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Compactly Generated Triangulated Categories and the Telescope Conjecture

Thesis for the degree of Philosophiae Doctor

Trondheim, October 2009

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ISBN 978-82-471-1774-3 (printed ver.)
ISBN 978-82-471-1775-0 (electronic ver.)
ISSN 1503-8181

Doctoral theses at NTNU, 2009:188

Printed by NTNU-trykk

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ACKNOWLEDGEMENTS

I would like to thank to my advisor Øyvind Solberg for many helpful discussions and suggestions. I am also grateful to the Research Council of Norway for supporting my research through the Storforsk-project "Homological and geometric methods in algebra", which was lead by Idun Reiten. It has been a great pleasure to work in the creative environment in Trondheim, as well as in Paderborn and Murcia, where I spent a few months in 2009. Parts of this work were written as joint papers with Jan Šaroch and Henning Krause, who deserve my thanks for their co-operative approach.

Last but not least, my personal thanks belong to Renata for her support during the whole period of working on the thesis.

Jan Šťovíček
Trondheim, September 2009

COMPACTLY GENERATED TRIANGULATED CATEGORIES AND THE TELESCOPE CONJECTURE (SURVEY)

Introduction

The thesis consists of this survey and four papers in various stages of the publication process:

(1) J. Šaroch and J. Šťovíček, *The countable telescope conjecture for module categories*, Adv. Math. **219** (2008) 1002-1036.
(2) J. Šťovíček, *Telescope conjecture, idempotent ideals, and the transfinite radical*, to appear in Trans. Amer. Math. Soc.
(3) H. Krause and J. Šťovíček, *The telescope conjecture for hereditary rings via Ext-orthogonal pairs*, preprint, arXiv:0810.1401.
(4) J. Šťovíček, *Locally well generated homotopy categories of complexes*, preprint, arXiv:0810.5684.

In the papers with a coauthor, the contributions of myself and the coauthor should be considered as equal.

There are two main reasons for introducing the thesis with this survey. First, the necessary concepts and results on which this thesis relies are scattered among several papers as one can see from the reference list. It seemed, therefore, convenient to collect all the necessary terms and facts and put them into the corresponding context, together with motivation for the research. Second, there are some results which have been proved here, in particular Theorem 3.6(3) which seems to be a new result.

1. Preliminaries

1.1. **Triangulated categories.** Triangulated categories are ubiquitous in modern homological algebra and homotopy theory. They were independently introduced by Verdier [51] and Puppe [42] in 1960's. An additive category $\mathcal{T}$ is called *triangulated* if:

(1) It has a distinguished autoequivalence. The image of an object X or a morphism f under this equivalence is often denoted by $X[1]$ or $f[1]$, respectively. By $X[n]$ or $f[n]$, we denote the n-fold application of the equivalence (or $|n|$-fold application of its quasi-inverse if $n < 0$) on X or f.
(2) It has a distinguished class of diagrams of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

called *triangles*, satisfying certain axioms.

The axioms are:

TR0: A diagram isomorphic to a triangle is again a triangle. Moreover, the diagram $X \xrightarrow{\mathbf{1}_X} X \longrightarrow 0 \longrightarrow X[1]$ is a triangle for each $X \in \mathcal{T}$.

TR1: For any morphism $f : X \to Y$ in $\mathcal{T}$, there is a triangle of the form
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

TR2: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a triangle if and only if $Y \xrightarrow{-g} Z \xrightarrow{-h} X[1] \xrightarrow{-f[1]} Y[1]$ is a triangle.

TR3: For any commutative diagram of the form
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1], \end{array}$$
where the rows are triangles, there is a (not necessarily unique) morphism $w : Z \to Z'$, which makes the diagram
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow & & {\scriptstyle u[1]}\downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1], \end{array}$$
commutative.

TR4: Given two composable morphisms $f_1 : X \to Y$ and $g_2 : Y \to Z$ and the composition $f_3 = g_2 \circ f_1$, it is possible to form a commutative diagram
$$\begin{array}{ccccccc} & & U & = & U & & \\ & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_4}\downarrow & & \\ X & \xrightarrow{f_1} & Y & \xrightarrow{g_1} & V & \xrightarrow{h_1} & X[1] \\ \| & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_4}\downarrow & & \| \\ X & \xrightarrow{f_3} & Z & \xrightarrow{g_3} & W & \xrightarrow{h_3} & X[1] \\ & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_4}\downarrow & & \downarrow{\scriptstyle f_1[1]} \\ & & U[1] & = & U[1] & \xrightarrow[f_2[1]]{} & Y[1] \end{array}$$
such that the two middle rows and the two middle columns are triangles.

Axiom [TR4] is usually called the *octahedral axiom*, because it can be depicted in the form of an octahedron; see [22, pg. 74]. An alternative but equivalent form of the axiom, which is used in [38], was introduced by Neeman in [34].

As the concept of a triangulated category is rather well known and have been already studied for half a century, we refer for basic properties of such categories for example to [38, §1], [11, §1] or [29]. The common intuition is that triangles share certain formal properties with short exact sequences in abelian categories. There are important differences from the abelian setting, though. First, as mentioned in [38] or [22], the class of triangles in a given triangulated category $\mathcal{T}$ is not intrinsic to the category $\mathcal{T}$, but it is rather an additional datum. Second, the morphism w in [TR3] is required to exist, but neither to be unique nor to be functorial. This may cause considerable problems in certain situations.

In connection with triangulated categories, it is natural to consider functors compatible with the triangulated structure. If $\mathcal{S}$ and $\mathcal{T}$ are triangulated categories, an additive functor $F : \mathcal{S} \to \mathcal{T}$ is called *triangulated* (or sometimes also *exact*) if it comes along with a natural isomorphism

$$\phi_X : F(X[1]) \longrightarrow (FX)[1]$$

for each object $X \in \mathcal{T}$ such that for each triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{S}$, we get a triangle

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\phi_X \circ Fh} (FX)[1]$$

in $\mathcal{T}$. The natural isomorphisms ϕ_X are usually not explicitly mentioned because they are often obvious from the context.

1.2. **Compactly generated triangulated categories.** For many practical purposes, a triangulated category is a too abstract and general notion to deduce a strong enough theory. Thus, one seeks after more axioms which are widely satisfied to build up a richer theory. One successful direction of this effort has lead to compactly and well generated triangulated categories. In this section we introduce the essential part of the theory and we refer for further details to [38] and [29]. Later in Section 2 we will present several examples. A part of the material in this section is also briefly reviewed in the article [49] in this volume.

Let us start with compactly generated triangulated categories. The concept has been implicitly known in algebraic topology since the stable homotopy category of spectra is compactly generated. The abstract axioms were given by Neeman in 1990's and used to considerably generalize and simplify proofs of some classical results; see [36, 37]. Since then, the concept has found many applications in algebra.

From now on, a triangulated category $\mathcal{T}$ is usually assumed to satisfy:

TR5: $\mathcal{T}$ has arbitrary infinite coproducts.

In such a case, a coproduct of triangles is automatically a triangle again; see [38, Proposition 1.2.1 and Remark 1.2.2]. Moreover, $\mathcal{T}$ necessarily has splitting idempotents by [38, Proposition 1.6.8]. Now we can give a formal definition.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. An object $C \in \mathcal{T}$ is called *compact* if for any family $(Y_i \mid i \in I)$ of objects of $\mathcal{T}$ the natural morphism

$$\coprod_i \mathrm{Hom}_{\mathcal{T}}(C, Y_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(C, \coprod_i Y_i)$$

is an isomorphism. Equivalently, any morphism $C \to \coprod Y_i$ factorizes through a finite subcoproduct.

The category $\mathcal{T}$ is said to be *compactly generated* if there is a set $\mathcal{C}$ of compact objects with the following property: If $X \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(C, X) = 0$ for each $C \in \mathcal{C}$, then $X = 0$.

To state some basic properties of compactly generated triangulated categories, we need one more definition:

Definition 1.2. Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{L}$ of $\mathcal{T}$ is called a *triangulated subcategory* if it is closed under applying the distinguished autoequivalence of $\mathcal{T}$ and taking triangle completions in the sense of [TR1]. A triangulated subcategory $\mathcal{L}$ is called *thick* if it is in addition closed under taking those direct summands which exist in $\mathcal{T}$.

Assume, moreover, $\mathcal{T}$ satisfies [TR5]. Then the subcategory $\mathcal{L}$ is called *localizing* if it is a triangulated subcategory which is closed under taking arbitrary coproducts.

Note that by [38, Proposition 1.6.8], any localizing subcategory of a [TR5] triangulated category $\mathcal{T}$ is thick. Now, we have the following useful properties:

Proposition 1.3. *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{C}$ a set of compact objects which generates $\mathcal{T}$ in the sense of Definition 1.1. Then the following assertions hold:*

1. *The smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$ is the whole of $\mathcal{T}$.*
2. *The full subcategory $\mathcal{T}^c$ of all compact objects of $\mathcal{T}$ coincides with the smallest thick subcategory containing $\mathcal{C}$.*

Before stating another crucial property, the so called Brown representability theorem, we first define the more general concept of well generated triangulated categories.

1.3. **Well generated triangulated categories.** It has turned out both in algebra and topology that many naturally occurring triangulated categories are not compactly generated triangulated categories, yet sharing many important properties with them. In an effort to get a better grasp of this phenomenon, Neeman defined well generated triangulated categories and motivated them in the introduction of [38] as well as by the results from [39].

Before giving the definition, we have to recall some very basic set theoretic concepts; we use [16] as the universal reference. An infinite cardinal number κ is called *regular* if κ cannot be obtained as a sum of a collection of less than κ cardinal numbers all of which are strictly smaller than κ. For example, the first infinite cardinal $\aleph_0$ is regular. It is also well known that the immediate successor of any infinite cardinal is regular. An infinite cardinal κ which is not regular is called *singular*. Here, the first limit cardinal $\aleph_\omega = \sup_{n\in\mathbb{N}} \aleph_n$ may serve as an example. Now we turn back to triangulated categories:

Definition 1.4. Let $\mathcal{T}$ be a triangulated category satisfying [TR5] and κ a regular cardinal number. An object $C \in \mathcal{T}$ is called *κ–small* provided that every morphism of the form

$$C \longrightarrow \coprod_{i\in I} Y_i$$

factorizes through a subcoproduct $\coprod_{i\in J} Y_i$ with $|J| < \kappa$.

The category $\mathcal{T}$ is called *κ-well generated* provided there is a set $\mathcal{C}$ of objects of $\mathcal{T}$ such that

(1) If $X \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(C, X) = 0$ for each $C \in \mathcal{C}$, then $X = 0$;
(2) Each $C \in \mathcal{C}$ is κ-small;
(3) For any morphism in $\mathcal{T}$ of the form $f : C \to \coprod_{i\in I} Y_i$ with $C \in \mathcal{C}$, there exists a family of morphisms $f_i : C_i \to Y_i$ such that $C_i \in \mathcal{C}$ for each $i \in I$ and f factorizes as

$$C \longrightarrow \coprod_{i\in I} C_i \xrightarrow{\amalg f_i} \coprod_{i\in I} Y_i.$$

Finally, $\mathcal{T}$ is called *well generated* if it is κ–well generated for some regular cardinal κ.

As mentioned in [49] in this volume, this definition differs from Neeman's original definition in [38, 8.1.7], but it is equivalent by [25, Lemmas 4 and 5]. Note also that $\aleph_0$-well generated triangulated categories are precisely compactly generated triangulated categories.

Now, we are in a position to state a crucial result, which has origin in the work of Brown [5]. For a different proof of the below statement and more references we also refer to [28, §§4.5 and 4.6]. Recall that a contravariant additive functor $F : \mathcal{T} \to \mathrm{Ab}$ is called *cohomological* if it sends each triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ to an exact sequence $F(X[1]) \xrightarrow{Fh} FZ \xrightarrow{Fg} FY \xrightarrow{Ff} FX$ of abelian groups.

Proposition 1.5 (Brown representability)**.** [38, 8.3.3] *Let $\mathcal{T}$ be a well generated triangulated category. Then any contravariant cohomological functor $F : \mathcal{T} \to Ab$ which takes coproducts to products is, up to isomorphism, of the form* $\mathrm{Hom}_{\mathcal{T}}(-, X)$ *for some $X \in \mathcal{T}$.*

As noticed by Neeman in [36, 37], this statement allows to give formally rather simple proofs for some classical results. One particular consequence of Brown representability is the existence of certain adjoint functors. Using a suitable set theoretic axiomatics (see the last two paragraphs at the end of [49, §1] in this volume), it is rather easy to prove the following:

Corollary 1.6. *Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories such that $\mathcal{S}$ is well generated. Then a triangulated functor $F : \mathcal{S} \to \mathcal{T}$ has a right adjoint if and only if F preserves coproducts.*

As an application, one has a nice theory for localization of well generated triangulated categories, part of which we will mention in Section 3. It comes from the work of Bousfield [4] in homotopy theory, and an algebraic presentation can be found in [38] and [29].

Finally, we will give two more consequences of Proposition 1.5. First, there is an analogue of Proposition 1.3(1):

Corollary 1.7. *Assume that $\mathcal{T}$ is a κ-well generated triangulated category for some regular cardinal κ, and that $\mathcal{C}$ is a set of objects of $\mathcal{T}$ as in Definition 1.4. The the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$ is the whole of $\mathcal{T}$.*

Now, consider the following condition on a triangulated category, which is dual to [TR5]:

TR5*: $\mathcal{T}$ has arbitrary infinite products.

Then we have by [38, Propositions 8.4.6 and 1.2.1]:

Corollary 1.8. *Any well generated triangulated category $\mathcal{T}$ satisfies* [TR5*]. *Moreover, a product of a family of triangles is always a triangle.*

2. Examples of triangulated categories

Now we are going to give examples of the concepts from the previous section. Our list is, however, meant to be more illustrative than exhaustive. We will focus on algebraic triangulated categories, that is, the stable categories of Frobenius exact categories; see [10] or [11, §1]. There are also other important classes of triangulated categories. One can consider the homotopy categories of closed model structures [14, §7] and their full triangulated subcategories. Such categories are called topological triangulated categories. Although all triangulated categories used in practice usually belong to one of these two families, there are as well some "exotic" examples which are neither algebraic nor topological [32].

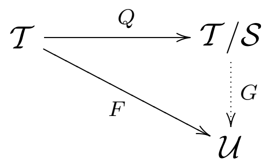
2.1. **Derived categories.** Probably the most well known algebraic representative of a compactly generated triangulated category is the unbounded derived category of a ring. Here we just sketch the construction and refer to [51], [10] or [11, §1] for more details.

We fix a ring R and denote by Mod-R the category of all right R-modules. Let $\mathbf{C}(\text{Mod-}R)$ be the (abelian) category of chain complexes over Mod-R and $\mathbf{K}(\text{Mod-}R)$ the homotopy category of complexes. That is, $\mathbf{K}(\text{Mod-}R)$ is the factor of $\mathbf{C}(\text{Mod-}R)$ modulo the ideal of all null-homotopic morphisms. It is well known that if we consider $\mathbf{C}(\text{Mod-}R)$ as an exact category in the sense of [19, Appendix A] such that the conflations are the componentwise split exact sequences of complexes, then $\mathbf{C}(\text{Mod-}R)$ is a Frobenius exact category and $\mathbf{K}(\text{Mod-}R)$ is its stable category, hence a triangulated category. However, $\mathbf{K}(\text{Mod-}R)$ is usually not well generated—see Section 4 or [49] in this volume. In order to get a compactly generated triangulated category, we will take a so-called Verdier quotient of $\mathbf{K}(\text{Mod-}R)$.

Definition 2.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full triangulated subcategory. Then the *Verdier quotient* of $\mathcal{T}$ by $\mathcal{S}$ is a triangulated category $\mathcal{T}/\mathcal{S}$ together with a triangulated functor

$$Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S},$$

with the following universal property. Whenever $\mathcal{U}$ is a triangulated category and $F : \mathcal{T} \to \mathcal{U}$ is a triangulated functor such that $FX = 0$ for each $X \in \mathcal{S}$, then there is a unique triangulated functor $G : \mathcal{T}/\mathcal{S} \to \mathcal{U}$ making the following diagram commutative:



If $\mathcal{T}$ satisfies [TR5], we sometimes require that $\mathcal{S}$ not only be a triangulated subcategory but rather a localizing subcategory of $\mathcal{T}$, since in this case $\mathcal{T}/\mathcal{S}$ also satisfies [TR5] and the functor Q preserves coproducts.

Note that Verdier quotients do always exist, see [38, Theorem 2.1.8], and the universal property guarantees their uniqueness. It may, however, happen in some cases that $\mathcal{T}/\mathcal{S}$ is strictly speaking not a category since some morphism spaces $\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$ may be proper classes rather than sets. We refer to [6] for an example.

Looking back at our case, we define the *unbounded derived category* of R, denoted by $\mathbf{D}(\text{Mod-}R)$, as the Verdier quotient

$$\mathbf{K}(\text{Mod-}R)/\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R).$$

Here, $\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R)$ is the full triangulated subcategory of $\mathbf{K}(\text{Mod-}R)$ whose objects are acyclic complexes, that is, those with all homologies vanishing. It follows from the work of Spaltenstein [48] (see also [3, Proposition 2.12]) that all morphism spaces in $\mathbf{D}(\text{Mod-}R)$ are sets, so no set-theoretic problems occur. Moreover, it is easy to see that R is a compact object in $\mathbf{D}(\text{Mod-}R)$ and the set

$$\mathcal{C} = \{R[i] \mid i \in \mathbb{Z}\}$$

generates $\mathbf{D}(\text{Mod-}R)$ in the sense of Definition 1.1. This follows using the natural isomorphisms $\text{Hom}_{\mathbf{D}(\text{Mod-}R)}(R[i], X) \cong H^{-i}(X)$. Proposition 1.3 together with [44, Proposition 6.3] and the standard way to compute direct summands via homotopy colimits as in [38, Proposition 1.6.8] yield the following well-known fact:

Proposition 2.2. *For any ring R, the derived category $\mathbf{D}(\text{Mod-}R)$ is compactly generated. Moreover, $X \in \mathbf{D}(\text{Mod-}R)$ is compact if and only if it is isomorphic to a bounded complex of finitely generated projective modules.*

One may also be interested in derived categories of more general abelian categories. In connection with geometric examples and categories of quasi-coherent sheaves, it is natural to consider Grothendieck categories. Recall that an abelian category is called *Grothendieck* if it has exact filtered colimits (i.e. it is an [AB5] abelian category) and a set of generators.

Given a Grothendieck category $\mathcal{G}$, we again define the derived category as $\mathbf{D}(\mathcal{G}) = \mathbf{K}(\mathcal{G})/\mathbf{K}_{\text{ac}}(\mathcal{G})$. In this case, $\mathbf{D}(\mathcal{G})$ is in general not compactly generated, but all morphism spaces in $\mathbf{D}(\mathcal{G})$ are still sets by [1, Corollary 5.6] and we have:

Proposition 2.3. [29, §7.7] *For any Grothendieck category $\mathcal{G}$, the unbounded derived category $\mathbf{D}(\mathcal{G})$ is well generated.*

In several interesting cases though, $\mathbf{D}(\mathcal{G})$ is in fact compactly generated. Neeman proved this in [37, Proposition 2.5] for $\mathcal{G} = \text{Qcoh}(\mathbb{X})$, where $\mathbb{X}$ is a quasi-compact separated scheme. We will also point out another result which will be useful in Section 3. Recall that following [7], we can define a special class of Grothendieck categories:

Definition 2.4. A Grothendieck category $\mathcal{G}$ is called *locally noetherian* if it has a set $\mathcal{C}$ of generators such that each $X \in \mathcal{C}$ is noetherian. That is, each $X \in \mathcal{C}$ satisfies the ascending chain condition on subobjects.

Then we obtain the following statement as a consequence of results from [27]:

Proposition 2.5. *Let $\mathcal{G}$ be a locally noetherian Grothendieck category of finite global dimension. Then $\mathbf{D}(\mathcal{G})$ is compactly generated. Moreover, an object is compact if and only if it is isomorphic to a bounded complex of noetherian objects of $\mathcal{G}$.*

Proof. Let $\mathcal{I}$ denote the class of all injective objects in $\mathcal{G}$. Then the natural functor $F : \mathbf{K}(\mathcal{I}) \to \mathbf{D}(\mathcal{G})$ obtained as the composition

$$\mathbf{K}(\mathcal{I}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{G}) \xrightarrow{Q} \mathbf{D}(\mathcal{G})$$

is an equivalence of triangulated categories. Indeed, [27, Proposition 3.6] states (using a somewhat different terminology) that the induced functor

$$\bar{F} : \mathbf{K}(\mathcal{I})/\big(\mathbf{K}(\mathcal{I}) \cap \mathbf{K}_{\mathrm{ac}}(\mathcal{G})\big) \longrightarrow \mathbf{D}(\mathcal{G})$$

is an equivalence triangulated categories. Invoking the assumption of $\mathcal{G}$ having finite global dimension, one immediately sees that $\mathbf{K}(\mathcal{I}) \cap \mathbf{K}_{\mathrm{ac}}(\mathcal{G}) = 0$ and $\bar{F} = F$. Having established that F is an equivalence, the statement of Proposition 2.5 follows directly from [27, Proposition 2.3]. □

We remark here that the derived category of a ring is rather close to being a universal example of an algebraic compactly generated triangulated category. More precisely, [28, Theorem 7.5(3)], which is a slightly refined version of the important theorem [20, 4.3] due to Keller, says that any algebraic compactly generated triangulated category is equivalent to the derived category of a small dg-category. Porta recently showed in [41, Theorem 5.2] that algebraic well generated triangulated categories are precisely Verdier quotients of such derived categories by localizing classes generated by a set of objects. We refer to the just mentioned papers for more details. It is also worth to mention that analogous statements for topological triangulated categories have been proved by Schwede and Shipley [47] and Heider [13].

2.2. **Other algebraic triangulated categories.** Although we know a general form of an algebraic well generated triangulated category now, the description as a Verdier quotient of the derived category of a small dg-category may be far too complicated to do any practical computations. It may be, therefore, much more convenient to study the categories of interest directly.

We will give a few examples. For a ring R, let Proj-R be the category of all projective right R-modules and Inj-R the category of all injective right R-modules. Recall also that a ring R is called *left coherent* if each finitely generated left ideal is finitely presented. Equivalently, R is left coherent if the category mod-R^{op} of all finitely presented left R-modules is abelian. Then we have the following statement:

Proposition 2.6. *Let R be a ring. Then:*

(1) *The homotopy category $\mathbf{K}(\text{Proj-}R)$ is $\aleph_1$-well generated. If R is left coherent, then $\mathbf{K}(\text{Proj-}R)$ is even compactly generated and, moreover, the full triangulated subcategory of compact objects is equivalent to $\mathbf{D}^b(\text{mod-}R^{\mathrm{op}})^{\mathrm{op}}$.*

(2) *If R is right noetherian, then the homotopy category* $\mathbf{K}(\text{Inj-}R)$ *is compactly generated. Moreover, the full triangulated subcategory of compact objects is equivalent to* $\mathbf{D}^b(\text{mod-}R)$.

