.*

But we can always find an equivalent tilting module T' (that is $T^\perp = (T')^\perp$) such that T' is $\mathcal{A}^{<\aleph_0}$-filtered. Indeed, let $0 \to R \to T_0 \to X_0 \to 0$ be a special $\mathcal{B}$-preenvelope of R and $0 \to X_i \to T_{i+1} \to X_{i+1} \to 0$ be a special $\mathcal{B}$-preenvelope of X_i for each $i < \omega$. Then $T' = (\bigoplus_{i<n} T_i) \oplus X_n$ is an n-tilting module equivalent to T, [1], *and we can always construct the special $\mathcal{B}$-preenvelopes so that X_i's are $\mathcal{A}^{<\aleph_0}$-filtered for all $i < \omega$,* [15].

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