Proof. (1) follows from [39, Theorem 1.1 and Proposition 7.14], which extend previous results from [17], while (2) is a special case of [27, Theorem 1.1]. □

This statement gives several interesting insights, for example in connection with the Grothendieck duality theorem, totally reflexive modules or relative homological algebra. We refer to [15, 18, 39] for more information.

Another natural example is the stable module category of a quasi-Frobenius ring. Recall that R is *quasi-Frobenius* if $\text{Proj-}R = \text{Inj-}R$. For instance, any self-injective artin algebra or, as a particular case, any group algebra of a finite group is quasi-Frobenius. In this case, the module category Mod-R together with the natural abelian exact structure is Frobenius, and the stable module category $\underline{\text{Mod}}\text{-}R$ is triangulated. Moreover, the following is an easy consequence of Proposition 1.3 (cf. also [24, §1.5]):

Proposition 2.7. *Let R be a quasi-Frobenius ring. Then $\underline{\text{Mod}}\text{-}R$ is a compactly generated triangulated category, and $X \in \underline{\text{Mod}}\text{-}R$ is compact if and only if $X \cong Y$ in $\underline{\text{Mod}}\text{-}R$ for some finitely generated R-module Y.*

This particular example is quite important for this thesis since it connects the telescope conjecture as introduced in Section 3 to homological algebra in module categories; see [30, Theorem 7.6 and Corollary 7.7]. This motivated the papers [46, 50], which are a part of this volume.

3. The telescope conjecture

When we speak of the telescope conjecture in the context of triangulated categories, we mean the following statement:

Telescope Conjecture. *Let $\mathcal{T}$ be a compactly generated triangulated category. Given a smashing localization functor L on $\mathcal{T}$, the kernel of L is generated by compact objects. That is, there is a set $\mathcal{C}$ of compact objects such that* $\operatorname{Ker} L$ *is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$.*

Remark. Before explaining the terminology, we point out a few facts. First, the conjecture as well as a substantial part of the terminology comes from algebraic topology. The conjecture itself was introduced in the work of Bousfield [4] and Ravenel [43]. In this context, the category $\mathcal{T}$ was the stable homotopy category of spectra. In this thesis, however, the main focus is put on algebraic triangulated categories.

Second, the telescope conjecture is known to fail in general, see [21] and also [31, §7] in this volume. One is, therefore, left to prove or disprove the conjecture for certain classes of compactly generated triangulated categories. As suggested to me by Claus Ringel, it may then be more precise to say that a given category $\mathcal{T}$ has or does not have the "telescope property".

Finally, although the conjecture itself is a rather abstract problem, its analysis for particular cases gives many insights. This is actually the major topic for this thesis and the included papers [46, 50, 31]. Some other applications, for example to lifting of complexes of modules over a morphism of rings, are mentioned in [26].

Now we can give the necessary definitions. The key point here is the concept of a localization functor.

Definition 3.1. Let $\mathcal{T}$ be a triangulated category. A triangulated endofunctor $L : \mathcal{T} \to \mathcal{T}$ is called a *localization functor* if there is a natural transformation $\eta : \mathrm{Id}_{\mathcal{T}} \to L$ such that

(1) $L\eta_X = \eta_{LX}$ for each $X \in \mathcal{T}$. That is, if we apply L on the morphism $\eta_X : X \to LX$, we get precisely the morphism $\eta_{LX} : LX \to L^2X$.
(2) $\eta_{LX} : LX \to L^2X$ is an isomorphism for each $X \in \mathcal{T}$.

Localization functors formalize a certain way to localize triangulated categories, which is often referred to as *Bousfield localization* nowadays. We refer to [29, §4.9] for more facts and examples. What we are going to make precise here is the connection to Verdier quotients. Let us adopt the following notation. By the kernel of L, we mean the full subcategory of $\mathcal{T}$ defined by

$$\operatorname{Ker} L = \{X \mid LX = 0\},$$

and by $\operatorname{Im} L$, we mean the essential image of L. That is, the closure of the actual image of L under taking isomorphic objects. Then we have the following statement.

Proposition 3.2. *Let $\mathcal{T}$ be a triangulated category.*

(1) *If $L : \mathcal{T} \to \mathcal{T}$ is a localization functor, then $\operatorname{Ker} L$ is a thick subcategory of $\mathcal{T}$, and there is a unique equivalence of triangulated categories $G : \mathcal{T}/\operatorname{Ker} L \longrightarrow \operatorname{Im} L$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\;Q\;} & \mathcal{T}/\operatorname{Ker} L \\ & \searrow^{L} & \downarrow{\scriptstyle G} \\ & & \operatorname{Im} L \end{array}$$

Moreover, the inclusion $\mathrm{inc}\colon \operatorname{Im} L \longrightarrow \mathcal{T}$ *is a (fully faithful) right adjoint to $L : \mathcal{T} \longrightarrow \operatorname{Im} L$.*

(2) *Assume that* $\mathcal{S}$ *is a thick subcategory of* $\mathcal{T}$ *such that the Verdier quotient* $Q : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ *admits a right adjoint* $R : \mathcal{T}/\mathcal{S} \to \mathcal{T}$*. Then* R *is a fully faithful triangulated functor and* $L = R \circ Q : \mathcal{T} \to \mathcal{T}$ *together with the unit of adjunction* $\eta : \mathrm{Id}_{\mathcal{T}} \to L$ *is a localization functor such that* $\operatorname{Ker} L = \mathcal{S}$.

Proof. (1) follows from [29, Proposition 4.11.1], while (2) is an immediate consequence of [29, Corollary 2.4.2]. Here, one has to take into account that Verdier quotients are in fact localizations and that adjoints of triangulated functors are triangulated; see the proof of [38, Theorem 2.1.8] and [38, Lemma 5.3.6], respectively. □

Rephrasing Proposition 3.2, we can say that up to equivalence, localization functors parametrize those Verdier quotients which have right adjoints. This not only has many formal advantages, for example all the morphism spaces in the Verdier quotient are always sets, but such adjoints indeed do very often exist. One general way to obtain them is Proposition 1.5 together with the well-known fact that the quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ has a right adjoint if and only if the inclusion $\mathcal{S} \to \mathcal{T}$ has a right adjoint. Using the existence of an adjoint, we also get the following easy corollary:

Corollary 3.3. *Let* $\mathcal{T}$ *be a triangulated category satisfying* [TR5] *and* $L : \mathcal{T} \to \mathcal{T}$ *a localization functor. Then* $\operatorname{Ker} L$ *is a localizing subcategory of* $\mathcal{T}$*, that is, it is closed under coproducts.*

The tricky part now is that even though the kernel of L is always closed under coproducts provided $\mathcal{T}$ satisfies [TR5], this does not mean yet that L preserves coproducts. In fact, we make this to a definition:

Definition 3.4. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. Then a localization functor $L : \mathcal{T} \to \mathcal{T}$ is called *smashing* if L preserves coproducts.

The reason for the word smashing is explained in Section 3.2. For compactly generated triangulated categories, we always have the following general way of constructing smashing localization functors:

Proposition 3.5. [4, 43] *Let* $\mathcal{T}$ *be a compactly generated triangulated category and* $\mathcal{C}$ *be a set of compact objects. If* $\mathcal{S}$ *is the smallest localizing subcategory of* $\mathcal{T}$ *containing* $\mathcal{C}$*, then the Verdier quotient* $Q : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ *has a right adjoint* $R : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ *and* $L = R \circ Q$ *is a smashing localization functor.*

Proof. The category $\mathcal{S}$ is easily seen to be compactly generated, so the inclusion $\mathcal{S} \to \mathcal{T}$ has a right adjoint by Corollary 1.6. Hence Q has a right adjoint R and $L = R \circ Q$ is a localization functor by Proposition 3.2. Finally, the fact that L is smashing follows from the fact that taking the functorial triangle in the sense of [29, §4.11] commutes with taking coproducts. □

Now, the telescope conjecture can be restated as follows: For a given compactly generated triangulated category $\mathcal{T}$, *all* smashing localization functors on $\mathcal{T}$ can be obtained, up to natural equivalence, as in Proposition 3.5.

It is very desirable to have this property for the following reason. If $\mathcal{T}$ is compactly generated and L is smashing, then the quotient $\mathcal{T}/\operatorname{Ker} L$ is again compactly generated by [29, Remark 5.5.2], so it is natural to ask how the category of compact objects looks like. If L comes up as in Proposition 3.5, the answer is rather straightforward. Denoting by $\mathcal{T}^c$ the category of all compact objects in $\mathcal{T}$, the category of compact objects in $\mathcal{T}/\operatorname{Ker} L$ is equivalent to the idempotent completion of the Verdier quotient

$$\mathcal{T}^c/(\mathcal{T}^c \cap \operatorname{Ker} L);$$

see [36, Theorem 2.1]. If the conjecture fails, one needs more involved theory as developed in [26].

3.1. **Known cases when the telescope conjecture holds.** As mentioned before, the telescope conjecture is not true in general. There are, however, many natural triangulated categories $\mathcal{T}$ for which the conjecture holds. We summarize the positive results known so far, some of which are original in this thesis, in a theorem:

Theorem 3.6. *The telescope conjecture holds for the following algebraic compactly generated triangulated categories:*

1. $\mathbf{D}(\text{Mod-}R)$ *where R is commutative noetherian;*
2. $\mathbf{D}(\text{Mod-}R)$ *where R is right hereditary;*
3. $\mathbf{D}(\mathcal{G})$ *where $\mathcal{G}$ is a locally noetherian hereditary Grothendieck category;*
4. $\underline{\text{Mod}}\text{-}kG$ *where k is a field and G a finite group;*
5. $\underline{\text{Mod}}\text{-}R$ *where R is a domestic standard self-injective algebra in the sense of* [23].

Proof. (1) is a result due to Neeman, [35, Theorem 3.3 and Corollary 4.4]. (2) is proved in this thesis in [31, Theorem A]. (4) is a result of Benson, Iyengar and Krause, [2, Theorem 11.12]. (5) is again proved in this thesis. It follows from [50, Theorem 19], using the fact that the infinite radical of the category of finitely generated modules over a domestic standard self-injective algebra is nilpotent, [23]. The proof relies on techniques developed in [46], also contained in this volume.

Finally, we prove (3) right here. Note that $\mathbf{D}(\mathcal{G})$ is compactly generated by Proposition 2.5. Suppose further that $L : \mathbf{D}(\mathcal{G}) \to \mathbf{D}(\mathcal{G})$ is a smashing localizing functor and let us set $\mathcal{X} = H^0(\operatorname{Ker} L)$ and $\mathcal{Y} = H^0(\operatorname{Im} L)$. It follows from [31, Proposition 2.6] that $(\mathcal{X}, \mathcal{Y})$ is a so called complete Ext-orthogonal pair for $\mathcal{G}$. That is, the following hold:

- $\mathcal{X} = \big\{X \in \mathcal{G} \mid (\forall Y \in \mathcal{Y})\big(\operatorname{Hom}_{\mathcal{G}}(X,Y) = 0 = \operatorname{Ext}^1_{\mathcal{G}}(X,Y)\big)\big\}$,
- $\mathcal{Y} = \big\{Y \in \mathcal{G} \mid (\forall X \in \mathcal{X})\big(\operatorname{Hom}_{\mathcal{G}}(X,Y) = 0 = \operatorname{Ext}^1_{\mathcal{G}}(X,Y)\big)\big\}$,

- for each $M \in \mathcal{G}$, there is an exact sequence

$$\varepsilon_M : \quad 0 \to Y_M \longrightarrow X_M \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

Note that by [31, Lemma 2.9] the sequences ε_M are unique and naturally functorial. Moreover, $\mathcal{Y}$ is by [31, Proposition 2.4] an abelian subcategory of $\mathcal{G}$ closed under taking coproducts. Hence, $\mathcal{Y}$ is closed under taking arbitrary filtered colimits and we have $\varepsilon_M = \varinjlim \varepsilon_{M_i}$ whenever $M = \varinjlim M_i$.

Now we proceed in a very similar way as in the proof of [31, Theorem 5.1] and claim that

$$\mathcal{Y} = \{Y \in \mathcal{G} \mid \operatorname{Hom}_{\mathcal{G}}(X, Y) = 0 = \operatorname{Ext}^1_{\mathcal{G}}(X, Y) \text{ for each } X \in \mathcal{C}\},$$

where $\mathcal{C}$ stands for the class of all noetherian objects of $\mathcal{G}$ contained in $\mathcal{X}$. Since $\mathcal{X}$ is closed under taking filtered colimits, it is sufficient to show that $\mathcal{X} \subseteq \varinjlim \mathcal{C}$. To this end, fix $M \in \mathcal{X}$. Recall that since $\mathcal{G}$ is locally noetherian, M is a directed union of its noetherian subobjects. More precisely, there is a direct system $(M_i \mid i \in I)$ such that $M = \varinjlim M_i$, each M_i is noetherian, and each morphism $M_i \to M_j$ for $i < j$ is a monomorphism. In particular, all the colimit morphisms $M_i \to M$ are monomorphisms. Since $\operatorname{Ext}^1_{\mathcal{G}}(-, Y)$ is right exact for each $Y \in \mathcal{Y}$, we easily deduce that $\operatorname{Ext}^1_{\mathcal{G}}(M_i, \mathcal{Y}) = 0$ for each $i \in I$. By the preceding paragraph, we know that $\varepsilon_M = \varinjlim \varepsilon_{M_i}$, so

$$\varinjlim X_{M_i} \xrightarrow{\sim} X_M \xrightarrow{\sim} M.$$

Using the same argument as in [31, Lemma 5.3], we can show that $Y_{M_i} = 0$ for each $i \in I$. Hence, the morphisms $X_{M_i} \to M_i$ are all monomorphisms and X_{M_i} are all noetherian. In particular, $X_{M_i} \in \mathcal{C}$ for each $i \in I$ and $\mathcal{X} \subseteq \varinjlim \mathcal{C}$. This proves the claim.

Finally, using the bijective correspondence between the localizing subcategories of $\mathbf{D}(\mathcal{G})$ and the extension closed abelian subcategories of $\mathcal{G}$ that are closed under coproducts, which is given in [31, Proposition 2.6], we deduce that $\operatorname{Ker} L$ is the smallest localizing class containing $\mathcal{C}$. We remind the reader that all objects of $\mathcal{C}$ are compact in $\mathbf{D}(\mathcal{G})$ by Proposition 2.5. Thus, the telescope conjecture holds for $\mathbf{D}(\mathcal{G})$. □

We add a few remarks regarding the theorem:

(1) As particular examples of $\mathcal{G}$ in Theorem 3.6(3), we can take $\mathcal{C} = \operatorname{Qcoh}(\mathbb{X})$ where $\mathbb{X}$ is either a smooth projective curve or a weighted projective line in the sense of [8]. In particular, the telescope conjecture holds also for $\mathbf{D}(\text{Mod-}R)$, where R is a quasi-tilted artin algebra; we refer to [12] for details.

(2) Examples for Theorem 3.6(2) can be found in [31, §4] and examples for Theorem 3.6(5) in [50, §6], both in this thesis.

(3) The proofs of Theorem 3.6(2) and (5) use strong connections between the triangulated category in question and Mod-R. In the first case this connection is formulated in [31, Theorem B] and in the second case in [46, Theorem 6.1]. In both cases, the study of the telescope conjecture revealed other results which may be of interest by itself.

(4) In [31, Example 7.7], we refine Keller's ideas from [21] to construct a commutative domain such that the telescope conjecture fails for $\mathbf{D}(\text{Mod-}R)$. The domain R is of global dimension 2 and each ideal of R is countably generated. This shows that the conditions in Theorem 3.6(1) and (2) cannot be easily relaxed. We do not know, however, whether there is a quasi-Frobenius ring R such that the telescope conjecture fails for $\underline{\text{Mod}}\text{-}R$.

3.2. **Other interpretations of the telescope conjecture.** To conclude the section, we will very briefly introduce other points of view which has helped or may help in the future to tackle the conjecture.

First, we point out a result by Krause [26], which says that the telescope conjecture is a problem about small categories. This is not at all obvious from the definition. Namely, let $\mathcal{T}$ be a triangulated compactly generated category and $\mathcal{T}^c$ the full subcategory of all compact objects. We recall that $\mathcal{T}^c$ is necessarily skeletally small as a consequence of Proposition 1.3(2).

We further recall that an ideal $\mathfrak{I}$ of $\mathcal{T}^c$ is a collection of morphisms of $\mathcal{T}^c$ which contains all zero morphisms, and it is closed under addition and under composition with arbitrary morphisms from left and right, whenever the operations are defined. Following [26], we can further define:

Definition 3.7. An ideal $\mathfrak{I}$ of $\mathcal{T}^c$ is called *exact* if

(1) $\mathfrak{I} = \mathfrak{I}^2$ (that is, for each $f \in \mathfrak{I}$, there are $g, h \in \mathfrak{I}$ such that $f = gh$),

(2) $\mathfrak{I}$ is *saturated*, that is, for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and any morphism $u : Y \to V$ in $\mathcal{T}^c$, the implication
$$u \circ f,\ g \in \mathfrak{I} \implies f \in \mathfrak{I}$$
holds, and

(3) $\mathfrak{I} = \mathfrak{I}[1]$.

Then, we have the following criterion, [26, Corollary to Theorem 1]:

Proposition 3.8. *The telescope conjecture holds for $\mathcal{T}$ if and only if each exact ideal $\mathfrak{I}$ of $\mathcal{T}^c$ is generated by idempotent morphisms. That is, for each such $\mathfrak{I}$ there must exist a set $\mathcal{C}$ of objects of $\mathcal{T}^c$ such that $f \in \mathfrak{I}$ if and only if f factors through some $C \in \mathcal{C}$.*

Another point of view is connected to the term "smashing" from Definition 3.4. It comes from homotopy theory, since there every smashing localization $L : \mathcal{T} \to \mathcal{T}$ of the stable homotopy category of spectra is of the form $L = - \wedge E$, where "$\wedge$" is the smash product and E is a suitable spectrum (cf. [29, Example 5.5.3]).

In the case of $\mathcal{T} = \mathbf{D}(\text{Mod-}R)$, the analogue of the smash product is usually the tensor product. Indeed, if $f : R \to S$ is a homological epimorphism of rings (see [9, §4] and also [31, §3] in this volume), then $L = - \otimes_R^{\mathbf{L}} S_R : \mathbf{D}(\text{Mod-}R) \longrightarrow \mathbf{D}(\text{Mod-}R)$ is a smashing localization functor. Here, we point out two facts:

(1) If R is right hereditary, *all* smashing localization functors are obtained in this way up to natural equivalence, [31, Theorem B].
(2) The counterexample to the telescope conjecture constructed by Keller [21] is of this form.

If we want to study smashing localizations in terms of derived tensor products more generally, however, we need to pass to homological epimorphisms of small dg-categories. This has been recently studied by Nicolás and Saorín in [40].

4. More on homotopy categories of complexes

Finally, we shortly introduce the results from [49] in this volume. Inspired by results like Proposition 2.6, one may ask which other homotopy categories of complexes are compactly generated or, more generally, well generated. Motivation for this, except for the telescope conjecture, can be the possibility to construct adjoint functors, see Corollary 1.6.

It turns out, however, that there is a crucial obstruction. Namely, if $\mathcal{G}$ is an additive category with coproducts, then $\mathbf{K}(\mathcal{G})$ being well-generated implies by [49, Theorem 2.5] that $\mathcal{G}$ has an additive generator. That is, there is $X \in \mathcal{G}$ such that $\mathcal{G} = \operatorname{Add} X$. Although this condition may look rather innocent at the first glance, it has rather strong consequences using model theoretic techniques. To point out a few examples:

- [49, Proposition 2.6] $\mathbf{K}(\text{Mod-}R)$ is well generated if and only if $\mathbf{K}(\text{Mod-}R)$ is compactly generated if and only if R is right pure semisimple. If R is an artin algebra, this is further equivalent to R being of finite representation type.
- [49, Theorem 5.2] $\mathbf{K}(\text{Flat-}R)$ is well generated if and only if R is right perfect. In this case $\text{Flat-}R = \text{Proj-}R$ and $\mathbf{K}(\text{Flat-}R)$ is $\aleph_1$-well generated by Proposition 2.6.

If we further analyze why $\mathbf{K}(\mathcal{G})$ fails to be well-generated, we learn that the main reason is often that $\mathbf{K}(\mathcal{G})$ is not generated by any set as a localizing subcategory of itself. Note that this is a necessary condition by Corollary 1.7. However, if $\mathcal{G}$ is "nice enough", for example $\mathcal{G} = \text{Mod-}R$ or $\mathcal{G} = \text{Qcoh}(\mathbb{X})$ for a quasi-compact quasi-separated scheme $\mathbb{X}$, then $\mathbf{K}(\mathcal{G})$ is locally well generated in the following sense (we refer to [49, Theorems 3.5 and 4.3] for precise statements):

Definition 4.1. A triangulated category $\mathcal{T}$ satisfying [TR5] is called *locally well generated* if, whenever $\mathcal{C}$ is a set of objects of $\mathcal{T}$ and $\mathcal{S}$ is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$, then $\mathcal{S}$ is well generated.

This fact, together with [49, Proposition 3.9], gives rather convenient criteria to produce examples of algebraic well generated and locally well generated triangulated categories.

However, if we look back at the motivation of constructing adjoint functors, there is a serious glitch. An adaptation of an example by Casacuberta and Neeman in [49, Example 3.7] shows that the Brown representability property may fail and some adjoints one would like to have may not exist for general locally well generated triangulated categories.

At the very least, this shows that the concept itself is not strong enough and one has to look for other means to construct adjoints. An important step in this direction has been recently made by Neeman [33] and an attempt for a more systematic approach is being developed in a joint project of myself and Saorín [45]. These results are, however, beyond the scope of this thesis.

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I.

THE COUNTABLE TELESCOPE CONJECTURE FOR MODULE CATEGORIES

(JOINT WITH JAN ŠAROCH)

Abstract

By the Telescope Conjecture for Module Categories, we mean the following claim: "Let R be any ring and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in Mod-R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $(\mathcal{A}, \mathcal{B})$ is of finite type."

We prove a modification of this conjecture with the word 'finite' replaced by 'countable'. We show that a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ of modules over an arbitrary ring R is generated by a set of strongly countably presented modules provided that $\mathcal{B}$ is closed under unions of well-ordered chains. We also characterize the modules in $\mathcal{B}$ and the countably presented modules in $\mathcal{A}$ in terms of morphisms between finitely presented modules, and show that $(\mathcal{A}, \mathcal{B})$ is cogenerated by a single pure-injective module provided that $\mathcal{A}$ is closed under direct limits. Then we move our attention to strong analogies between cotorsion pairs in module categories and localizing pairs in compactly generated triangulated categories.

This paper has been published in Adv. Math. **219** (2008), 1002–1036.

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Motivated by the paper [30] of Krause and Solberg, the first author with Lidia Angeleri Hügel and Jan Trlifaj started in [4] an investigation of the *Telescope Conjecture for Module Categories* (TCMC) stated as follows (see Section 1 for unexplained terminology):

Telescope Conjecture for Module Categories. Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in Mod-R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod-}R)$.

The term 'Telescope Conjecture' is used here because the particular case of TCMC when R is a self-injective artin algebra and $(\mathcal{A}, \mathcal{B})$ is a projective cotorsion pair was shown in [30] to be equivalent to the following telescope conjecture for compactly generated triangulated categories (in this case—for the stable module category over R) which

Date: May 11, 2008.

2000 *Mathematics Subject Classification.* 16E30, 18E30 (primary), 03C60, 16D90, 18G25, 20K40 (secondary).

Key words and phrases. cotorsion pair, deconstruction, coherent functor, triangulated category, Telescope conjecture.

Both authors supported by the grants GAUK 301-10/252216, GAČR 201/05/H005, and the research project MSM 0021620839.

Second author supported also by the Research Council of Norway through Storforsk-project Homological and geometric methods in algebra.

originates in works of Bousfield [12] and Ravenel [38] and has been extensively studied by Krause in [29, 27]:

Telescope Conjecture for Triangulated Categories. Every smashing localizing subcategory of a compactly generated triangulated category is generated by compact objects.

Under some restrictions on homological dimensions of modules in the cotorsion pair $(\mathcal{A}, \mathcal{B})$, TCMC is known to hold. The first author and co-authors showed in [4] that the conclusion of TCMC amounts to saying that the given cotorsion pair is of finite type. If all modules in $\mathcal{A}$ have finite projective dimension, then the cotorsion pair is tilting [42], hence of finite type [9]. If R is a right noetherian ring and $\mathcal{B}$ consists of modules of finite injective dimension, then $(\mathcal{A}, \mathcal{B})$ is of finite type, too [4]. Therefore, TCMC holds true for example for any cotorsion pair over a ring with finite global dimension. Unfortunately, the interesting connection with triangulated categories introduced in [30] works for self-injective artin algebras, where the only cotorsion pairs satisfying the former conditions are the trivial ones.

The aim of this paper is twofold. First, we prove the Countable Telescope Conjecture in Theorem 3.5: any cotorsion pair satisfying the hypotheses of TCMC is of countable type—that is, the class $\mathcal{B}$ is the Ext^1-orthogonal class to the class of all (strongly) countably presented modules from $\mathcal{A}$. This is a weaker version of TCMC. We will also show that this result easily implies a more direct argument for a large part of the proof that all tilting classes are of finite type [7, 8, 42, 9].

The second goal is to systematically analyze analogies between approximation theory for cotorsion pairs and results about localizations in compactly generated triangulated categories. Considerable efforts have been made on both sides. Cotorsion pairs were introduced by Salce in [40] where he noticed a homological connection between special preenvelopes and precovers—or left and right approximation in the terminology of [6]. In [16], Eklof and Trlifaj proved that any cotorsion pair generated by a set of modules provides for these approximations. This turns out to be quite a usual case and the related theory with many applications is explained in the recently issued monograph [19]. Localizations of triangulated categories have, on the other hand, motivation in algebraic topology. The telescope conjecture above was introduced by Bousfield [12, 3.4] and Ravenel [38, 1.33]. Compactly generated triangulated categories and their localizations were studied by Neeman [34, 35] and Krause [29, 27]. Even though the telescope conjecture is known to be false for general triangulated categories [26], it is still open for the important and topologically motivated stable homotopy category as well as for stable module categories over self-injective artin algebras.

Although it should not be completely unexpected that there are some analogies between the two settings, as the derived unbounded category is triangulated compactly generated and provides a suitable language for homological algebra, the extent to which the analogies work is rather surprising. Roughly speaking, it is sufficient to replace an Ext^1-group in a module category by a Hom-group in a triangulated category, and we obtain a valid result. However, there are also substantial differences here—for instance special precovers and preenvelopes provided by cotorsion pairs are, unlike adjoint functors coming from localizations, not functorial.

In Section 4, we prove in Theorem 4.9 that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair meeting the assumptions of TCMC, then $\mathcal{B}$ is defined by finite data in the sense that it is the Ext^1-orthogonal class to a certain ideal of maps between finitely presented modules. Moreover, we characterize the countably generated modules in $\mathcal{A}$ as direct limits of systems of maps from this ideal (Theorem 4.8). In Section 5, we prove in Theorem 5.13 that $\mathcal{A} = \operatorname{Ker} \mathrm{Ext}^1(-, E)$ for a single pure-injective module E.

Finally, in Section 6, we give the triangulated category analogues of all of the main results for module categories. Some of them come from our analysis, while the others were originally proved by Krause in [29] and served as a source of inspiration for this paper.

Acknowledgements. The authors would like to thank Jan Trlifaj for reading parts of this text and giving several valuable comments, and also to Øyvind Solberg for stimulating discussions and helpful suggestions.

1. Preliminaries

Throughout this paper, R will always stand for an associative ring with unit, and all modules will be (unital) right R-modules. We call a module *strongly countably presented* if it has a projective resolution consisting of countably generated projective modules. *Strongly finitely presented* modules are defined in the same manner with the word ‘countably’ replaced by ‘finitely’. We denote the class of all modules by Mod-R and the class of all strongly finitely presented modules by mod-R.

We note that the notation mod-R is often used in the literature for the class of *finitely presented modules*; that is, the modules M possessing a presentation $P_1 \to P_0 \to M \to 0$ where P_0 and P_1 are finitely generated and projective. We have digressed a little from this de-facto standard for the sake of keeping our notation simple, and we believe that this should not cause much confusion. We remind that if R is a right coherent ring, then the class of strongly finitely presented modules coincides with the class of finitely presented ones. Moreover, one typically restricts oneself to coherent rings in various applications.

1.1. **Continuous directed sets and associated filters.** Let $(I, \leq)$ be a partially ordered set and λ be an infinite regular cardinal. We say that I is λ-*complete* if every well-ordered ascending chain $(i_\alpha \mid \alpha < \tau)$ of elements from I of length $< \lambda$ has a supremum in I. If this is the case, we call a subset $J \subseteq I$ λ-*closed* if, whenever such a chain is contained in J, its supremum is in J as well. For instance for any set X, the power set $\mathfrak{P}(X)$ ordered by inclusion is λ-complete and the set $\mathfrak{P}^{<\lambda}(X)$ of all subsets of X of cardinality $< \lambda$ is λ-closed in $\mathfrak{P}(X)$.

Recall that a subset $J \subseteq I$ is called *cofinal* if for every $i \in I$ there is $j \in J$ such that $i \leq j$. Note that if I is a totally ordered set, then the cofinal subsets of I are precisely the unbounded ones.

¿From now on, we assume that $(I, \leq)$ is a directed set. If $(M_i, f_{ji} : M_i \to M_j \mid i, j \in I \ \& \ i \leq j)$ is a direct system of modules, we call it λ-*continuous* if the index set I is λ-complete and for each well-ordered ascending chain $(i_\alpha \mid \alpha < \tau)$ in I of length $< \lambda$ we have

$$M_{\sup i_\alpha} = \varinjlim_{\alpha < \tau} M_{i_\alpha}.$$

It is well-known that every module is the direct limit of a direct system of finitely presented modules. But if we want the direct system to be λ-continuous, we have to pass to $< \lambda$-presented modules in general. The following lemma is a slight modification of [24, Proposition 7.15].

Lemma 1.1. *Let M be any module and λ an infinite regular cardinal. Then M is the direct limit of a λ-continuous direct system of $< \lambda$-presented modules.*

Proof. Fix a free presentation

$$R^{(X)} \xrightarrow{f} R^{(Y)} \to M \to 0$$

of M and let I be the following set:

$$\big\{(X', Y') \in \mathfrak{P}(X) \times \mathfrak{P}(Y) \mid |X'| + |Y'| < \lambda \ \& \ f\big[R^{(X')}\big] \subseteq R^{(Y')}\big\}.$$

It is straightforward to check that I with the partial ordering by inclusion in both components is directed and λ-complete. If we now define M_i as the cokernel of the map

$$f \restriction R^{(X')} : R^{(X')} \to R^{(Y')}$$

for every $i = (X', Y') \in I$, it is easy to check that $(M_i \mid i \in I)$ together with the natural maps forms a λ-continuous direct system with M as its direct limit. □

For every directed set I, there is an *associated filter* $\mathfrak{F}_I$ on $(\mathfrak{P}(I), \subseteq)$; namely the one with a basis consisting of the upper sets $\uparrow i = \{j \in I \mid$

$j \geq i\}$ for all $i \in I$. That is

$$\mathfrak{F}_I = \{X \subseteq I \mid (\exists i \in I)(\uparrow i \subseteq X)\}.$$

Recall that a filter $\mathfrak{F}$ on a power set is called *λ-complete* if any intersection of less than λ elements from $\mathfrak{F}$ is again in $\mathfrak{F}$.

Lemma 1.2. *Let $(I, \leq)$ be a λ-complete directed set. Then any subset $J \subseteq I$ such that $|J| < \lambda$ has an upper bound in I. In particular, the associated filter $\mathfrak{F}_I$ is λ-complete, and it is a principal filter if and only if $(I, \leq)$ has a (unique) maximal element.*

Proof. We can well-order J; that is $J = \{j_\alpha \mid \alpha < \tau\}$ for some $\tau < \lambda$. Then we construct by induction a chain $(k_\alpha \mid \alpha < \tau)$ in I such that $k_0 = j_0$ and k_α is a common upper bound for j_α and $\sup_{\beta<\alpha} k_\beta$. Then $\sup_{\beta<\tau} k_\beta$ is clearly an upper bound for J. The rest is also easy. □

1.2. **Filtrations and cotorsion pairs.** Given a module M and an ordinal number σ, an ascending chain $\mathcal{F} = (M_\alpha \mid \alpha \leq \sigma)$ of submodules of M is called a *filtration of* M if $M_0 = 0$, $M_\sigma = M$ and $\mathcal{F}$ is *continuous*—that is, $\bigcup_{\alpha<\beta} M_\alpha = M_\beta$ for each limit ordinal $\beta \leq \sigma$.

Furthermore, let a class $\mathcal{C} \subseteq$ Mod-R be given. Then $\mathcal{F}$ is said to be a *$\mathcal{C}$-filtration* if it has the extra property that each its consecutive factor $M_{\alpha+1}/M_\alpha$, $\alpha < \sigma$, is isomorphic to a module from $\mathcal{C}$. A module M is called *$\mathcal{C}$-filtered* if it admits (at least one) $\mathcal{C}$-filtration.

Let us turn our attention to cotorsion pairs now. By a *cotorsion pair* in Mod-R, we mean a pair $(\mathcal{A}, \mathcal{B})$ of classes of right R-modules such that $\mathcal{A} = \operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{B})$ and $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{A}, -)$. We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* provided that $\mathcal{A}$ is closed under kernels of epimorphisms or, equivalently, $\mathcal{B}$ is closed under cokernels of monomorphisms.

If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, then the class $\mathcal{A}$ is always closed under arbitrary direct sums and contains all projective modules. Dually, the class $\mathcal{B}$ is closed under direct products and it contains all injective modules. Also, every class of modules $\mathcal{C}$ determines two distinguished cotorsion pairs—the cotorsion pair *generated* by $\mathcal{C}$, that is the one with the right-hand class $\mathcal{B}$ equal to $\operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{C}, -)$, and dually the cotorsion pair *cogenerated*[1] by $\mathcal{C}$—the one with the left-hand class $\mathcal{A}$ equal to $\operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{C})$. We say that $(\mathcal{A}, \mathcal{B})$ is of *finite* or *countable type* if it is generated by a set of strongly finitely or strongly countably presented modules, respectively.

We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* if for every module $M \in$ Mod-R, there is a short exact sequence $0 \to B \to A \to M \to 0$ such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The map $A \to M$ is then called a *special $\mathcal{A}$-precover* of M. It is well-known that this condition is equivalent to

[1]It may cause some confusion that the meaning of the terms *generated* and *cogenerated* is sometimes swapped in the literature. Our terminology follows the monograph [19].

the dual one saying that $\mathcal{B}$ provides for *special* $\mathcal{B}$*-preenvelopes*; thus, for every $M \in \text{Mod-}R$ there is in this case also a short exact sequence $0 \to M \to B' \to A' \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$.

Finally, a cotorsion pair is said to be *projective* in the sense of [10] if it is hereditary, complete, and $\mathcal{A} \cap \mathcal{B}$ is precisely the class of all projective modules. It is an easy exercise to prove that $(\mathcal{A}, \mathcal{B})$ is projective if and only if it is complete and $\mathcal{B}$ contains all projective modules and has the "two out of three" property—that is: all three modules in a short exact sequence are in $\mathcal{B}$ provided that two of them are in $\mathcal{B}$. To conclude the discussion of terminology concerning cotorsion pairs, we recall that projective cotorsion pairs over self-injective artin algebras are (with a slightly different but equivalent definition) called *thick* in [30].

1.3. **Definable classes and coherent functors.** We will also need the notion of a definable class of modules. First recall that a covariant additive functor from Mod-R to the category of abelian groups is called *coherent* if it commutes with arbitrary products and direct limits. The following important characterization was obtained by Crawley-Boevey:

Lemma 1.3. [13, §2.1, Lemma 1] *A functor* $F : \text{Mod-}R \to \text{Ab}$ *is coherent if and only if it is isomorphic to* $\operatorname{Coker} \operatorname{Hom}_R(f, -)$ *for some homomorphism* $f : X \to Y$ *between finitely presented modules* X *and* Y*.*

A class $\mathcal{C} \subseteq \text{Mod-}R$ is called *definable* if it satisfies one of the following three equivalent conditions:

(1) $\mathcal{C}$ is closed under taking arbitrary products, direct limits, and pure submodules;
(2) $\mathcal{C}$ is defined by vanishing of some set of coherent functors;
(3) $\mathcal{C}$ is defined in the first order language of R-modules by satisfying some implications $\varphi(\bar{x}) \to \psi(\bar{x})$ where $\varphi(\bar{x})$ and $\psi(\bar{x})$ are primitive positive formulas.

Primitive positive formulas (pp-formulas for short) are first-order language formulas of the form $(\exists \bar{y})(\bar{x}A = \bar{y}B)$ for some matrices A, B over R. For this paper, the most important consequence of (3) is that definable classes are closed under taking elementarily equivalent modules since they are definable in the first-order language. This in particular implies the well-known fact that a definable class is determined by the pure-injective modules it contains since any module is elementarily equivalent to its pure-injective hull. For equivalence between the three definitions and more details, we refer to [37], [13, §2.3], and [45, Section 1].

1.4. **Inverse limits and the Mittag-Leffler condition.** The computation of Ext groups can sometimes be reduced to the computation of the derived functors of inverse limit. We will recall this here only for countable inverse systems. For more details on the topic see [44, §3.5]. Let

$$\cdots \to H_{n+1} \xrightarrow{h_n} H_n \to \cdots \to H_2 \xrightarrow{h_1} H_1 \xrightarrow{h_0} H_0$$

be a countable inverse system of abelian groups—a tower in the terminology of [44]. Then its inverse limit $\varprojlim H_n$ and the *first derived functor of the inverse limit*, $\varprojlim^1 H_n$, can be computed using the exact sequence

$$0 \to \varprojlim H_n \to \prod H_n \xrightarrow{\Delta} \prod H_n \to \varprojlim{}^1 H_n \to 0$$

where $\Delta((x_n)_{n<\omega}) = (x_n - h_n(x_{n+1}))_{n<\omega}$. The first derived functor is closely related to the fact that inverse limit is not exact—it is only left exact. Using the exact sequence above and the snake lemma, one easily observes that, given a countable inverse system of short exact sequences $0 \to H_n \to K_n \to L_n \to 0$, there is a canonical long exact sequence

$$0 \to \varprojlim H_n \to \varprojlim K_n \to \varprojlim L_n \to \varprojlim{}^1 H_n \to \varprojlim{}^1 K_n \to \varprojlim{}^1 L_n \to 0$$

In particular, $\varprojlim^1$ is right exact on countable inverse systems.

In practice, one is often interested whether or not $\varprojlim^1 H_n = 0$. To decide this can sometimes be tedious, but there is a useful tool—the notion of Mittag-Leffler inverse systems. Given a countable inverse system of abelian groups $(H_n, h_n \mid n < \omega)$ as above, we say that it is *Mittag-Leffler* if for each n the descending chain

$$H_n \supseteq h_n(H_{n+1}) \supseteq \cdots \supseteq h_n h_{n+1} \cdots h_{k-1}(H_k) \supseteq \cdots$$

is stationary. This occurs, for example, if all the maps h_n are onto. The following important result gives a connection to $\varprojlim^1$:

Proposition 1.4. *Let $(H_n, h_n \mid n < \omega)$ be a countable inverse system of abelian groups. Then the following hold:*

(1) [44, Proposition 3.5.7] *If (H_n, h_n) is Mittag-Leffler, then $\varprojlim^1 H_n = 0$.*
(2) [2, Theorem 1.3] *(H_n, h_n) is Mittag-Leffler if and only if $\varprojlim^1 H_n^{(\omega)} = 0$.*

We will also use a related notion of T-nilpotency. We say that $(H_n, h_n)_{n<\omega}$ is *T-nilpotent* if for each n there exists $k > n$ such that the composition $H_k \to H_n$ is zero.

2. Filter-closed classes and factorization systems

We start with analyzing properties of modules lying in $\operatorname{Ker} \operatorname{Ext}^1_R(-, \mathcal{G})$ for a class $\mathcal{G}$ closed under arbitrary direct products and unions of well-ordered chains. We will always assume in this case that $\mathcal{G}$ is closed under isomorphic images and that $0 \in \mathcal{G}$, since the trivial module could be viewed as a product of an empty system. As an application to keep in mind, such classes occur as right-hand classes of cotorsion pairs satisfying the hypotheses of TCMC.

Definition 2.1. Let $\mathfrak{F}$ be a filter on the power set $\mathfrak{P}(X)$ for some set X, and let $\{M_x \mid x \in X\}$ be a set of modules. Set $M = \prod_{x\in X} M_x$. Then the $\mathfrak{F}$-*product* $\Sigma_{\mathfrak{F}}M$ is the submodule of M such that

$$\Sigma_{\mathfrak{F}}M = \{m \in M \mid z(m) \in \mathfrak{F}\}$$

where for an element $m = (m_x \mid x \in X) \in M$, we denote by $z(m)$ its zero set $\{x \in X \mid m_x = 0\}$.

The module $M/\Sigma_{\mathfrak{F}}M$ is then called an $\mathfrak{F}$-*reduced product*. Note that for $a, b \in M$, we have an equality $\bar{a} = \bar{b}$ in the $\mathfrak{F}$-reduced product if and only if a and b agree on a set of indices that is in the filter $\mathfrak{F}$.

In the case that $M_x = M_y$ for every pair of elements $x, y \in X$, we speak of an $\mathfrak{F}$-*power* and an $\mathfrak{F}$-*reduced power* (of the module M_x) instead of an $\mathfrak{F}$-product and an $\mathfrak{F}$-reduced product, respectively.

Finally, a nonempty class of modules $\mathcal{G}$ is called *filter-closed*, if it is closed under arbitrary $\mathfrak{F}$-products (for any set X and an arbitrary filter $\mathfrak{F}$ on $\mathfrak{P}(X)$).

Lemma 2.2. *Let $\mathcal{G}$ be a class of modules closed under arbitrary direct products and unions of well-ordered chains. Then $\mathcal{G}$ is filter-closed.*

Proof. It is just a matter of straightforward induction to prove that the closure under unions of well-ordered chains implies closure under arbitrary directed unions—see for instance [1, Corollary 1.7] which is easily adapted for unions. Moreover, any $\mathfrak{F}$-product is just the directed union of products of the modules with indices from the complementary sets to those belonging to $\mathfrak{F}$. □

In the next few paragraphs, we will show that filter-closedness of $\mathcal{G}$ forces existence of certain factoring systems inside modules from $\operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{G})$. Let us note that the following lemma presents the crucial technical step in proving the Countable Telescope Conjecture.

Lemma 2.3. *Let $\mathcal{G}$ be a filter-closed class of modules. Let λ be an* uncountable *regular cardinal and $(M, f_i \mid i \in I)$ be a direct limit of a λ-continuous direct system $(M_i, f_{ji} \mid i \leq j)$ indexed by a set I and consisting of $< \lambda$-generated modules.*

Assume that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$*. Then there is a λ-closed cofinal subset $J \subseteq I$ such that every homomorphism from M_j to B factors through f_j whenever $j \in J$ and $B \in \mathcal{G}$.*

Proof. Suppose that the claim of the lemma is not true. Then the set

$$S = \{i \in I \mid (\exists B_i \in \mathcal{G})(\exists g_i \in \operatorname{Hom}_R(M_i, B_i)) \\ (g_i \text{ does not factor through } f_i)\} \quad (*)$$

must intersect every λ-closed cofinal subset of I (so S is a generalized stationary set, in an obvious sense). For each $i \in S$, choose some

$B_i \in \mathcal{G}$ and $g_i : M_i \to B_i$ whose existence is claimed in $(*)$. For the indices $i \in I \setminus S$, let B_i be an arbitrary module from $\mathcal{G}$ and $g_i : M_i \to B_i$ be the zero map. Put $B = \prod_{i \in I} B_i$.

Now, define a homomorphism $h_{ji} : M_i \to B_j$ for each pair $i, j \in I$ in the following way: $h_{ji} = g_j \circ f_{ji}$ if $i \le j$ and $h_{ji} = 0$ otherwise. This family of maps gives rise to a canonical homomorphism $h : \bigoplus_{k \in I} M_k \to B$. More precisely, if we denote by $\pi_j : B \to B_j$ the projection to the j-th component and by $\nu_i : M_i \to \bigoplus_{k \in I} M_k$ the canonical inclusion of the i-th component, h is (unique) such that $\pi_j \circ h \circ \nu_i = h_{ji}$. Note that for every $i, j \in I$ such that $i \le j$, the set $\{k \in I \mid h_{ki} = h_{kj} \circ f_{ji}\}$ is in the associated filter $\mathfrak{F}_I$ since it contains $\uparrow j$. Hence, if we denote by φ the canonical pure epimorphism $\bigoplus_{i \in I} M_i \to M = \varinjlim_{i \in I} M_i$ (that is such that $\varphi \circ \nu_i = f_i$ for all $i \in I$), there is a well-defined homomorphism u from M to the $\mathfrak{F}_I$-reduced product $B/\Sigma_{\mathfrak{F}_I} B$ making the following diagram commutative (ρ denotes the canonical projection):

$$\begin{array}{ccccc} B & \xrightarrow{\rho} & B/\Sigma_{\mathfrak{F}_I} B & \longrightarrow & 0 \\ {\scriptstyle h}\uparrow & & {\scriptstyle u}\uparrow & & \\ \bigoplus_{i \in I} M_i & \xrightarrow{\varphi} & M & \longrightarrow & 0. \end{array}$$

We have $\Sigma_{\mathfrak{F}_I} B \in \mathcal{G}$ since $\mathcal{G}$ is filter-closed. Hence, using the assumption that $\operatorname{Ext}^1_R(M, \Sigma_{\mathfrak{F}_I} B) = 0$, we can factorize u through ρ to get some $g \in \operatorname{Hom}_R(M, B)$ such that $u = \rho \circ g$. Since the M_i are all $< \lambda$-generated and $\mathfrak{F}_I$ is λ-complete by Lemma 1.2, we obtain (for every $i \in I$) that "$h \circ \nu_i$ coincides with $g \circ \varphi \circ \nu_i = g \circ f_i$ on a set from the filter", that is:

$$\{k \in I \mid \pi_k \circ g \circ f_i = \pi_k \circ h \circ \nu_i\} \in \mathfrak{F}_I. \qquad (**)$$

Let us define J as follows:

$$J = \{i \in I \mid (\forall k \ge i)(\pi_k \circ g \circ f_i = g_k \circ f_{ki})\}.$$

Then clearly, g_i factors through f_i for every $i \in J$ (just by applying the definition of J for $k = i$). Hence certainly $J \cap S = \varnothing$.

To obtain a contradiction and finish the proof of the lemma, it is now enough to show that J is λ-closed cofinal. The fact that J is λ-closed follows easily by λ-continuity of the direct system $(M_i, f_{ji} \mid i \le j)$. So we are left to prove that J is cofinal in I. But by $(**)$ and the definition of $\mathfrak{F}_I$, we can find for every $i \in I$ an element $s(i) \in I$ such that $s(i) \ge i$ and

$$(\forall k \ge s(i))(\pi_k \circ g \circ f_i = \pi_k \circ h \circ \nu_i). \qquad (\Delta)$$

Recall that $\pi_k \circ h \circ \nu_i = h_{ki} = g_k \circ f_{ki}$. Now, if we fix any $i' \in I$, we can define $j_0 = i'$, $j_{n+1} = s(j_n)$ for all $n \ge 0$, and $j = \sup_{n<\omega} j_n$. Then clearly $j \ge i'$, and it is easy to check that $j \in J$ using the $\aleph_1$-continuity of the direct system $(M_i, f_{ji} \mid i \le j)$. $\square$

An important consequence follows by applying Lemma 2.3 to the case when the class $\mathcal{G}$ cogenerates every module. This is for instance always the case when $\mathcal{G}$ is a right-hand class of a cotorsion pair, since then all injective modules are inside $\mathcal{G}$.

Proposition 2.4. *Let $\mathcal{G}$ be a cogenerating filter-closed class of modules. Then for any uncountable regular cardinal λ and any module M such that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$, there is a family $\mathcal{C}_\lambda$ of $< \lambda$-presented submodules of M such that*

1. *$\mathcal{C}_\lambda$ is closed under unions of well-ordered ascending chains of length $< \lambda$,*
2. *every subset $X \subseteq M$ such that $|X| < \lambda$ is contained in some $N \in \mathcal{C}_\lambda$, and*
3. *$\operatorname{Ext}^1_R(M/N, \mathcal{G}) = 0$ for every $N \in \mathcal{C}_\lambda$.*

Proof. By Lemma 1.1, there is a λ-continuous direct system $(M_i, f_{ji} \mid i \le j)$ of $< \lambda$-presented modules indexed by a set I such that M together with some maps $f_i : M_i \to M$ forms its direct limit. Now, the data $\mathcal{G}$, λ, $(M, f_i \mid i \in I)$, $(M_i, f_{ji} \mid i \le j)$ and I fits exactly to Lemma 2.3. Hence, there is a λ-closed cofinal subset $J \subseteq I$ such that for every $j \in J$, every homomorphism from M_j to a module in $\mathcal{G}$ factors through f_j. But the fact that $\mathcal{G}$ is a cogenerating class implies that f_j is injective. Thus, we can view the modules M_j for $j \in J$ as submodules of M, and the maps f_j and f_{ji} as inclusions. Let us define

$$\mathcal{D} = \{M_j \mid j \in J\}$$

and let $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$ under unions of well-ordered chains of length $< \lambda$. Observe, that $(\mathcal{D}, \subseteq)$ is a directed poset since J is a cofinal subset of the directed set I. Using Lemma 1.2, we easily deduce that $\overline{\mathcal{D}}$ is directed, too. Now, we can view the modules in $\overline{\mathcal{D}}$ together with inclusions between them as a λ-continuous direct system indexed by $\overline{\mathcal{D}}$ itself. Hence, we can apply Lemma 2.3 for the second time to get a λ-closed cofinal subset $\mathcal{C}_\lambda$ of $\overline{\mathcal{D}}$ such that every homomorphism from a module $N \in \mathcal{C}_\lambda$ to a module in $\mathcal{G}$ extends to M.

The latter property together with the fact that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$ immediately implies (3). The property (1) is just another way to say that $\mathcal{C}_\lambda$ is λ-closed in $\overline{\mathcal{D}}$. For (2), first notice that $\bigcup \mathcal{C}_\lambda = M$ since $\mathcal{C}_\lambda$ is cofinal in $\overline{\mathcal{D}}$. Hence, if $X \subseteq M$ is a subset of cardinality $< \lambda$, there is a subset $\mathcal{M} \subseteq \mathcal{C}_\lambda$ of cardinality $< \lambda$ such that every $x \in X$ is contained in some $N' \in \mathcal{M}$. Finally, Lemma 1.2 provides us with an upper bound $N \in \mathcal{C}_\lambda$ for $\mathcal{M}$, and clearly $X \subseteq N$. □

In Lemma 2.3, the assumption of λ being uncountable is essential. We can, nevertheless, obtain a weaker but important result using the same technique for $\lambda = \omega$ and $(I, \le) = (\omega, \le)$. Lemma 2.5 actually says that, for $B \in \mathcal{G}$, the inverse system of groups $(\operatorname{Hom}_R(M_i, B), \operatorname{Hom}_R(f_{ji}, B) \mid i \le j < \omega)$ is Mittag-Leffler, and the stationary indices determined by s are common over all $B \in \mathcal{G}$. In this terminology, a proof of the lemma is mostly contained in the proof of [8, Theorems 2.5 and 3.7].

We give a different proof here and we do this for two main reasons: First, the statement about common stationary indices has an important interpretation in the first-order theory of modules and is missing in [8]. Second, we show that the Mittag-Leffler property is a part of a common framework which works for both countable and uncountable systems.

Lemma 2.5. *Let $\mathcal{G}$ be a class of modules closed under countable direct sums. Let $(M, f_i \mid i < \omega)$ be a direct limit of a countable direct system $(M_i, f_{ji} \mid i \leq j < \omega)$ consisting of finitely generated modules.*

Assume that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then there is a strictly increasing function $s : \omega \to \omega$ such that for each $B \in \mathcal{G}$, $i < \omega$ and $c : M_i \to B$ the following holds: If c factors through $f_{s(i)i}$, then it factors through f_{ni} for all $n \geq s(i)$.

Proof. We will show that it is possible to construct the values $s(i)$ by induction on i. Suppose by way of contradiction that there is some $i < \omega$ for which we cannot define $s(i)$. This can only happen if for each $j \geq i$, there is a homomorphism $g_j : M_j \to B_j$ such that $B_j \in \mathcal{G}$, and $g_j \circ f_{ji}$ does not factor through f_{ni} for some $n > j$. For $j < i$ let g_j be zero maps and $B_j \in \mathcal{G}$ be arbitrary. Put $B = \prod_{j<\omega} B_j$.

Now, we follow the proof of Lemma 2.3 (with ω in place of I and λ) starting with the second paragraph and ending just after the definition of $(**)$. Note that the corresponding notion of $\aleph_0$-completeness is void, $\mathfrak{F}_\omega$ is the Fréchet filter on ω, and the $\mathfrak{F}_\omega$-product $\Sigma_{\mathfrak{F}_\omega} B$ is just the direct sum $\bigoplus_{j<\omega} B_j$.

By the same argument as for (Δ) in the proof of Lemma 2.3 and with the same notation as there, there is some $s' \geq i$ such that

$$(\forall k \geq s')(\pi_k \circ g \circ f_i = \pi_k \circ h \circ \nu_i)$$

holds and $\pi_k \circ h \circ \nu_i = h_{ki} = g_k \circ f_{ki}$ for each $k \geq s'$. But this contradicts the fact implied by the choice of g_k that $g_k \circ f_{ki}$ does not factor through f_i. □

Let us remark that we have actually proved a little more than we stated in Lemma 2.5—we have constructed $s : \omega \to \omega$ such that if $c : M_i \to B$ factors through $f_{s(i)i}$, then it factors through $f_i : M_i \to M$. The motivation for the seemingly more complicated statement of the lemma should become clear in the following paragraphs.

If the modules M_i in the direct system from the lemma above are finitely presented instead of finitely generated, we have a statement about factorization through maps between finitely presented modules. Which in other words means that some coherent functors vanish and the Mittag-Leffler property is preserved within the smallest definable class containing $\mathcal{G}$. This is made precise by the following lemma.

Lemma 2.6. *Let $\mathcal{G}$ be a class of modules closed under countable direct sums and $\mathcal{D}$ be the smallest definable class containing $\mathcal{G}$. Let $(M, f_i \mid i < \omega)$ be a direct limit of a direct system $(M_i, f_{ji} \mid i \leq j < \omega)$ consisting of finitely presented modules.*

Assume that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then there is a strictly increasing function $s : \omega \to \omega$ such that for each $D \in \mathcal{D}$, $i < \omega$ and $c : M_i \to D$ the following holds: If c factors through $f_{s(i)i}$, then it factors through f_{ni} for all $n \geq s(i)$.

Proof. By restating the conclusion of Lemma 2.5, we get that $\operatorname{Im}\operatorname{Hom}_R(f_{s(i)i}, D) = \operatorname{Im}\operatorname{Hom}_R(f_{ni}, D)$ for each $D \in \mathcal{G}$ and $i \leq s(i) \leq n < \omega$. It is also straightforward to check that $F = \operatorname{Im}\operatorname{Hom}_R(f_{s(i)i}, -)/\operatorname{Im}\operatorname{Hom}_R(f_{ni}, -)$ is a coherent functor. Hence we have $\operatorname{Im}\operatorname{Hom}_R(f_{s(i)i}, D) = \operatorname{Im}\operatorname{Hom}_R(f_{ni}, D)$ also for each $D \in \mathcal{D}$ and the claim follows. □

Note also that instead of vanishing of the coherent functors in the proof above, we can equivalently consider that certain implications between pp-formulas are satisfied [13, §2.1], thus reformulating the proof in a more model theoretic way.

Now, we can prove a crucial statement similar to [8, Theorem 2.5]:

Proposition 2.7. *Let $\mathcal{G}$ be a class of modules closed under countable direct sums, and let M be a countably presented module such that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then $\operatorname{Ext}^1_R(M, D) = 0$ for every D isomorphic to a pure submodule of a product of modules from $\mathcal{G}$.*

Proof. Let D be a pure submodule of $\prod_k B_k$ for some $B_k \in \mathcal{G}$. Since M is countably presented, it can be considered as a direct limit of a countable chain of finitely presented modules $M_i, i < \omega$, as in the assumptions of Lemma 2.6. Hence $(\operatorname{Hom}_R(M_i, D), \operatorname{Hom}_R(f_{ji}, D) \mid i \leq j < \omega)$ is Mittag-Leffler since any definable class is closed under taking products and pure submodules.

Then we continue as in the proof of [8, Theorem 2.5]. Since $\operatorname{Ext}^1_R\big(M, \prod_k B_k\big) = 0$ by assumption, we have the exact sequence

$$\operatorname{Hom}_R\big(M, \prod_k B_k\big) \xrightarrow{h} \operatorname{Hom}_R\big(M, \big(\prod_k B_k\big)/D\big) \to \operatorname{Ext}^1_R(M, D) \to 0,$$

and so it suffices to show that h is an epimorphism. This easily follows from Proposition 1.4 applied on the inverse system $(\operatorname{Hom}_R(M_i, D), \operatorname{Hom}_R(f_{ji}, D) \mid i \leq j < \omega)$. Indeed, we see that $\varprojlim^1_i \operatorname{Hom}_R(M_i, D) = 0$ and obtain the exact sequence

$$\varprojlim_i \operatorname{Hom}_R\big(M_i, \prod_k B_k\big) \to \varprojlim_i \operatorname{Hom}_R\big(M_i, \big(\prod_k B_k\big)/D\big) \to 0.$$

It remains to use the basic fact that contravariant Hom-functors take colimits to limits. □

3. Countable type

In this section, we prove the main result of our paper—the Countable Telescope Conjecture for Module Categories. But before doing this, we introduce a fairly simplified version of Shelah's Singular Compactness Theorem. It is based on [15, Theorem IV.3.7]. In the terminology there, systems witnessing strong λ-"freeness" correspond to the λ-dense systems defined below.

A reader acquainted with the full-fledged compactness theorem for filtrations of modules proved in [15, XII.1.14 and IV.3.7] or [14] may well skip Lemma 3.2. We state and prove the lemma for the sake of completeness, and also because we are using only a fragment of the full compactness theorem, and it makes the proof of the Countable Telescope Conjecture more transparent.

Definition 3.1. Let M be a module and λ be a regular uncountable cardinal. Then a set $\mathcal{C}_\lambda$ of $< \lambda$-generated submodules of M is called a λ-*dense system in* M if

1. $0 \in \mathcal{C}_\lambda$,
2. $\mathcal{C}_\lambda$ is closed under unions of well-ordered ascending chains of length $< \lambda$, and
3. every subset $X \subseteq M$ such that $|X| < \lambda$ is contained in some $N \in \mathcal{C}_\lambda$.

Lemma 3.2 (Simplified Shelah's Singular Compactness Theorem)**.** *Let κ be a singular cardinal, M a κ-generated module, and let μ be a cardinal such that* $\operatorname{cf} \kappa \leq \mu < \kappa$. *Suppose we are given a λ-dense system, $\mathcal{C}_\lambda$, in M for each regular λ such that $\mu < \lambda < \kappa$. Then there is a filtration $(M_\alpha \mid \alpha \leq \operatorname{cf} \kappa)$ of M and a continuous strictly increasing chain of cardinals $(\kappa_\alpha \mid \alpha < \operatorname{cf} \kappa)$ cofinal in κ such that $M_\alpha \in \mathcal{C}_{\kappa_\alpha^+}$ for each $\alpha < \operatorname{cf} \kappa$.*

Proof. We will start with choosing the chain $(\kappa_\alpha \mid \alpha < \operatorname{cf} \kappa)$. In fact, we can choose any such chain provided that $\mu \leq \kappa_0$, just to make sure that $\mathcal{C}_{\kappa_\alpha^+}$ is always available. Let us fix one such chain $(\kappa_\alpha \mid \alpha < \operatorname{cf} \kappa)$.

Next, let $(X_\alpha \mid \alpha < \operatorname{cf} \kappa)$ be an ascending chain of subsets of M such that $\bigcup_{\alpha < \operatorname{cf} \kappa} X_\alpha$ generates M and $|X_\alpha| = \kappa_\alpha$ for each $\alpha < \operatorname{cf} \kappa$. Then, we can by induction construct a (not necessarily continuous) chain $(N_\alpha^0 \mid \alpha < \operatorname{cf} \kappa)$ of submodules of M such that $N_\alpha^0 \in \mathcal{C}_{\kappa_\alpha^+}$ and $X_\alpha \cup \bigcup_{\beta < \alpha} N_\beta^0 \subseteq N_\alpha^0$ for every $\alpha < \operatorname{cf} \kappa$. Since N_α is κ_α-generated, we can fix for each α a generating set Y_α^0 of N_α^0 together with some enumeration $Y_\alpha^0 = \{y_{\alpha,\gamma}^0 \mid \gamma < \kappa_\alpha\}$. Next, we proceed by induction on $n < \omega$ and construct for each $n > 0$ chain of modules $(N_\alpha^n \mid \alpha < \operatorname{cf} \kappa)$ and sets $Y_\alpha^n = \{y_{\alpha,\gamma}^n \mid \gamma < \kappa_\alpha\}$ such that

1. $(N_\alpha^n \mid \alpha < \operatorname{cf} \kappa)$ is a (not necessarily continuous) chain of submodules of M,

(2) $N^n_\alpha \in \mathcal{C}_{\kappa^+_\alpha}$ and $N^n_\alpha \supseteq \{y^{n-1}_{\zeta,\gamma} \mid \alpha \leq \zeta < \operatorname{cf}\kappa \ \&\ \gamma < \kappa_\alpha\} \cup \bigcup_{\beta<\alpha} N^n_\beta$, and

(3) $Y^n_\alpha = \{y^n_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\}$ is a fixed enumeration of some set of generators of N^n_α, for each $\alpha < \operatorname{cf}\kappa$.

For each $n < \omega$, we clearly can construct such a chain and sets by induction on α. Note in particular that we have always $N^{n-1}_\alpha \subseteq N^n_\alpha$ since $Y^{n-1}_\alpha = \{y^{n-1}_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\} \subseteq N^n_\alpha$ by (2). Hence, if we define $M_\alpha = \bigcup_{n<\omega} N^n_\alpha$, we clearly have $M_\alpha \in \mathcal{C}_{\kappa^+_\alpha}$ for each $\alpha < \operatorname{cf}\kappa$. Also, $\bigcup_{\alpha<\operatorname{cf}\kappa} M_\alpha = M$ since $X_\alpha \subseteq N^0_\alpha \subseteq M_\alpha$ for each α. We claim that the chain $(M_\alpha \mid \alpha < \operatorname{cf}\kappa)$ is continuous. To see this, fix for this moment a limit ordinal $\alpha < \operatorname{cf}\kappa$. Then clearly $M_\alpha \supseteq \bigcup_{\beta<\alpha} M_\beta$. On the other hand, for a given $n > 0$ and $\beta < \alpha$, we have $\{y^{n-1}_{\alpha,\gamma} \mid \gamma < \kappa_\beta\} \subseteq N^n_\beta$ by (2). Therefore, $Y^{n-1}_\alpha \subseteq \bigcup_{\beta<\alpha} N^n_\beta$ and also $N^{n-1}_\alpha \subseteq \bigcup_{\beta<\alpha} N^n_\beta$ by (3). Hence $M_\alpha \subseteq \bigcup_{\beta<\alpha} M_\beta$ and the claim is proved. Now, if we change M_0 for the zero module and put $M_{\operatorname{cf}\kappa} = M$, $(M_\alpha \mid \alpha \leq \operatorname{cf}\kappa)$ becomes a filtration with the desired properties. □

While Lemma 3.2 or Shelah's Singular Compactness Theorem give us some information about the structure of a module with enough dense systems for a singular number of generators, we can prove a rather straightforward lemma which takes care of regular cardinals.

Lemma 3.3. *Let κ be a regular uncountable cardinal, M be a κ-generated module and $\mathcal{C}_\kappa$ be a κ-dense system in M. Then there is a filtration $(M_\alpha \mid \alpha \leq \kappa)$ of M such that $M_\alpha \in \mathcal{C}_\kappa$ for each $\alpha < \kappa$.*

Proof. Let us fix an enumeration $\{m_\gamma \mid \gamma < \kappa\}$ of generators of M. We will construct the filtration by induction. Put $M_0 = 0$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \kappa$. For $\alpha = \beta + 1$, we can find $M_\alpha \in \mathcal{C}_\kappa$ such that $M_\beta \cup \{m_\beta\} \subseteq M_\alpha$, using (3) from Definition 3.1. □

Before stating and proving the main result, we need a technical lemma about filtrations which has been studied in [17, 41, 43], and whose origins can be traced back to an ingenious idea of P. Hill [22].

Lemma 3.4. [43, Theorem 6]. *Let $\mathcal{S}$ be a set of countably presented modules and M be a module possessing an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{F}$ of submodules of M such that:*

(1) *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.*

(2) *$\mathcal{F}$ is closed under arbitrary sums and intersections.*

(3) *For each $N, P \in \mathcal{F}$ such that $N \subseteq P$, the module P/N is $\mathcal{S}$-filtered.*

(4) *For each $N \in \mathcal{F}$ and a countable subset $X \subseteq M$, there is $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is countably presented.*

Now, we are in a position to prove the Countable Telescope Conjecture.

Theorem 3.5 (Countable Telescope Conjecture)**.** *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair of R-modules such that $\mathcal{B}$ is closed under unions of well-ordered chains. Then*

(1) *$\mathfrak{C}$ is generated by a set of strongly countably presented modules,*
(2) *$\mathfrak{C}$ is complete, and*
(3) *$\mathcal{B}$ is a definable class.*

Proof. (1). First, we claim that $\mathfrak{C}$ is generated by a representative set $\mathcal{S}$ of the class of all countably presented modules from $\mathcal{A}$. To do this, in view of Eklof's Lemma ([19, Lemma 3.1.2] or [16, Lemma 1]), it is enough to prove that every module $M \in \mathcal{A}$ has an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$.

We will prove this by induction on the minimal cardinal κ such that M is κ-presented. If κ is finite or countable, then we are done since M itself is isomorphic to a module from $\mathcal{S}$. Assume that κ is uncountable. By our assumption and Lemma 2.2, the class $\mathcal{B}$ is filter-closed and cogenerating. Hence, we can fix for each regular uncountable $\lambda \leq \kappa$ a family $\mathcal{C}_\lambda$ of $< \lambda$-presented modules given by Proposition 2.4 used with $\mathcal{G} = \mathcal{B}$. Note that we can without loss of generality assume that $\mathcal{C}_\lambda$ is a λ-dense system, since we always can add the zero module to $\mathcal{C}_\lambda$ without changing its properties. Then, we can use Lemma 3.3 if κ is regular, and Lemma 3.2 if κ is singular to obtain a filtration $(L_\beta \mid \beta \leq \tau)$ of M such that for each $\beta < \tau$

(i) L_β is $< \kappa$-presented, and
(ii) $M/L_\beta \in \mathcal{A}$.

We also have $L_{\beta+1}/L_\beta \in \mathcal{A}$ since it is a kernel of the projection $M/L_\beta \to M/L_{\beta+1}$ and $\mathfrak{C}$ is hereditary. Thus, each of the modules $L_{\beta+1}/L_\beta$ has an $\mathcal{S}$-filtration by the inductive hypothesis, so we can refine the filtration $(L_\beta \mid \beta \leq \tau)$ to an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of M and the claim is proved.

Let us note that for the induction step at singular cardinals κ, we can alternatively use the full version of Shelah's Singular Compactness Theorem, considering $\mathcal{S}$-filtered modules as "free" (cf. [15, XII.1.14 and IV.3.7] or [14]).

It is still left to show that all modules in $\mathcal{S}$ are actually strongly countably presented. Note that it is enough to prove that every countably generated module $M \in \mathcal{A}$ is countably presented. If we prove this, we can take for every module $N \in \mathcal{S}$ a presentation $0 \to K \to R^{(\omega)} \to N \to 0$ with K a countably generated module. Since $\mathfrak{C}$ is hereditary, we have $K \in \mathcal{A}$. Now, if K is countably presented, it must be isomorphic to a module from $\mathcal{S}$ again, and we can proceed by induction to construct a free resolution of N consisting of countably generated free modules.

So fix $M \in \mathcal{A}$ countably generated. Then M is $\mathcal{S}$-filtered by the arguments above. Hence, we can consider the family $\mathcal{F}$ given by Lemma 3.4

for M. To finish our proof, we use (4) from this lemma with $N = 0$ and X a countable set of generators of M as parameters.

(2). This follows from (1) by [19, Theorem 3.2.1].

(3). Note that $\mathcal{B}$ is always closed under arbitrary direct products. It is closed under infinite direct sums too since these are precisely $\mathfrak{F}$-products corresponding to Fréchet filters $\mathfrak{F}$. Then $\mathcal{B}$ is closed under pure submodules by (1) and Proposition 2.7. Further, $\mathcal{B}$ is closed under pure epimorphic images and, therefore, also under arbitrary direct limits since $\mathfrak{C}$ is hereditary. Hence $\mathcal{B}$ is definable. □

Remark. We can actually prove a little more than we state in Theorem 3.5. Notice that the proof of (1) and (2) works also for any hereditary cotorsion pair cogenerated (as a cotorsion pair) by some cogenerating (in the module category) filter-closed class $\mathcal{G}$.

To conclude this section, we will discuss the relation of Theorem 3.5 to tilting theory. In fact, it turns out that the countable type and definability of tilting classes is a rather easy consequence of Theorem 3.5. This allows us to give a more direct argumentation for most of the proof of the fact that all tilting classes are of finite type [8, 9].

Recall that $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ is called a *tilting cotorsion pair* if $\mathfrak{T}$ is hereditary, $\mathcal{A}$ consists of modules of finite projective dimension, and $\mathcal{B}$ is closed under direct sums. In this case, $\mathcal{B}$ is said to be a *tilting class.*

Theorem 3.6. *Let R be a ring and $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair. Then $\mathfrak{T}$ is generated by a set of strongly countably presented modules and $\mathcal{B}$ is definable.*

Proof. Notice that since $\mathcal{A}$ is closed under direct sums, there is $n < \omega$ such that projective dimension of any module from $\mathcal{A}$ is at most n. We will prove the theorem by induction on this n.

If the $n = 0$, then $\mathcal{B} = \text{Mod-}R$ and the statement follows trivially. Let $n > 0$. Then it is easy to see that the class $\mathcal{D} = \operatorname{Ker}\operatorname{Ext}^2_R(\mathcal{A}, -)$ is tilting and in the corresponding tilting cotorsion pair $(\mathcal{C}, \mathcal{D})$, all modules in $\mathcal{C}$ have projective dimension $< n$ (cf. [4, Lemma 4.8]). Thus $\mathcal{D}$ is definable by the inductive hypothesis. In particular, it is closed under pure submodules. By a simple dimension shifting argument, one observes that $\mathcal{B}$ is closed under pure-epimorphic images. Since, by our assumption, $\mathcal{B}$ is closed under direct sums, it follows that $\mathcal{B}$ is closed under arbitrary direct limits. Thus we may apply Theorem 3.5 to $\mathfrak{T}$ to finish the proof. □

4. Definability

In this section, we will give a description of which coherent functors define the class $\mathcal{B}$ of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ satisfying the hypotheses of TCMC. Our aim is twofold: First, vanishing of a coherent functor on a module M translates to the fact that a certain implication between pp-formulas is satisfied in M, [13, §2.1]. So there is a clear model-theoretic motivation. Second, proving that the cotorsion pair is of finite type amounts to showing that $\mathcal{B}$ is defined by a family of coherent functors of the form $\operatorname{Coker}\operatorname{Hom}_R(f,-)$ where $f : X \to Y$ is an inclusion of $X \in \text{mod-}R$ into a finitely generated projective module Y. The projectivity of Y is essential here: it implies that $Y \in \mathcal{A}$ which in turn means that the functor $\operatorname{Coker}\operatorname{Hom}_R(f,-)$ vanishes on all modules from $\mathcal{B}$ if and only if $Y/X \in \mathcal{A}$. Compare this with Remark (*ii*) at the end of the section.

Even though the finite type question still remains open, we will describe a family of coherent functors defining $\mathcal{B}$ in Theorem 4.9—this can be viewed as a counterpart of [29, Theorem A (3)] for module categories. We will also characterize the countably presented modules from the class $\mathcal{A}$ in Theorem 4.8. In both tasks, the key role is played by the ideal $\mathfrak{I}$ of the category mod-R consisting of the morphisms which, when considered in Mod-R, factor through some module from $\mathcal{A}$.

For the whole section, let R be a *right coherent* ring; that is, finitely (and also countably) presented modules are precisely the strongly finitely (countably) presented ones, respectively. We will deal with countable direct systems of finitely generated modules of the form:

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \to \cdots \to C_n \xrightarrow{f_n} C_{n+1} \to \cdots .$$

Here, we write for simplicity f_n instead of $f_{n+1,n}$. We start with recalling some important preliminary results whose proofs are essentially in [8] and [2]:

Lemma 4.1. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of R-modules. Let M be a module such that $\operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$. Then $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$.*

Proof. The proof here is in fact a part of the proof of [8, Theorem 5.1]. If we apply the functor $\operatorname{Hom}_R(-,M)$ to the canonical presentation

$$0 \to \bigoplus C_n \xrightarrow{\phi} \bigoplus C_n \to \varinjlim C_n \to 0$$

of the countable direct limit $\varinjlim C_n$, we get exactly the first three terms of the exact sequence defining the first derived functor of inverse limit of the system $(H_n \mid n < \omega)$, where $H_n = \operatorname{Hom}_R(C_n, M)$:

$$0 \to \varprojlim H_n \to \prod H_n \xrightarrow{\Delta} \prod H_n \to \varprojlim{}^1 H_n \to 0$$

Since $\operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$, the map $\Delta = \operatorname{Hom}_R(\phi, M)$ is surjective. Hence $\varprojlim^1 H_n = 0$. □

Corollary 4.2. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of finitely generated modules. Let M be a module such that $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = 0$. Then the inverse system $(\operatorname{Hom}_R(C_n, M), \operatorname{Hom}_R(f_n, M))_{n<\omega}$ is Mittag-Leffler.*

Proof. This follows either immediately from Lemma 2.5 for $\mathcal{G} = \{N \mid N \cong M^{(\omega)}\}$, or from Proposition 1.4. Note that in both cases we use the fact that all modules C_n are finitely generated. $\square$

The following lemma gives us information about a syzygy of a countable direct limit of finitely presented modules and it will be useful for computation.

Lemma 4.3. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of finitely presented modules. Then there exists a countable direct system*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D_2 & \xrightarrow{i_2} & P_2 & \xrightarrow{p_2} & C_2 & \longrightarrow & 0 \\
 & & {\scriptstyle g_1}\uparrow & & {\scriptstyle s_1}\uparrow & & {\scriptstyle f_1}\uparrow & & \\
0 & \longrightarrow & D_1 & \xrightarrow{i_1} & P_1 & \xrightarrow{p_1} & C_1 & \longrightarrow & 0 \\
 & & {\scriptstyle g_0}\uparrow & & {\scriptstyle s_0}\uparrow & & {\scriptstyle f_0}\uparrow & & \\
0 & \longrightarrow & D_0 & \xrightarrow{i_0} & P_0 & \xrightarrow{p_0} & C_0 & \longrightarrow & 0
\end{array}$$

of short exact sequences of finitely presented modules such that P_n is projective and s_n is split mono for each $n < \omega$. In particular, $\varinjlim P_n$ is projective.

Proof. We will construct the short exact sequences by induction on n. For $n = 0$, let $0 \to D_0 \overset{i_0}{\to} P_0 \overset{p_0}{\to} C_0 \to 0$ be a short exact sequence with P_0 projective finitely generated. Then D_0 is finitely generated, hence finitely presented since we are working over a right coherent ring. If $0 \to D_n \overset{i_n}{\to} P_n \overset{p_n}{\to} C_n \to 0$ has already been constructed, let $q : Q \to C_{n+1}$ be an epimorphism such that Q is a finitely generated projective module. Now define $P_{n+1} = P_n \oplus Q$, $s_n : P_n \to P_{n+1}$ as the canonical inclusion, and $p_{n+1} = (f_n p_n, q)$. Then $D_{n+1} = \operatorname{Ker} p_{n+1}$ is finitely presented and g_n is determined by the commutative diagram above. The last assertion is clear. $\square$

Next, we will need a generalized version of Auslander's well-known lemma. It says that $\operatorname{Ext}^1_R(\varinjlim C_i, M) \cong \varprojlim \operatorname{Ext}^1_R(C_i, M)$ whenever M is a pure-injective module. Note that for a countable direct system $(C_n, f_n)_{n<\omega}$, the fact that M is pure-injective implies that $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$. To see this, we will again use the fact that after applying $\operatorname{Hom}_R(-, M)$ on the canonical pure-exact sequence

$$0 \to \bigoplus C_i \overset{\phi}{\to} \bigoplus C_i \to \varinjlim C_i \to 0, \qquad (\dagger)$$

we get first three terms of the exact sequence

$$0 \to \varprojlim H_n \to \prod H_n \xrightarrow{\Delta} \prod H_n \to \varprojlim{}^1 H_n \to 0$$

where $H_n = \mathrm{Hom}_R(C_n, M)$. But if M is pure-injective, then applying $\mathrm{Hom}_R(-, M)$ on (†) yields an exact sequence and consequently $\varprojlim^1 \mathrm{Hom}_R(C_i, M) = 0$. It turns out that the latter condition is sufficient for $\mathrm{Ext}^1_R(-, M)$ to turn a direct limit into an inverse limit over a right coherent ring:

Lemma 4.4. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system and let M be a module such that $\varprojlim^1 \mathrm{Hom}_R(C_i, M) = 0$. Then $\mathrm{Ext}^1_R(\varinjlim C_i, M) \cong \varprojlim \mathrm{Ext}^1_R(C_i, M)$.*

Proof. Consider the direct system of short exact sequences $0 \to D_n \xrightarrow{i_n} P_n \xrightarrow{p_n} C_n \to 0$ given by Lemma 4.3. After applying $\mathrm{Hom}_R(-, M)$, we get an inverse system of exact sequences

$$0 \to \mathrm{Hom}_R(C_n, M) \xrightarrow{p_n^*} \mathrm{Hom}_R(P_n, M) \xrightarrow{i_n^*}$$
$$\xrightarrow{i_n^*} \mathrm{Hom}_R(D_n, M) \xrightarrow{\delta_n} \mathrm{Ext}^1_R(C_n, M) \to 0.$$

By assumption, the following short sequence is exact:

$$0 \to \varprojlim \mathrm{Hom}_R(C_n, M) \to \varprojlim \mathrm{Hom}_R(P_n, M) \to \varprojlim \mathrm{Im}\, i_n^* \to 0.$$

On the other hand, it follows from Proposition 1.4 that $\varprojlim^1 \mathrm{Hom}_R(P_n, M) = 0$ since $(\mathrm{Hom}_R(P_n, M), \mathrm{Hom}_R(s_n, M))_{n<\omega}$ is a countable inverse system with all the maps (split) epic. Moreover, $\varprojlim^1 \mathrm{Im}\, i_n^* = 0$ since $\varprojlim^1$ is right exact on countable inverse systems. Hence, the following sequence is also exact:

$$0 \to \varprojlim \mathrm{Im}\, i_n^* \to \varprojlim \mathrm{Hom}_R(D_n, M) \to \varprojlim \mathrm{Ext}^1_R(C_n, M) \to 0.$$

Putting everything together, we have obtained the following diagram with canonical maps and exact rows:

$$\begin{array}{ccccccc} \varprojlim \mathrm{Hom}_R(P_n, M) & \longrightarrow & \varprojlim \mathrm{Hom}_R(D_n, M) & \longrightarrow & \varprojlim \mathrm{Ext}^1_R(C_n, M) & \longrightarrow & 0 \\ \cong \uparrow & & \cong \uparrow & & & & \\ \mathrm{Hom}(\varinjlim P_n, M) & \longrightarrow & \mathrm{Hom}(\varinjlim D_n, M) & \longrightarrow & \mathrm{Ext}^1_R(\varinjlim C_n, M) & \longrightarrow & 0 \end{array}$$

It follows that $\mathrm{Ext}^1_R(\varinjlim C_n, M) \cong \varprojlim \mathrm{Ext}^1_R(C_n, M)$. □

Now, we will focus on T-nilpotent inverse systems. It is clear that every T-nilpotent countable inverse system is Mittag-Leffler. It turns out that the converse is true precisely when the inverse limit of the system vanishes. This is made precise by the following lemma:

Lemma 4.5. *Let $(H_n, h_n)_{n<\omega}$ be a countable inverse system of abelian groups. Then the following are equivalent:*

(1) *$(H_n, h_n)_{n<\omega}$ is T-nilpotent,*

(2) $(H_n, h_n)_{n<\omega}$ *is Mittag-Leffler and* $\varprojlim H_n = 0$.

Proof. (1) $\implies$ (2) follows easily from the definitions. Let us prove (2) $\implies$ (1). For each $m < \omega$, let $s(m) > m$ be minimal such that the chain

$$H_m \supseteq h_m(H_{m+1}) \supseteq \cdots \supseteq h_m h_{m+1} \cdots h_{n-1}(H_n) \supseteq \cdots$$

is constant for $n \geq s(m)$ and let $\rho_m : \varprojlim H_n \to H_m$ be the limit map for each m. It follows easily that $s(m) \leq s(m')$ for $m < m'$. We will prove by induction that $\operatorname{Im} \rho_m = \operatorname{Im} h_m h_{m+1} \cdots h_{s(m)-1}$. Together with the assumption that $\varprojlim H_n = 0$, this will imply the T-nilpotency. Let us fix $x_m \in \operatorname{Im} h_m h_{m+1} \cdots h_{s(m)-1}$. All we need to do is to construct by induction a sequence of elements $(x_n)_{m<n<\omega}$ such that $x_n \in \operatorname{Im} h_n h_{n+1} \cdots h_{s(n)-1} \subseteq H_n$ and $x_{n-1} = h_{n-1}(x_n)$ for each $n > m$. Suppose we have already constructed x_{n-1} for some n. Then, by the chain condition, there is $y \in H_{s(n)}$ such that $h_{n-1} h_n \cdots h_{s(n)-1}(y) = x_{n-1}$. We can put $x_n = h_n \cdots h_{s(n)-1}(y)$. □

We are in a position now to give a connection between vanishing of Ext^i_R and the chain conditions mentioned above (the Mittag-Leffler condition and T-nilpotency). We state the connection in the following key lemma:

Lemma 4.6. *Let* $(C_n, f_n)_{n<\omega}$ *be a countable direct system of finitely presented modules and let* M *be an arbitrary module. Consider the following conditions:*

(1) $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim C_n, M^{(\omega)}) = 0$.

(2) *The inverse system* $(\operatorname{Hom}_R(C_n, M), \operatorname{Hom}_R(f_n, M))_{n<\omega}$ *is Mittag-Leffler and* $(\operatorname{Ext}^1_R(C_n, M), \operatorname{Ext}^1_R(f_n, M))_{n<\omega}$ *is T-nilpotent.*

(3) $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = 0$.

Then (1) implies (2) and (2) implies (3).

Proof. (1) $\implies$ (2). Assume $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim C_n, M^{(\omega)}) = 0$. Then the inverse system $(\operatorname{Hom}_R(C_n, M), \operatorname{Hom}_R(f_n, M))_{n<\omega}$ is Mittag-Leffler by Corollary 4.2. By Proposition 1.4 we have $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$, and subsequently it follows by Lemma 4.4 that

$$\varprojlim \operatorname{Ext}^1_R(C_n, M) \cong \operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$$

Next, let $0 \to D_n \to P_n \to C_n \to 0$ be the countable direct system given by Lemma 4.3. Since

$$\operatorname{Ext}^1_R(\varinjlim D_n, M^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim C_n, M^{(\omega)}) = 0$$

by dimension shifting, the inverse system $(\operatorname{Hom}_R(D_n, M))_{n<\omega}$ is also Mittag-Leffler by Corollary 4.2. Then $(\operatorname{Ext}^1_R(C_n, M))_{n<\omega}$ is Mittag-Leffler as well, since an epimorfic image of a Mittag-Leffler inverse system is Mittag-Leffler again, [20, Proposition 13.2.1]. Thus, $(\operatorname{Ext}^1_R(C_n, M))_{n<\omega}$ is T-nilpotent by Lemma 4.5.

(2) $\implies$ (3). Clearly, condition (2) implies that $(\mathrm{Hom}_R(C_n, M^{(\omega)}))_{n<\omega}$ is Mittag-Leffler and $(\mathrm{Ext}^1_R(C_n, M^{(\omega)}))_{n<\omega}$ is T-nilpotent. Hence

$$\mathrm{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = \varprojlim \mathrm{Ext}^1_R(C_n, M^{(\omega)}) = 0$$

by Lemmas 4.4 and 4.5. □

With the previous lemma in mind, a natural question arises when $\mathrm{Ext}^1_R(f, M)$ is a zero map for a homomorphism $f : X \to Y$ between finitely presented modules. It is possible to characterize such maps f when $\mathrm{Ext}^1_R(f, M) = 0$ as M runs over all modules in the right-hand class of a complete cotorsion pair. We state this precisely in Lemma 4.7. In view of [30], the lemma can be viewed as a module-theoretic counterpart of [29, Lemmas 3.4 (3) and 3.8].

Lemma 4.7. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in* Mod-R *and let $f : X \to Y$ be a homomorphism between R-modules. Then the following are equivalent:*

(1) $\mathrm{Ext}^1_R(f, B) = 0$ *for every $B \in \mathcal{B}$,*
(2) *f factors through some module in $\mathcal{A}$.*

Proof. (1) $\implies$ (2). Let $0 \to B \to A \to Y \to 0$ be a special $\mathcal{A}$-precover of Y and consider the following pull-back diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

Then the upper row splits by assumption and f factors through A.

(2) $\implies$ (1). This is easy, since the assumption that f factors through some $A \in \mathcal{A}$ implies that $\mathrm{Ext}^1_R(f, B)$ factors through $\mathrm{Ext}^1_R(A, B) = 0$ for each $B \in \mathcal{B}$. □

Now, we can characterize countably presented modules in the left-hand class of a cotorsion pair satisfying the hypotheses of TCMC. Actually, we state the theorem more generally, for cotorsion pairs satisfying somewhat weaker conditions. Recall that by Theorem 3.5, every cotorsion pair satisfying the hypotheses of TCMC is complete.

Theorem 4.8. *Let R be a right coherent ring and $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair with $\mathcal{B}$ closed under (countable) direct sums. Denote by $\mathfrak{I}$ the ideal of all morphisms in* mod-R *which factor through some module from $\mathcal{A}$. Then the following are equivalent for a countably presented module M:*

(1) $M \in \mathcal{A}$,
(2) *M is a direct limit of a countable system $(C_n, f_n)_{n<\omega}$ of finitely presented modules such that $f_n \in \mathfrak{I}$ for every n and $(\mathrm{Hom}_R(C_n, B), \mathrm{Hom}_R(f_n, B))_{n<\omega}$ is Mittag-Leffler for each $B \in \mathcal{B}$.*

If, in addition, $\mathcal{A}$ is closed under (countable) direct limits, then these conditions are further equivalent to:

(3) *M is a direct limit of a countable system $(C_n, f_n)_{n<\omega}$ of finitely presented modules such that $f_n \in \mathfrak{I}$ for every n.*

Proof. (1) $\Longrightarrow$ (2). Let us fix (any) countable system $(D_n, g_n)_{n<\omega}$ of finitely presented modules such that $M = \varinjlim D_n$. Assume $M \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $B^{(\omega)} \in \mathcal{B}$ and $\operatorname{Ext}^1_R(\varinjlim D_n, B^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim D_n, B^{(\omega)}) = 0$ by assumption. So the inverse system $(\operatorname{Hom}_R(D_n, B), \operatorname{Hom}_R(g_n, B))_{n<\omega}$ is Mittag-Leffler and the system $(\operatorname{Ext}^1_R(D_n, B), \operatorname{Ext}^1_R(g_n, B))_{n<\omega}$ is T-nilpotent for each $B \in \mathcal{B}$ by Lemma 4.6.

Now, we will by induction construct a strictly increasing sequence $n_0 < n_1 < \cdots$ of natural numbers such that the compositions

$$f_i = g_{n_{i+1}-1} \dots g_{n_i+1} g_{n_i} : D_{n_i} \to D_{n_{i+1}}$$

satisfy $\operatorname{Ext}^1_R(f_i, B) = 0$ for each $i < \omega$ and $B \in \mathcal{B}$. Let us start with $n_0 = 0$. For the inductive step, assume that n_i has already been constructed. If there is some $l > n_i$ such that $\operatorname{Ext}^1_R(g_{l-1} \dots g_{n_i+1} g_{n_i}, B) = 0$ for each $B \in \mathcal{B}$, we are done since we can put $n_{i+1} = l$. If this was not the case, there would be some $B_l \in \mathcal{B}$ for each $l > n_i$ such that $\operatorname{Ext}^1_R(g_{l-1} \dots g_{n_i+1} g_{n_i}, B_l) \neq 0$. But this would imply that $(\operatorname{Ext}^1_R(D_n, \bigoplus_{l>n_i} B_l))_{n<\omega}$ is not T-nilpotent, a contradiction.

Finally, we can just put $C_i = D_{n_i}$ and observe using Lemma 4.7 that $f_i \in \mathfrak{I}$ for each $i < \omega$.

(2) $\Longrightarrow$ (1). This follows directly from Lemma 4.6, since the inverse system $(\operatorname{Ext}^1_R(C_n, B), \operatorname{Ext}^1_R(f_n, B))_{n<\omega}$ is clearly T-nilpotent for each $B \in \mathcal{B}$ (see Lemma 4.7).

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). For each n, write f_n as a composition of the form $C_n \overset{u_n}{\to} A_n \overset{v_n}{\to} C_{n+1}$ with $A_n \in \mathcal{A}$. In this way, we get a direct system

$$C_0 \overset{u_0}{\to} A_0 \overset{v_0}{\to} C_1 \overset{u_1}{\to} A_1 \overset{v_1}{\to} C_2 \overset{u_2}{\to} \cdots .$$

Now, $\varinjlim_{n<\omega} C_n = \varinjlim_{n<\omega} A_n$. Hence $M \in \mathcal{A}$ since $\mathcal{A}$ is closed under countable direct limits. □

The preceding theorem allows us to characterize modules in the right-hand class of a cotorsion pair satisfying the assumptions of TCMC. Again, we state the following theorem for more general cotorsion pairs than those in question for TCMC. Note that for projective cotorsion pairs over self-injective artin algebras, the following statement is a consequence of [30, Corollary 7.7] and [29, Theorem A].

Theorem 4.9. *Let R be a right coherent ring and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *with $\mathcal{B}$ closed under unions of well-ordered chains. Denote by $\mathfrak{I}$ the ideal of all morphisms in* mod-R *which factor through some module from $\mathcal{A}$. Then the following are equivalent:*

(1) $B \in \mathcal{B}$,
(2) $\operatorname{Ext}^1_R(f, B) = 0$ *for each* $f \in \mathfrak{I}$.

Proof. (1) $\implies$ (2). This is clear, since in this case, for each $f \in \mathfrak{I}$, the map $\operatorname{Ext}^1_R(f, B)$ factors through $\operatorname{Ext}^1_R(A, B) = 0$ for some $A \in \mathcal{A}$.

(2) $\implies$ (1). Recall that the cotorsion pair is of countable type and complete by Theorem 3.5. Moreover, every countably presented module in $\mathcal{A}$ can be expressed as a direct limit of a direct system $(C_n, f_n)_{n<\omega}$ with all the morphisms f_n in $\mathfrak{I}$ by Theorem 4.8.

Let us define a class of modules $\mathcal{C}$ as

$$\mathcal{C} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(f, M) = 0 \text{ for each } f \in \mathfrak{I}\}$$

By definition $\mathcal{B} \subseteq \mathcal{C}$.

Note that since every $f \in \mathfrak{I}$ is a morphism between strongly finitely presented modules, say $f : X \to Y$, and it is not difficult to see that the functors $\operatorname{Ext}^1_R(X, -)$ and $\operatorname{Ext}^1_R(Y, -)$ are coherent in this case, so is the functor $F_f = \operatorname{Im} \operatorname{Ext}^1_R(f, -)$. Hence $\mathcal{C}$ is a definable class as it is defined by vanishing of the functors F_f where f runs through a representative set of morphisms from $\mathfrak{I}$. In particular, this means that showing $\mathcal{C} \subseteq \mathcal{B}$ reduces just to showing that every *pure-injective* module $M \in \mathcal{C}$ is already in $\mathcal{B}$, since definable classes are determined by the pure-injective modules they contain.

To this end, assume that $M \in \mathcal{C}$ is pure-injective and $A \in \mathcal{A}$ is countably presented. Then $A = \varinjlim C_n$ where $(C_n, f_n)_{n<\omega}$ is a direct system such that $f_n \in \mathfrak{I}$ for each n. In particular, $\operatorname{Ext}^1_R(f_n, M) = 0$ by assumption and

$$\operatorname{Ext}^1_R(A, M) = \operatorname{Ext}^1_R(\varinjlim C_n, M) \cong \varprojlim \operatorname{Ext}^1_R(C_n, M) = 0$$

by Auslander's lemma. Finally, since $(\mathcal{A}, \mathcal{B})$ is of countable type and A was arbitrary, it follows that $M \in \mathcal{B}$. $\square$

Remark. (i) Countable type of the cotorsion pair considered in Theorem 4.9 together with Lemma 3.4 imply that when defining $\mathfrak{I}$, we may assume that the modules from $\mathcal{A}$ through which the maps $f \in \mathfrak{I}$ are required to factorize are all countably presented.

(ii) To determine which implication of pp-formulas corresponds to the coherent functor F_f from the proof of Theorem 4.9, we build the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i_X} & F_X & \xrightarrow{p_X} & X & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & L & \xrightarrow{i_Y} & F_Y & \xrightarrow{p_Y} & Y & \longrightarrow & 0
\end{array}$$

with F_X, F_Y finitely generated free, K, L finitely presented, s a split embedding and i, i_X, i_Y inclusions. Now, an equivalent statement to $F_f(M) = 0$ is that every homomorphism from K into M which extends to L must extend to F_X as well, and this can be routinely translated to an implication between two pp-formulas to be satisfied in M. If we denote by H the pushout of i and i_X, and by h the pushout map $L \to H$, then the latter actually means that $\operatorname{Coker}\operatorname{Hom}_R(h, M) = 0$. Thus, $\operatorname{Coker}\operatorname{Hom}_R(h, -)$ is a coherent functor which may be equivalently used instead of F_f when defining $\mathcal{B}$.

5. Direct limits and pure-epimorphic images

In the cases when TCMC holds true, the class $\mathcal{A}$ of any cotorsion pair $(\mathcal{A}, \mathcal{B})$ meeting its assumptions must be closed under pure-epimorphic images. Indeed, in this setting, we have $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod-}R)$ and the latter class is closed under pure-epimorphic images by the well-known result of Lenzing (cf. [32] or [19, Lemma 1.2.9]). In this section, we prove that the hypotheses of TCMC do always imply that $\mathcal{A}$ is closed under pure-epimorphic images. As a consequence, we prove that every complete cotorsion pair with both classes closed under arbitrary direct limits is cogenerated by a single pure-injective module—this can be viewed as a module-theoretic counterpart of [29, Theorem C].

Note that the first part—to make sure that $\mathcal{A}$ is closed under pure-epimorphic images—is the crucial one. For projective cotorsion pairs over self-injective algebras which satisfy the hypotheses of TCMC, this property follows by analysis of the proofs in [29] and [30]. But when proving this in a more general setting, one obstacle appears. Namely, complete cotorsion pairs provide us with approximations (special precovers and preenvelopes) which are not functorial in general. Therefore, implementing the rather simple underlying idea—expressing each module in $\mathcal{A}$ in terms of direct limits of $\mathcal{A}$-precovers of finitely presented modules and proving that this transfers to pure-epimorphic images—requires several technical steps. In particular, we need special indexing sets for our direct systems which we call *inverse trees*.

We start with a preparatory lemma. Recall that for an ordinal number α, we denote by $|\alpha|$ the cardinality of α when viewed as the set of all smaller ordinals.

Definition 5.1. A direct system $(M_i, f_{ji} \mid i, j \in I \ \& \ i \leq j)$ of R-modules is said to be *continuous* if $(M_k, f_{kj} \mid j \in J)$ is the direct limit of the system $(M_i, f_{ji} \mid i, j \in J \ \& \ i \leq j)$ whenever J is a directed subposet of I and k is a supremum of J in I.

Lemma 5.2. *Let κ be an infinite cardinal and M be a κ-presented module. Then M is a direct limit of a continuous well-ordered system $(M_\alpha, f_{\beta\alpha} \mid \alpha \leq \beta < \kappa)$ such that for all $\alpha < \kappa$, M_α is $|\alpha|$-presented.*

Proof. We can start as in Lemma 1.1. Let

$$\bigoplus_{\beta<\kappa} x_\beta R \xrightarrow{g} \bigoplus_{\gamma<\kappa} y_\gamma R \to M \to 0$$

be a free presentation of M. For each $\alpha < \kappa$, let X_α be the subset of all ordinals $\beta < \alpha$ such that $f(x_\beta) \in \bigoplus_{\gamma<\alpha} y_\gamma R$. If we define M_α as the cokernel of the restriction $\bigoplus_{\beta\in X_\alpha} x_\beta R \to \bigoplus_{\gamma<\alpha} y_\gamma R$ of g, it is easy to see that the direct system $(M_\alpha \mid \alpha < \kappa)$ together with the natural maps has the properties we require. □

For a set X, we will denote by X^* the set of all finite strings over X, that is, all functions $u : n \to X$ for $n < \omega$. We will denote strings by letters $u, v, w, \ldots$ and write them as sequences of elements of X, which we will denote by Greek letters for a reason which will be clear soon. For example, we write $u = \alpha_0\alpha_1 \ldots \alpha_{n-1}$. When u, v are strings, we denote by uv their *concatenation*, we define the *length* of a string u in the usual way and denote it by $\ell(u)$, and we identify strings of length 1 with elements in X. The empty string is denoted by $\varnothing$. Note that the set X^* together with the concatenation operation is nothing else than the free monoid over X.

Definition 5.3. Let κ be an infinite cardinal and κ^* be the free monoid over κ. Let us equip $\kappa^* \setminus \{\varnothing\}$ with a partial order in the following way: If $u = \alpha_0\alpha_1 \ldots \alpha_{n-1}$ and $v = \beta_0\beta_1 \ldots \beta_{m-1}$, we put $u \le v$ if

(1) $n \ge m$,
(2) $\alpha_0\alpha_1 \ldots \alpha_{m-2} = \beta_0\beta_1 \ldots \beta_{m-2}$, and
(3) $\alpha_{m-1} \le \beta_{m-1}$ as ordinal numbers.

Then an *inverse tree* over κ is the subposet of $(\kappa^* \setminus \{\varnothing\}, \le)$ defined as

$$I_\kappa = \big\{\alpha_0\alpha_1 \ldots \alpha_{n-1} \mid \\ \big(\forall i \le n-2\big)\big(\alpha_i \text{ is infinite, non-limit \& } \alpha_{i+1} < |\alpha_i|\big)\big\}.$$

For convenience, given a non-empty string $u = \alpha_0\alpha_1 \ldots \alpha_{n-1} \in \kappa^*$, we define the *tail* of u, denoted by $t(u)$, to be the last symbol α_{n-1} of u, and the *rank* of u, $\mathrm{rk}(u)$, to be the cardinal number $|\alpha_{n-1}|$. Notice that in this terminology, the tail of a string $u \in I_\kappa$ is allowed to be a limit or finite ordinal.

Having defined inverse trees, we can start collecting basic properties of the partial ordering:

Lemma 5.4. *Let $(I_\kappa, \le)$ be an inverse tree, and let v and $u = \beta_0 \ldots \beta_{m-2}\beta_{m-1}$ be two elements of I_κ such that $v < u$. Then there is $w \in I_\kappa$ such that $v \le w < u$ and one of the following cases holds true:*

(1) *There is an ordinal $\gamma < \beta_{m-1}$ such that $w = \beta_0\beta_1 \ldots \beta_{m-2}\gamma$.*
(2) *There is an ordinal $\gamma < |\beta_{m-1}|$ such that $w = \beta_0\beta_1 \ldots \beta_{m-2}\beta_{m-1}\gamma$.*

Proof. This follows easily from the definition. Notice that (2) can only hold if $\beta_{m-1} = t(u)$ is infinite and non-limit. □

As an immediate corollary, we will see that the properties of $u \in I_\kappa$ with respect to the ordering depend very much on the tail (and rank) of u:

Corollary 5.5. *Let $u = \alpha_0 \dots \alpha_{n-2}\alpha_{n-1} \in I_\kappa$. Then the following hold in $(I_\kappa, \leq)$:*

(1) *If $t(u) = 0$, then u is a minimal element.*
(2) *If $t(u)$ is non-zero finite, then u has a unique immediate predecessor.*
(3) *If $t(u)$ is an infinite non-limit ordinal, then $u = \sup\{u\gamma \mid \gamma < \mathrm{rk}(u)\}$.*
(4) *If $t(u)$ is a limit ordinal, then $u = \sup\{\alpha_0 \dots \alpha_{n-2}\gamma \mid \gamma < t(u)\}$.*

We have seen that an element $u \in I_\kappa$ can be expressed as a supremum of a chain of strictly smaller elements if and only if $\mathrm{rk}(u)$ is infinite. If so, this chain depends on whether $t(u)$ is a limit ordinal or not. We will prove in the next lemma that as far as we are concerned with continuous direct systems indexed with I_κ, this expression of u as a supremum is essentially unique.

Lemma 5.6. *Let $u \in I_\kappa$ be of infinite rank and C be the chain as in Corollary 5.5* (3) *or* (4) *such that $u = \sup C$ in I_κ. Let $J \subseteq I_\kappa$ be a directed subposet of I_κ such that $u = \sup J$ in I_κ and $u \notin J$. Then $C \cap J$ is cofinal in J.*

Proof. Choose some $j \in J$ of the least possible length. Since J is directed, u is the supremum of the upper set $\uparrow j = \{i \in J \mid i \geq j\}$, too. By the definition of the ordering and the fact that j has been taken of the least possible length, we see that each $i \in (\uparrow j)$ is of the form $\beta_0\beta_1 \dots \beta_{m-2}\gamma_i$ where $\beta_0, \beta_1, \dots, \beta_{m-2}$ are fixed and $\gamma_i < |\beta_{m-2}|$. Thus $u = \beta_0\beta_1 \dots \beta_{m-2}$ provided that $\sup\{\gamma_i \mid i \in (\uparrow j)\} = |\beta_{m-2}|$ (case (3)), and $u = \beta_0\beta_1 \dots \beta_{m-2}\beta_{m-1}$ if $\beta_{m-1} = \sup\{\gamma_i \mid i \in (\uparrow j)\} < |\beta_{m-2}|$ (case (4)). Hence, $\uparrow j \subseteq C \cap J$ by assumption, and $C \cap J$ is cofinal in J since $\uparrow j$ is. □

So far, we have studied elements strictly smaller than a given $u \in I_\kappa$. But, we will also need to look "upwards":

Lemma 5.7. *Let $(I_\kappa, \leq)$ be an inverse tree. Then*

(1) *For each $u \in I_\kappa$, the upper set $\uparrow u = \{w \in I_\kappa \mid w \geq u\}$ is well-ordered.*
(2) *$(I_\kappa, \leq)$ is directed.*
(3) *Every non-empty bounded subset $X \subseteq I_\kappa$ has a supremum in I_κ.*

Proof. (1). It follows from the definition that $\uparrow u$ is a totally ordered subset of I_κ. If $X \subseteq (\uparrow u)$ is nonempty, then the longest string $u \in X$ with the minimum tail $t(u)$ is the least element in X. Hence, $\uparrow u$ is well-ordered.

(2). Let $u = \alpha_1 \dots \alpha_{n-1}$, $v = \beta_1 \dots \beta_{m-1}$ be elements in I_κ. Then $\max\{\alpha_1, \beta_1\}$, viewed as a string of length 1, is greater than both u and v.

(3). Suppose $X \subseteq I_\kappa$ is non-empty and has an upper bound $u \in I_\kappa$. In other words, $u \in Y$ for $Y = \bigcap_{w \in X} (\uparrow w)$. But since for any $v \in X$ clearly $Y \subseteq (\uparrow v)$, there must be the least element in Y, which is by definition the supremum of X. □

In view of the preceding lemma, we can introduce the following definition:

Definition 5.8. Let $(I_\kappa, \leq)$ be an inverse tree and $u = \alpha_0 \dots \alpha_{n-2}\alpha_{n-1} \in I_\kappa$. Then the *successor* of u in I_κ is defined as $\mathrm{s}(u) = \alpha_0 \dots \alpha_{n-2}\beta$ where $\beta = \alpha + 1$ is the ordinal successor of α. Similarly, if $t(u) = \alpha_{n-1}$ is non-limit and non-zero, we define the *predecessor* of u as $\mathrm{p}(u) = \alpha_0 \dots \alpha_{n-2}\gamma$ where $\gamma = \alpha - 1$ is the ordinal predecessor of α.

Note that by Lemma 5.7, $\mathrm{s}(u)$ is the unique immediate successor of u in $(I_\kappa, \leq)$. On the other hand, even if $\mathrm{p}(u)$ is defined, there still may be other elements in I_κ less than u that are incomparable with $\mathrm{p}(u)$—see Lemma 5.4. We can summarize our observations in a figure showing "neighbourhoods" of elements $u \in I_\kappa$ depending on $t(u)$, where $w \in \kappa^*$ is the string obtained from u by removing its last symbol:

$t(u)$ infinite and non-limit	$t(u)$ limit
$\mathrm{p}(u) \longrightarrow u \longrightarrow \mathrm{s}(u)$ $u\gamma \longrightarrow u(\gamma+1) \cdots\!\!> u$	$w\gamma \longrightarrow w(\gamma+1) \cdots\!\!> u \longrightarrow \mathrm{s}(u)$

This picture also shows the motivation for calling $(I_\kappa, \leq)$ an inverse tree. From each $u \in I_\kappa$, there is exactly one possible way towards greater elements, while when traveling in I_κ down the ordering, there are many branches. The rank zero elements of I_κ can be viewed as leaves. Just the root is missing—it is easy to see that I_κ has no maximal element.

Next, we will turn our attention back to modules. We shall see that each infinitely presented module is the direct limit of a special direct system indexed by an inverse tree.

Lemma 5.9. *Let κ be an infinite cardinal and M be a κ-presented module. Then M is the direct limit of a continuous direct system $(M_u, f_{vu} \mid u, v \in I_\kappa \;\&\; u \leq v)$ indexed by the inverse tree I_κ and such that M_u is $\mathrm{rk}(u)$-presented for each $u \in I_\kappa$.*

Proof. We will construct the direct system by induction on $\ell(u)$ using Lemma 5.2. If $\ell(u) = 1$, then u can be viewed as an ordinal number $< \kappa$ and we just use the modules M_u and morphisms f_{vu} obtained for M by Lemma 5.2.

Suppose we have defined M_u and f_{vu} for all $u, v \in I_\kappa$ with $\ell(u), \ell(v) \leq n$. Let $v \in I_\kappa$ be arbitrary with $\ell(v) = n$ and such that $t(v)$ is infinite and non-limit. Then by using Lemma 5.2 for M_v, we obtain a well-ordered continuous system $(M_\alpha^v, f_{\beta\alpha}^v \mid \alpha \leq \beta < \mathrm{rk}(v))$, and we set $M_{v\alpha} = M_\alpha^v$ and $f_{v\beta,v\alpha} = f_{\beta\alpha}^v$ for all $\alpha \leq \beta < \mathrm{rk}(v)$. Finally, the morphisms $f_{v,v\alpha}$, $\alpha < \mathrm{rk}(v)$, will be defined as the colimit maps $M_\alpha^v \to M_v$, and the rest of the morphisms $f_{u,v\alpha}$ just by taking the appropriate compositions.

The correctness of this construction is ensured by the properties of I_κ proved above, and the fact that $(M_u \mid u \in I_\kappa)$ is continuous is taken care of by Lemma 5.6. □

The crucial fact about inverse trees is that, under the assumptions of TCMC, they allow us to construct for each module a continuous direct system of special precovers:

Lemma 5.10. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with both classes closed under direct limits, κ be an infinite cardinal, and M be a κ-presented module. Then there is a continuous direct system of short exact sequences $0 \to B_u \overset{\iota_u}{\to} A_u \overset{\pi_u}{\to} M_u \to 0$ indexed by I_κ such that $B_u \in \mathcal{B}$, $A_u \in \mathcal{A}$, M_u is $\mathrm{rk}(u)$-presented for each $u \in I_\kappa$, and M is the direct limit of the modules M_u.*

Proof. We start with the continuous direct system $(M_u, f_{vu} \mid u, v \in I_\kappa \;\&\; u \leq v)$ given by Lemma 5.9 and construct the exact sequences for each $u \in I_\kappa$ by transfinite induction on $t(u)$.

For each $u \in I_\kappa$ of finite rank, we choose a special $\mathcal{A}$-precover,

$$0 \to B_u \overset{\iota_u}{\to} A_u \overset{\pi_u}{\to} M_u \to 0,$$

of M_u, and if $t(u) > 0$, we find appropriate morphisms $g_{u\mathrm{p}(u)} : A_{\mathrm{p}(u)} \to A_u$ and $h_{u\mathrm{p}(u)} : B_{\mathrm{p}(u)} \to B_u$ using the precover property for the map $f_{u\mathrm{p}(u)} \circ \pi_{\mathrm{p}(u)}$.

Suppose that α is a limit ordinal and the sequences $0 \to B_u \overset{\iota_u}{\to} A_u \overset{\pi_u}{\to} M_u \to 0$ and the maps between them have been constructed for all $u \in I_\kappa$ with $t(u) < \alpha$. Then for each $v \in I_\kappa$ with $t(v) = \alpha$, we define the exact sequence $0 \to B_v \overset{\iota_v}{\to} A_v \overset{\pi_v}{\to} M_v \to 0$ as the direct limit of the direct system of already constructed short exact sequences $0 \to B_w \overset{\iota_w}{\to} A_w \overset{\pi_w}{\to} M_w \to 0$ where w runs over the chain given by Corollary 5.5 (4) used for v. By assumption, we get $A_v \in \mathcal{A}$ and $B_v \in \mathcal{B}$.

Finally, suppose that $\alpha = \delta + 1$ for some infinite δ and we have constructed the exact sequences for all $u \in I_\kappa$ such that $t(u) \leq \delta$. Similarly as above, we define for each $v \in I_\kappa$ with $t(v) = \alpha$ the exact sequence $0 \to B_v \overset{\iota_v}{\to} A_v \overset{\pi_v}{\to} M_v \to 0$ as the direct limit of the direct system of short exact sequences $0 \to B_{v\beta} \overset{\iota_{v\beta}}{\to} A_{v\beta} \overset{\pi_{v\beta}}{\to} M_{v\beta} \to 0$ where β runs over all ordinal numbers $< \mathrm{rk}(v)$. The morphisms $g_{v\mathrm{p}(v)} : A_{\mathrm{p}(v)} \to A_v$ and

$h_{v\mathrm{p}(v)} : B_{\mathrm{p}(v)} \to B_v$ can be defined again by the precover property and the rest of the morphisms by obvious compositions. This concludes the construction.

The fact that the direct system of the exact sequences just constructed is well-defined and continuous follows from the lemmas above, in particular from Lemmas 5.4 and 5.6. □

Before stating one of the main results in this section, let us recall that a cotorsion pair satisfying the assumptions of TCMC is complete by Theorem 3.5 (2), thus it fits the setting of the following theorem.

Theorem 5.11. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with both classes closed under direct limits. Then $\mathcal{A}$ is closed under pure epimorphic images.*

Proof. Let M be a pure epimorphic image of a module from $\mathcal{A}$. We can assume that M is not finitely presented since otherwise M is trivially in $\mathcal{A}$. Hence, Lemma 5.10 gives us a continuous direct system $0 \to B_u \xrightarrow{\iota_u} A_u \xrightarrow{\pi_u} M_u \to 0$ indexed by I_κ for some κ, and the direct limit $0 \to B \xrightarrow{\iota} A \xrightarrow{\pi} M \to 0$ of this system is a special $\mathcal{A}$-precover of M. It follows from our assumption on M that π is a pure epimorphism.

Now, M is also the direct limit of some direct system $(K_i, k_{ji} \mid i \preceq j)$ consisting of finitely presented modules and indexed by some poset $(J, \preceq)$. We claim that although there is no obvious relation between the direct systems $(M_u \mid u \in I_\kappa)$ and $(K_i \mid i \in J)$, the following holds: For each $i \in J$, there is $s(i) \in J$ such that $i \prec s(i)$ and $k_{s(i)i}$ factors through A_u for some $u \in I_\kappa$ of finite rank.

To this end, denote for all $i \in J$ by $k_i : K_i \to M$ the colimit maps and fix an arbitrary $i \in J$. Then k_i can be factorized through π since K_i is finitely presented and π is pure. Moreover, since $A = \varinjlim_{I_\kappa} A_u$, there is $u_1 \in I_\kappa$ such that k_i factors through A_{u_1}. If $\mathrm{rk}(u_1)$ is finite, we put $u = u_1$. If not, A_{u_1} is by Corollary 5.5 the direct limit of a direct system consisting of some modules A_v with $t(v) < t(u_1)$. Hence, k_i further factors through A_{u_2} for some $u_2 \in I_\kappa$ such that $t(u_2) < t(u_1)$. If the rank of u_2 is finite, we put $u = u_2$. Otherwise, we construct in a similar way u_3 such that $t(u_3) < t(u_2)$, and so forth. Since there are no infinite descending sequences of ordinals, we must arrive at some $u = u_n$ of finite rank after finitely many steps.

Hence, there must be $u_i \in I_\kappa$ of finite rank such that k_i factors through $\pi \circ g_{u_i} = f_{u_i} \circ \pi_{u_i}$ where $g_{u_i} : A_{u_i} \to A$ and $f_{u_i} : M_{u_i} \to M$ are the colimit maps. That is, $k_i = f_{u_i} \circ \pi_{u_i} \circ e_i$ for some $e_i : K_i \to A_{u_i}$ and, since M_{u_i} is finitely presented by Lemma 5.10, f_{u_i} further factors as $k_{j_i} \circ d_{u_i}$ for some $d_{u_i} : M_{u_i} \to K_{j_i}$ and $j_i \in J$ such that $j_i \succ i$. Together, we have $k_i = k_{j_i} \circ d_{u_i} \circ \pi_{u_i} \circ e_i$. Thus, using the fact that K_i is finitely presented and well-known properties of direct limits, there must exist some $s(i) \succeq j_i$ such that $k_{s(i)i} = k_{s(i)j_i} \circ d_{u_i} \circ \pi_{u_i} \circ e_i$, and the claim is proved.

Now set $\tilde{J} = J \times \{0,1\}$ and define $(\tilde{J}, \preceq)$ as the poset generated by the relations $(i,0) \preceq (j,0)$ and $(i,0) \preceq (i,1) \preceq (s(i),0)$ where $i,j \in J, i \preceq j$. Further, for such i,j, put $K_{(i,0)} = K_i$, $K_{(i,1)} = A_{u_i}$, $k_{(j,0),(i,0)} = k_{ji}$, $k_{(i,1),(i,0)} = e_i$, and $k_{(s(i),0),(i,1)} = k_{s(i)j_i} \circ d_{u_i} \circ \pi_{u_i}$, using the same notation as above. In this way, defining the remaining morphisms as the appropriate compositions, we obtain the system $(K_x, k_{yx} \mid x, y \in \tilde{J}\ \&\ x \preceq y)$ which is easily seen to be direct, it has M as its direct limit, and $(K_{(i,1)} \mid i \in J)$ forms a cofinal subsystem. Therefore, M is a direct limit of this cofinal subsystem, which clearly consists of modules from $\mathcal{A}$.

□

Now, we can prove the crucial statement regarding cogeneration of cotorsion pairs by a single pure-injective module. To this end, we need the following notion from [37, Section 9.4]: A pure-injective module N is said to be an *elementary cogenerator* if every pure-injective direct summand of a module elementarily equivalent to $N^{\aleph_0}$ is a direct summand of some power of N. Further recall that the *dual module* M^d of a module M is defined as $M^d = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. It is a well-known fact that any module M is an elementary submodel in its double dual M^{dd} as well as in any reduced $\mathfrak{F}$-power $M^I/\Sigma_{\mathfrak{F}} M^I$ provided that $\mathfrak{F}$ is an ultrafilter on $\mathfrak{P}(I)$ (cf. Definition 2.1, these reduced powers are called *ultrapowers*).

Proposition 5.12. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with $\mathcal{B}$ closed under direct limits. Then there exists a pure-injective module E such that the class $\mathrm{Ker}\,\mathrm{Ext}^1_R(-,E)$ coincides with the class of all pure-epimorphic images of modules from $\mathcal{A}$. Moreover, E can be taken of the form $\prod_{k \in K} E_k$, with E_k indecomposable for each $k \in K$.*

Proof. First of all, since $\mathcal{B}$ is closed under direct products and direct limits, it is closed under ultrapowers as well. Thence $M \in \mathcal{B}$ implies by Frayne's Theorem that $N \in \mathcal{B}$ provided that N is a pure-injective direct summand of a module elementarily equivalent to M. In particular, $\mathcal{B}$ is closed under taking double dual modules.

If we denote by $(\mathcal{D}, \mathcal{E})$ the cotorsion pair cogenenerated by the class of all pure-injective modules from $\mathcal{B}$, then $\mathcal{D}$ is exactly the class of all pure-epimorphic images of modules from $\mathcal{A}$ (cf. [5, Lemmas 2.1 and 2.2]; here, the completeness of $(\mathcal{A}, \mathcal{B})$ and $\mathcal{B}$ being closed under *double duals* are actually needed).

By [37, Corollary 9.36], for every module M there exists an elementary cogenerator elementarily equivalent to M. Thus, by the first paragraph, we may consider a representative set $\mathcal{S}$ consisting of elementary cogenerators in $\mathcal{B}$ such that any module in $\mathcal{B}$ is elementarily equivalent to a module from $\mathcal{S}$. Now define E to be the direct product of all modules from $\mathcal{S}$. To finish the main part of our proof, it is enough to show that any pure-injective module from $\mathcal{B}$ is in $\mathrm{Prod}(E)$, the class of all direct summands of powers of E. This is sufficient since then the left-hand class of the cotorsion pair cogenerated by $\{E\}$ will coincide with $\mathcal{D}$.

Let, therefore, $M \in \mathcal{B}$ be a pure-injective module and $N \in \mathcal{S}$ be a module elementarily equivalent to M. By [37, Proposition 2.30], M is a pure submodule (hence a direct summand) in a module elementarily equivalent to $N^{\aleph_0}$. Thus M is a direct summand of some power of N by the definition of elementary cogenerator.

To prove the moreover statement, first recall that, by a well-known result of Fischer, $E = PE(\bigoplus_{j\in J} E_j) \oplus F$ where PE stands for pure-injective hull, E_j is indecomposable pure-injective for each $j \in J$, and F has no indecomposable direct summands; it may happen that J is empty or $F = 0$. By [37, Corollary 4.38], F is a direct summand of a direct product, say $\prod_{l\in L} E_l$, of indecomposable pure-injective direct summands of modules elementarily equivalent to E. According to the first paragraph, $E_l \in \mathcal{B}$ for every $l \in L$. It follows that $PE(\bigoplus_{j\in J} E_j) \oplus \prod_{l\in L} E_l$ cogenerates the same cotorsion pair as E does. Further, $PE(\bigoplus_{j\in J} E_j)$ is a direct summand in $\prod_{j\in J} E_j$ and the latter module is in $\mathcal{B}$ since it is elementarily equivalent to $PE(\bigoplus_{j\in J} E_j) \in \mathcal{B}$. (Here, we use the fact that the direct sum is an elementary submodel in its pure-injective hull as well as in the direct product.) Thus, again, $\prod_{k\in J\cup L} E_k$ cogenerates the same cotorsion pair as E did. □

We are in a position to state the main result of this section. It is in fact an immediate consequence of the previous statements.

Theorem 5.13. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with both classes closed under direct limits. Then $\mathfrak{C}$ is cogenerated by a direct product of indecomposable pure-injective modules.*

Proof. This follows easily by Theorem 5.11 and Proposition 5.12. □

Remark. (1). Note that if R is an artin algebra or, more generally, a semi-primary ring and $(\mathcal{A}, \mathcal{B})$ is a projective cotorsion pair satisfying the hypotheses of TCMC, it follows from [31, Corollary 4.5] that the class $\mathcal{B}$ is also of the form $\operatorname{Ker}\operatorname{Ext}^1_R(-, N)$ for a pure-injective module N.

(2). The distinction between closure under direct limits and closure under pure-epimorphic images is rather subtle. The two notions often coincide, but no example of a (hereditary) cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ closed under direct limits and *not* closed under pure-epimorphic images is known to the authors as yet.

6. Compactly generated triangulated categories

In this section, we compare the results we have obtained above with the work of Krause on smashing localizations of triangulated categories in [29, 27]. As mentioned before, there is a bijective correspondence between smashing localizing pairs in the stable module category and certain cotorsion pairs in the usual module category which works for self-injective artin algebras [30]. However, as we want to indicate now, there are strong analogues of both settings well beyond where the correspondence from [30] works. First, we will recall some necessary terminology.

Let $\mathcal{T}$ be a *triangulated category* which admits arbitrary (set indexed) coproducts. We will not define this concept here since it is well-known and the definition is rather complicated, but we refer for example to [18, IV], [21] or [25, §3]. We say that an object $C \in \mathcal{T}$ is *compact* if the canonical map $\bigoplus_i \operatorname{Hom}_{\mathcal{T}}(C, X_i) \to \operatorname{Hom}_{\mathcal{T}}(C, \coprod_i X_i)$ is an isomorphism for any family $(X_i)_{i \in I}$ of objects of $\mathcal{T}$. Here, we will denote coproducts in $\mathcal{T}$ by the symbol $\coprod$ to distinguish them from direct sums of abelian groups. Let us denote by $\mathcal{T}_0$ the full subcategory of $\mathcal{T}$ formed by the compact objects. The category $\mathcal{T}$ is then called *compactly generated* if

(1) $\mathcal{T}_0$ is equivalent to a small category.
(2) Whenever $X \in \mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(C, X) = 0$ for all $C \in \mathcal{T}_0$, then $X = 0$.

As an important example here, let R be a *quasi-Frobenius ring*, that is a ring for which projective and injective modules coincide, and let $\underline{\text{Mod}}$-R be the *stable category*, that is the quotient of Mod-R modulo the projective modules. Then $\underline{\text{Mod}}$-R is triangulated [21] and compactly generated [29, §1.5]. Moreover, compact objects are precisely those isomorphic in $\underline{\text{Mod}}$-R to finitely generated R-modules. Other examples of compactly generated triangulated categories are unbounded derived categories of module categories and the stable homotopy category.

Let $\mathcal{X}$ be a full triangulated subcategory of $\mathcal{T}$. Then $\mathcal{X}$ is called *localizing* if $\mathcal{X}$ is closed under forming coproducts with respect to $\mathcal{T}$. We call $\mathcal{X}$ *strictly localizing* if the inclusion $\mathcal{X} \to \mathcal{T}$ has a right adjoint. Finally, $\mathcal{X}$ is said to be *smashing* if the right adjoint preserves coproducts. Note that being a smashing subcategory is stronger than being strictly localizing, which in turn is stronger than being a localizing subcategory.

A localizing subcategory $\mathcal{X} \subseteq \mathcal{T}$ is *generated* by a class $\mathcal{C}$ of objects in $\mathcal{T}$ if it is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$. Notice that $\mathcal{T}$ itself is generated by $\mathcal{T}_0$ as a localizing subcategory (cf. [39, §5] or [35, Theorem 2.1]).

As in [30], we define $(\mathcal{X}, \mathcal{Y})$ to be a *localizing pair* if $\mathcal{X}$ is a strictly localizing subcategory of $\mathcal{T}$ and $\mathcal{Y} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, -)$. The objects in $\mathcal{Y}$ are then called $\mathcal{X}$*-local*. Note that this definition makes sense also for non-compactly generated triangulated categories and with this in mind, $(\mathcal{X}, \mathcal{Y})$ is a localizing pair in $\mathcal{T}$ if and only if $(\mathcal{Y}, \mathcal{X})$ is a localizing pair in $\mathcal{T}^{op}$. Moreover, the class $\mathcal{X}$ is smashing if and only if the class $\mathcal{Y}$ of all $\mathcal{X}$-local objects is closed under coproducts.

There is a useful analogue of countable direct limits in a triangulated category, called a homotopy colimit. Let

$$X_0 \overset{\varphi_0}{\to} X_1 \overset{\varphi_1}{\to} X_2 \overset{\varphi_2}{\to} \cdots$$

be a sequence of maps in $\mathcal{T}$. A *homotopy colimit* of the sequence, denoted by $\underrightarrow{\mathrm{hocolim}}\, X_i$, is by definition an object X which occurs in the triangle

$$\coprod_{i<\omega} X_i \overset{\Phi}{\to} \coprod_{i<\omega} X_i \to X \to \coprod_{i<\omega} X_i[1] \qquad (\ddagger)$$

where the i-th component of the map Φ is the composite

$$X_i \overset{\binom{\mathrm{id}}{-\varphi_i}}{\to} X_i \amalg X_{i+1} \overset{j}{\to} \coprod_{i<\omega} X_i$$

and j is the split monomorphism to the coproduct. Note that a homotopy colimit is unique up to a (non-unique) isomorphism. As an easy but important fact, we point up that when applying the functor $\mathrm{Hom}_{\mathcal{T}}(-, Z)$ on ($\ddagger$) for any $Z \in \mathcal{T}$, we get an exact sequence

$$\begin{aligned} 0 \leftarrow \underleftarrow{\lim}^1 \mathrm{Hom}_{\mathcal{T}}(X_i, Z) \leftarrow \\ \leftarrow \prod \mathrm{Hom}_{\mathcal{T}}(X_i, Z) \overset{\Phi^*}{\leftarrow} \prod \mathrm{Hom}_{\mathcal{T}}(X_i, Z) \leftarrow \\ \leftarrow \underleftarrow{\lim}\, \mathrm{Hom}_{\mathcal{T}}(X_i, Z) \leftarrow 0 \end{aligned}$$

where $\Phi^* = \mathrm{Hom}_{\mathcal{T}}(\Phi, Z)$ and $\underleftarrow{\lim}^1$ is the first derived functor of inverse limit.

Having recalled the terminology, we also recall the crucial correspondence between cotorsion pairs and localizing pairs shown in [30]:

Theorem 6.1. *Let R be a self-injective artin algebra,* Mod-R *the category of all right R-modules and* $\underline{\mathrm{Mod}}$-R *the stable category. Then the assignment*

$$(\mathcal{A}, \mathcal{B}) \to (\underline{\mathcal{A}}, \underline{\mathcal{B}})$$

gives a bijective correspondence between projective cotorsion pairs in Mod-R *and localizing pairs in* $\underline{\mathrm{Mod}}$-R*. Moreover, the following hold:*

1. $\underline{\mathcal{A}}$ *is smashing in* $\underline{\mathrm{Mod}}$-R *if and only if both $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits in* Mod-R.
2. $\underline{\mathcal{A}}$ *is generated, as a localizing subcategory in* $\underline{\mathrm{Mod}}$-R*, by a set of compact objects if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair of finite type in* Mod-R.

Proof. This is an immediate consequence of [30, Theorem 7.6 and Corollary 7.7] and [4, Corollary 4.6]. □

We have proved in Theorem 3.5 that any cotorsion pair $(\mathcal{A}, \mathcal{B})$ coming from a smashing localizing pair is of countable type. We show that it is possible to state a similar countable type result for $\underline{\mathrm{Mod}}$-R purely in the language of triangulated categories.

Definition 6.2. Let $\mathcal{T}$ be a compactly generated triangulated category. We call an object $X \in \mathcal{T}$ *countable* if it is isomorphic to the homotopy colimit of a sequence of maps $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots$ between compact objects. Furthermore, let $\mathcal{T}_\omega$ stand for the full subcategory of $\mathcal{T}$ formed by all countable objects.

Note that $\mathcal{T}_\omega$ is skeletally small. Now we can state the following theorem:

Theorem 6.3. *Let R be a self-injective artin algebra and $\mathcal{T} = \underline{\mathrm{Mod}}\text{-}R$ the stable category of right R-modules. Then every smashing subcategory of $\mathcal{T}$ is generated, as a localizing subcategory of $\mathcal{T}$, by a set of countable objects.*

We postpone the proof until after a few preparatory observations and lemmas. First note that countable objects in $\underline{\mathrm{Mod}}\text{-}R$ for a self-injective algebra R are precisely those isomorphic in $\underline{\mathrm{Mod}}\text{-}R$ to countably generated modules from Mod-R, see [39, Lemma 4.3].

Next, we recall a technical statement concerning vanishing of derived functors of inverse limits. We recall that $\varprojlim^k$ stands for the k-th derived functor of inverse limit and, for convenience, we let $\aleph_{-1} = 1$.

Lemma 6.4. [33] *Let R be a ring and I be a directed set whose smallest cofinal subset has cardinality $\aleph_\alpha$, where α is an ordinal number or -1. Put*

$$d = \sup\{k < \omega \mid \varprojlim^k N_i \neq 0 \text{ for some } (N_i)_{i \in I^{op}}\}$$

where $(N_i)_{i\in I^{op}}$ stands for an inverse system of right R-modules indexed by I^{op}. Then $d = \alpha+1$ if α is finite and $d = \omega$ if α is an infinite ordinal number.

The latter lemma has important consequences for direct limits that are "small enough". Recall that given a class $\mathcal{C}$ of modules, we denote by $\mathrm{Add}\,\mathcal{C}$ the class of all direct summands of arbitrary direct sums of modules in $\mathcal{C}$.

Lemma 6.5. *Let R be a ring and $(M_i)_{i\in I}$ be a direct system of R-modules such that $|I| < \aleph_\omega$. Then there is an exact sequence:*

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to \varinjlim M_i \to 0,$$

where n is a non-negative integer and $X_j \in \mathrm{Add}\,\{M_i \mid i \in I\}$ for all $j = 0, \dots, n$.

Proof. Consider the canonical presentation of $\varinjlim M_i$:

$$\cdots \xrightarrow{\delta_2} \bigoplus_{i_0<i_1<i_2} M_{i_0 i_1 i_2} \xrightarrow{\delta_1} \bigoplus_{i_0<i_1} M_{i_0 i_1} \xrightarrow{\delta_0} \bigoplus_{i_0 \in I} M_{i_0} \to \varinjlim M_i \to 0,$$

where $M_{i_0 i_1 \dots i_k} = M_{i_0}$ for all k-tuples $i_0 < i_1 < \cdots < i_k$ of elements of I. This is an exact sequence and it follows from [23] that

$$\varprojlim^k \mathrm{Hom}_R(M_i, Y) = \mathrm{Ker}\,\mathrm{Hom}_R(\delta_k, Y) / \,\mathrm{Im}\,\mathrm{Hom}_R(\delta_{k-1}, Y)$$

for any R-module Y and any $k \geq 0$ (we let $\delta_{-1} = 0$ here). If we take the smallest n such that $|I| \leq \aleph_n$ and $Y = \operatorname{Ker}\delta_n$, it follows from Lemma 6.4 that the inclusion

$$0 \to \operatorname{Ker}\delta_n \to \bigoplus_{i_0<i_1<\cdots<i_{n+1}} M_{i_0 i_1 \dots i_{n+1}}$$

splits since $\varprojlim^{n+2} \operatorname{Hom}_R(M_i, Y) = 0$ in this case. The claim of the lemma follows immediately. □

Corollary 6.6. *Let R be a quasi-Frobenius ring and let $\underline{\mathcal{A}}$ be a localizing subcategory of $\underline{\mathrm{Mod}}$-R. Assume that $(M_i)_{i\in I}$ is a direct system in* Mod-R *such that $|I| < \aleph_\omega$ and M_i is an object of $\underline{\mathcal{A}}$ for each $i \in I$. Then also $\varinjlim M_i$ is an object of $\underline{\mathcal{A}}$.*

Proof. Note that any localizing subcategory is closed under direct summands [11]. Then the claim follows immediately from the preceding lemma when taking into account that triangles in $\underline{\mathrm{Mod}}$-R correspond to short exact sequences in Mod-R and that the canonical functor Mod-$R \to \underline{\mathrm{Mod}}$-$R$ preserves coproducts. □

Now we are in a position to prove the theorem.

Proof of Theorem 6.3. Let $\underline{\mathcal{A}}$ be a smashing subcategory of $\mathcal{T} = \underline{\mathrm{Mod}}$-$R$ and let $(\mathcal{A}, \mathcal{B})$ be the corresponding projective cotorsion pair in Mod-R with $\mathcal{B}$ closed under direct limits given by Theorem 6.1. Then by Theorem 3.5, there is a set $\mathcal{S}$ of countably generated R-modules that generates the cotorsion pair.

Let us denote by $\mathcal{L}$ the localizing subcategory of $\mathcal{T}$ generated by $\mathcal{S}$, viewed as set of (countable) objects of $\mathcal{T}$. We claim that then for each $X \in \mathcal{T}$, there is a triangle $X \xrightarrow{w_X} B_X \to L_X \to X[1]$ in $\mathcal{T}$ such that $B_X \in \underline{\mathcal{B}}$ and $L_X \in \mathcal{L}$.

Let us assume for a moment that we have proved the claim and let $A \in \underline{\mathcal{A}}$. If we consider the shifted triangle $L_A[-1] \xrightarrow{f} A \xrightarrow{w_A} B_A \to L_A$, then clearly $w_A = 0$ and f is split epi. Hence, A is a direct summand of $L_A[-1]$ and consequently, since $\mathcal{L}$ is closed under direct summands by [11], $A \in \mathcal{L}$. Thus, $\underline{\mathcal{A}} = \mathcal{L}$ and the theorem follows.

Therefore, it remains to prove the claim. Let $X \in \mathcal{T}$. If we view X as an R-module, we can construct a special $\mathcal{B}$-preenvelope $0 \to X \to B_X \to L_X \to 0$ following the lines of [19, Theorem 3.2.1]: We construct a well-ordered continuous chain

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots$$

indexed by ordinal numbers such that $B_0 = X$ and $B_{\alpha+1}$ is a *universal extension* of B_α by modules from $\mathcal{S}$. That is, there is an exact sequence of the form:

$$0 \to B_\alpha \to B_{\alpha+1} \to \bigoplus_{j\in J_\alpha} Y_j \to 0,$$

where Y_j is isomorphic to a module from $\mathcal{S}$ for each $j \in J_\alpha$ and the connecting homomorphisms $\delta_Z : \mathrm{Hom}_R(Z, \bigoplus_{j\in J} Y_j) \to \mathrm{Ext}^1_R(Z, B_\alpha)$ are surjective for all $Z \in \mathcal{S}$. In particular, $\mathrm{Ext}^1_R(Z, -)$ applied on $B_\alpha \subseteq B_\beta$ for any $\alpha < \beta$ gives the zero map. Since all the modules in $\mathcal{S}$ are countably presented, any morphism $\Omega(Z) \to B_{\aleph_1}$ in Mod-R, where $Z \in \mathcal{S}$, factors through the inclusion $B_\alpha \subseteq B_{\aleph_1}$ for some $\alpha < \aleph_1$. It follows that $\mathrm{Ext}^1_R(Z, B_{\aleph_1}) = 0$ for each $Z \in \mathcal{S}$; hence $B_{\aleph_1} \in \mathcal{B}$. Now, if we set $L_\alpha = B_\alpha/X$ for each α, we have a well-ordered continuous chain

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_\alpha \subseteq \cdots$$

such that $L_{\alpha+1}/L_\alpha \cong B_{\alpha+1}/B_\alpha \in \mathrm{Add}\,\mathcal{S}$. It follows from Eklof's Lemma ([19, Lemma 3.1.2] or [16, Lemma 1]) that $L_\alpha \in \mathcal{A}$ for each ordinal α. Hence, $0 \to X \to B_{\aleph_1} \to L_{\aleph_1} \to 0$ is a special $\mathcal{B}$-preenvelope of X.

Now let us focus on the corresponding triangle $X \to B_{\aleph_1} \to L_{\aleph_1} \to X[1]$ in $\mathcal{T}$. Clearly $B_{\aleph_1} \in \underline{\mathcal{B}}$. Moreover, it follows by a straightforward transfinite induction on α that $L_\alpha \in \mathcal{L}$ for each $\alpha \leq \aleph_1$. For $\alpha = 0$, obviously $L_0 = 0 \in \mathcal{L}$. To pass from α to $\alpha + 1$, we use the fact that the third term in the triangle $L_\alpha \to L_{\alpha+1} \to \coprod_{j\in J_\alpha} Y_j \to L_\alpha[1]$ is in $\mathrm{Add}\,\mathcal{S}$. Finally, limit steps are taken care of by Corollary 6.6. The claim is proved and so is the theorem. □

Inspired by Theorem 6.3, we can ask the following question:

Question (Countable Telescope Conjecture). Let $\mathcal{T}$ be an arbitrary compactly generated triangulated category. Is every smashing localizing subcategory of $\mathcal{T}$ generated by a set of countable objects?[2]

In this context, it is a natural question if one can characterize the countable objects in a smashing subcategory of a triangulated category. That is, we are looking for a triangulated category analogue of Theorem 4.8. It turns out that there is an analogous statement that holds for any compactly generated triangulated category.

Theorem 6.7. *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{X}$ be a smashing subcategory of $\mathcal{T}$. Denote by $\mathfrak{I}$ the ideal of all morphisms between compact objects which factor through some object in $\mathcal{X}$. Then the following are equivalent for a countable object $X \in \mathcal{T}$:*

(1) *$X \in \mathcal{X}$,*
(2) *X is the homotopy colimit of a countable direct system (X_n, φ_n) of compact objects such that $\varphi_n \in \mathfrak{I}$ for every n.*

Proof. (1) $\implies$ (2). Since X is countable, we have $X = \underrightarrow{\mathrm{hocolim}}\, Y_n$ where (Y_n, ψ_n) is a direct system of compact objects (not necessarily from $\mathcal{X}$). Let Z be an $\mathcal{X}$-local object and let $\tilde{Z} = \coprod_{i<\omega} Z_i$, where

[2] An affirmative and far more general answer to this question was given by Krause in [28, §7.4] after submission of this paper.

$Z_i = Z$ for each $i < \omega$. By assumption, $\tilde{Z}$ is also $\mathcal{X}$-local. If we apply $\mathrm{Hom}_{\mathcal{T}}(-, \tilde{Z})$ on the triangle $\coprod_n Y_n \xrightarrow{\Phi} \coprod_n Y_n \to X \to \coprod_n Y_n[1]$, we see that $\mathrm{Hom}_{\mathcal{T}}(\Phi, \tilde{Z})$ is an isomorphism. Hence we get:

$$\varprojlim \mathrm{Hom}_{\mathcal{T}}(Y_n, \tilde{Z}) = 0 = \varprojlim{}^1 \mathrm{Hom}_{\mathcal{T}}(Y_n, \tilde{Z}).$$

Note also that $\mathrm{Hom}_{\mathcal{T}}(Y_n, \tilde{Z})$ is canonically isomorphic to $\mathrm{Hom}_{\mathcal{T}}(Y_n, Z)^{(\omega)}$ for each $n < \omega$ since all the Y_n are compact. Consequently, the inverse system

$$(\mathrm{Hom}_{\mathcal{T}}(Y_n, Z), \mathrm{Hom}_{\mathcal{T}}(\psi_n, Z))_{n<\omega}$$

is Mittag-Leffler by Proposition 1.4 and T-nilpotent by Lemma 4.5. Since the class of all $\mathcal{X}$-local objects is closed under coproducts, we infer, as in the proof of Theorem 4.8, that there are some bounds for T-nilpotency common for all $\mathcal{X}$-local objects Z. In other words, there is a cofinal subsystem $(Y_{n_k}, \varphi_k \mid k < \omega)$ of the direct system (Y_n, ψ_n) such that $\mathrm{Hom}_{\mathcal{T}}(\varphi_k, Z) = 0$ for all $k < \omega$ and $\mathcal{X}$-local objects Z. Note that $X \cong \underrightarrow{\mathrm{hocolim}}_k Y_{n_k}$ since the homotopy colimit does not change when passing to a cofinal subsystem, [36, Lemma 1.7.1].

Finally, if φ is a morphism in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\varphi, Z) = 0$ whenever Z is $\mathcal{X}$-local, then φ factors through an object in $\mathcal{X}$ by [29, Lemmas 3.4 and 3.8]. Hence, $\varphi_k \in \mathfrak{I}$ for each k and we can just put $X_k = Y_{n_k}$.

(2) $\Longrightarrow$ (1). If X and (X_n, φ_n) are as in the assumption, then, by Lemma 4.5,

$$\varprojlim \mathrm{Hom}_{\mathcal{T}}(X_n, Z) = 0 = \varprojlim{}^1 \mathrm{Hom}_{\mathcal{T}}(X_n, Z)$$

whenever Z is $\mathcal{X}$-local. Thus, if we consider the triangle $\coprod_n X_n \xrightarrow{\Phi} \coprod_n X_n \to X \to \coprod_n X_n[1]$ defining X, then $\mathrm{Hom}_{\mathcal{T}}(\Phi, Z)$ is an isomorphism. For a similar reason, $\mathrm{Hom}_{\mathcal{T}}(\Phi[1], Z)$ is an isomorphism, and consequently $\mathrm{Hom}_{\mathcal{T}}(X, Z) = 0$ for all $\mathcal{X}$-local objects Z. In other words: $X \in \mathcal{X}$. □

Triangulated category analogues of Theorems 4.9 and 5.13, the remaining main results of this paper, have been proved by Krause in [29]. We include the corresponding statements from [29] here to underline how straightforward the translation is. Let us start with Theorem 4.9—actually, [29, Theorem A] served as an inspiration for it:

Theorem 6.8. [29, Theorem A] *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{X}$ be a smashing subcategory of $\mathcal{T}$. Denote by $\mathfrak{I}$ the ideal of all morphisms between compact objects which factor through some object in $\mathcal{X}$. Then the following are equivalent for $Y \in \mathcal{T}$:*

(1) *Y is $\mathcal{X}$-local,*
(2) *$\mathrm{Hom}_{\mathcal{T}}(f, Y) = 0$ for each $f \in \mathfrak{I}$.*

We conclude the paper with an analogue of Theorem 5.13. Let us first recall that one defines pure-injective objects in a compactly generated triangulated category $\mathcal{T}$ as follows (see [29]): Let us call a morphism

$X \to Y$ in $\mathcal{T}$ a *pure monomorphism* if the induced map $\mathrm{Hom}_{\mathcal{T}}(C, X) \to \mathrm{Hom}_{\mathcal{T}}(C, Y)$ is a monomorphism for every compact objects C. An object X is then called *pure-injective* if every pure monomorphism $X \to Y$ splits. As for module categories, the isomorphism classes of indecomposable pure-injective objects form a set which we call a spectrum of $\mathcal{T}$. The following has been proved in [29]:

Theorem 6.9. [29, Theorem C] *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{X}$ be a smashing subcategory of $\mathcal{T}$. Then $X \in \mathcal{X}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for each indecomposable pure-injective $\mathcal{X}$-local object Y.*

For stable module categories over self-injective artin algebras, the correspondence via Theorem 6.1 works especially well because of the following result from [29]:

Proposition 6.10. [29, Proposition 1.16] *Let R be a quasi-Frobenius ring and X be a right R-module. Then X is a pure-injective module if and only if X is a pure-injective object in $\underline{\mathrm{Mod}}$-R.*

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II.

TELESCOPE CONJECTURE, IDEMPOTENT IDEALS, AND THE TRANSFINITE RADICAL

Abstract

We show that for an artin algebra Λ, the telescope conjecture for module categories is equivalent to certain idempotent ideals of modΛ being generated by identity morphisms. As a consequence, we prove the conjecture for domestic standard selfinjective algebras and domestic special biserial algebras. We achieve this by showing that in any Krull-Schmidt category with local d.c.c. on ideals, any idempotent ideal is generated by identity maps and maps from the transfinite radical.

This paper has been accepted in Trans. Amer. Math. Soc.

TELESCOPE CONJECTURE, IDEMPOTENT IDEALS, AND THE TRANSFINITE RADICAL

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Abstract. We show that for an artin algebra Λ, the telescope conjecture for module categories is equivalent to certain idempotent ideals of $\operatorname{mod} \Lambda$ being generated by identity morphisms. As a consequence, we prove the conjecture for domestic standard selfinjective algebras and domestic special biserial algebras. We achieve this by showing that in any Krull-Schmidt category with local d.c.c. on ideals, any idempotent ideal is generated by identity maps and maps from the transfinite radical.

Introduction

The aim of this paper is to further develop and apply connections between seemingly rather different topics in algebra:

(1) localizations of triangulated compactly generated categories;
(2) theory of cotorsion pairs and induced approximations;
(3) the structure of idempotent ideals in a module category;
(4) representation type of a finite dimensional algebra.

The main motivation for this paper was point (1), the study of so called smashing localizations in triangulated compactly generated categories. There is an important conjecture, the telescope conjecture, which roughly says that any smashing localization of a compactly generated triangulated category comes from a set of compact objects. For an extensive study of this problem and explanation of the terminology we refer to work by Krause [18, 16]. Even though the conjecture is known to be false in this generality—see [14] for a simple algebraic counterexample—it is not resolved for many particular important settings. Such special solutions would still have significant consequences. In the case of unbounded derived categories of rings, this is discussed in [16].

Date: April 22, 2008.

2000 *Mathematics Subject Classification.* 18E35 (primary), 16E30, 16G60 (secondary).

Key words and phrases. Telescope conjecture, idempotent ideals, transfinite radical.

The author was supported by the Research Council of Norway through Storforsk-project Homological and geometric methods in algebra, and also by the grant GAČR 201/05/H005 and the research project MSM 0021620839.

In this paper, we will focus on another setting. Let R be a quasi-Frobenius ring (that is, the projective and injective left modules coincide), and $\underline{\mathrm{Mod}}\, R$ be the stable module category of left R-modules. Then $\underline{\mathrm{Mod}}\, R$ is a triangulated compactly generated category in the sense of [18, 16]. If, moreover, R is a self-injective artin algebra, the telescope conjecture has been translated by Krause and Solberg [20] to a statement about modules, or more precisely about certain cotorsion pairs of modules. The precise statements and explanation of terminology are given below. Recently, a positive solution to the telescope conjecture for stable module categories over finite group algebras was annouced by the authors of [4]. Their methods are, however, closely tied to group algebras and do not allow direct generalization to other self-injective artin algebras. We will develop an alternative approach.

The above mentioned version of the telescope conjecture for cotorsion pairs of modules from [20, §7] makes sense not only for self-injective artin algebras, but in fact for any associative ring with unit, leading to a problem in homological algebra which is of interest by itself (cf. [2, 25]). Even though one loses the translation to triangulated categories, similarities between the new and the original settings are striking and have been analyzed more in detail in [25].

In the present paper, we further develop the approach from [25] and show that the telescope conjecture for module categories depends on the structure of certain idempotent ideals of the category of finitely presented modules. This is another analogy to so called exact ideals from [16]. Further, we prove that the structure of idempotent ideals in the category of finitely presented modules over an artin algebra, as well as in many other categories studied by representation theory, heavily depends on idempotent ideals inside the radical. In particular, if there are no non-zero idempotent ideals in the radical, we get a positive answer to the telescope conjecture.

The condition of no non-zero idempotent ideals in the radical of the module category seems to be closely related to the domestic representation type. These notions were proved to coincide for special biserial algebras by Schröer [27, 24]. A stronger but closely related condition when the infinite radical is nilpotent was studied by several authors, see for example [15, 28, 5, 6]. Our main interest in the existing results stems from the fact that they provide us with non-trivial examples of artin algebras over which the telescope conjecture for module categories holds. Some of them, coming from a paper by Skowroński and Kerner [15], are self-injective, thus allowing us to go all the way back and get a statement about smashing localizations of their stable module categories.

Another condition which seems to be closely related to both the domestic representation type and vanishing of the transfinite radical is that of the Krull-Gabriel dimension of an artin algebra being an ordinal number. The concept of the Krull-Gabriel dimension of a ring R can be interpreted as a measure for complexity of both the category $\mathrm{fp}(\mathrm{mod}\, R, \mathrm{Ab})$ of finitely presented additive functors $\mathrm{mod}\, R \to \mathrm{Ab}$, and the lattice of primitive positive formulas over R. Using a result from [19], we prove that the telescope conjecture for module categories holds true if the Krull-Gabriel dimension of the artin algebra in question is an ordinal number.

The author would like to thank Øyvind Solberg for several helpful discussions. The author is also grateful to Otto Kerner for his comments on idempotent ideals in the radical of some module categories and communicating the unpublished result by Dieter Vossieck mentioned in Section 2.

1. Preliminaries

In this text, Λ will always be an artin algebra and all modules will be left Λ-modules. Let us denote by $\mathrm{Mod}\,\Lambda$ the category of all modules and by $\mathrm{mod}\,\Lambda$ the full subcategory of finitely generated modules. Some results in this paper will be proved for more general categories: Krull-Schmidt categories with local d.c.c. on ideals as defined in Section 3. This setting includes $\mathrm{mod}\,\Lambda$, derived bounded categories, categories of coherent sheaves, and other categories of representation theoretic significance. A reader who is not interested in the full generality can, nevertheless, read the corresponding statements as if they were stated for $\mathrm{mod}\,\Lambda$.

A *cotorsion pair* in $\mathrm{Mod}\,\Lambda$ is a pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of $\mathrm{Mod}\,\Lambda$ such that $\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1_\Lambda(-, \mathcal{B})$ and $\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_\Lambda(\mathcal{A}, -)$. A cotorsion pair is called *hereditary* if in addition $\mathrm{Ext}^i_\Lambda(\mathcal{A}, \mathcal{B}) = 0$ for all $i \geq 2$. This paper deals with the telescope conjecture for module categories (TCMC) as formulated in [20, Conjecture 7.9]. Actually, we slightly alter the assumptions—we require the cotorsion pair in question to be hereditary (since the cotorsion pairs of interest in [20] always are) and relax the condition that [20] imposes on the class $\mathcal{A}$ of the cotorsion pair. We state the conjecture as follows:

Conjecture (A). *Let Λ be an artin algebra and let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in $\mathrm{Mod}\,\Lambda$ such that $\mathcal{B}$ is closed under taking filtered colimits. Then every module in $\mathcal{A}$ is a colimit of a filtered system of finitely generated modules from $\mathcal{A}$.*

Note that, in view of [1, Theorem 1.5], we can equivalently replace filtered colimits by direct limits in the statement above. We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathrm{Mod}\,\Lambda$ is of *finite type* if $\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_\Lambda(\mathcal{S}, -)$ for a set $\mathcal{S}$ of finitely generated modules. Similarly, we define $(\mathcal{A}, \mathcal{B})$ to be of *countable type* if we can take $\mathcal{S}$ to be a set of countably generated modules. With this definition we can for any particular algebra Λ equivalently restate Conjecture (A) as follows, see [2, Corollary 4.6]:

Conjecture (B). *Let Λ be an artin algebra and let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* $\operatorname{Mod}\Lambda$ *such that $\mathcal{B}$ is closed under taking direct limits. Then $(\mathcal{A}, \mathcal{B})$ is of finite type.*

As a tool to handle the conjectures, we will need the notion of an ideal of an additive category. Let $\mathcal{C}$ be a skeletally small additive category. A class $\mathfrak{I}$ of morphisms in $\mathcal{C}$ is called a (2-sided) *ideal* of $\mathcal{C}$ if $\mathfrak{I}$ contains all zero morphisms, and it is closed under addition and under composition with arbitrary morphisms from left and right, whenever the operations are defined. Let us denote $\mathfrak{I}(X,Y) = \mathfrak{I} \cap \operatorname{Hom}_{\mathcal{C}}(X,Y)$. Note that if $\mathcal{C} = \operatorname{mod}\Lambda$ then $\mathfrak{I}(X,Y)$ is always a k-submodule of $\operatorname{Hom}_\Lambda(X,Y)$ where k is the centre of Λ. Since $\mathcal{C}$ was assumed to be skeletally small, ideals of $\mathcal{C}$ form a set.

We say that an additive category $\mathcal{C}$ is a *Krull-Schmidt category* if it is skeletally small, every indecomposable object of $\mathcal{C}$ has a local endomorphism ring, and every object of $\mathcal{C}$ (uniquely) decomposes as a finite coproduct of indecomposables. As an example to keep in mind, we can put $\mathcal{C} = \operatorname{mod}\Lambda$. For Krull-Schmidt categories there is a prominent ideal called the *radical*—it is the ideal generated by all non-invertible morphisms between indecomposable objects. We denote this ideal by $\operatorname{rad}_{\mathcal{C}}$ and if $\mathcal{C} = \operatorname{mod}\Lambda$ we use the abbreviated notation $\operatorname{rad}_\Lambda$. Let us recall the well known fact that $\operatorname{rad}_{\mathcal{C}}$ contains no identity morphisms and, clearly, it is the maximal ideal with this property. Here and also later in this paper we, of course, mean no identity morphisms of non-zero objects since zero morphisms are in any ideal by definition.

Following an idea in [23], we can inductively define transfinite powers $\mathfrak{I}^\alpha$ for any ideal $\mathfrak{I}$ and any ordinal number α. Let $\mathfrak{I}^0$ be the ideal of all morphisms in $\mathcal{C}$ and $\mathfrak{I}^1 = \mathfrak{I}$. For a natural number $n \geq 1$, we define $\mathfrak{I}^n$ as usual as the ideal generated by all compositions of n-tuples of morphisms from $\mathfrak{I}$. If α is a limit ordinal, we define $\mathfrak{I}^\alpha = \bigcap_{\beta<\alpha} \mathfrak{I}^\beta$. If α is infinite non-limit, then uniquely $\alpha = \beta + n$ for some limit ordinal β and natural number $n \geq 1$, and we set $\mathfrak{I}^\alpha = (\mathfrak{I}^\beta)^{n+1}$. Note that since we assume that $\mathcal{C}$ is skeletally small, the decreasing chain

$$\mathfrak{I}^0 \supseteq \mathfrak{I}^1 \supseteq \mathfrak{I}^2 \supseteq \cdots \supseteq \mathfrak{I}^\alpha \supseteq \mathfrak{I}^{\alpha+1} \supseteq \dots$$

stabilizes for cardinality reasons. Let us denote $\mathfrak{I}^* = \bigcap_\alpha \mathfrak{I}^\alpha$, the minimum of the chain.

We will focus mostly on the case when $\mathfrak{I} = \operatorname{rad}_{\mathcal{C}}$. In this case we call $\operatorname{rad}^*_{\mathcal{C}}$ the *transfinite radical* of $\mathcal{C}$. Notice that not necessarily $\operatorname{rad}^*_{\mathcal{C}} = 0$, even when $\mathcal{C} = \operatorname{mod}\Lambda$ for an artin algebra Λ—see the next section or [23, 27]. The main goal of this paper is to prove that TCMC formulated as Conjecture (B) holds true over those artin algebras for which $\operatorname{rad}^*_\Lambda = 0$. This applies in particular to:

- [15] standard selfinjective algebras of domestic representation type;
- [27] special biserial algebras of domestic representation type.

Recall that a finite dimensional algebra over an algebraically closed field is of *domestic representation type* if there is a natural number N such that for each dimension d, all but finitely many indecomposable modules of dimension d belong to at most N one-parametric families.

2. Transfinite radical

Let $\mathcal{C}$ be an additive category. We call an ideal $\mathfrak{I}$ of $\mathcal{C}$ *idempotent* if $\mathfrak{I} = \mathfrak{I}^2$. Equivalently, $\mathfrak{I}$ is idempotent if and only if for each $f \in \mathfrak{I}$ there are $g, h \in \mathfrak{I}$ such that $f = gh$. Using idempotency, we can give the following characterization of the transfinite radical:

Lemma 1. *Let $\mathcal{C}$ be a Krull-Schmidt category. Then $\operatorname{rad}^*_{\mathcal{C}}$ is the unique maximal idempotent ideal of $\mathcal{C}$ which does not contain any identity morphisms.*

Proof. We use the same (just more verbose) proof as the one given for [19, 8.10] for module categories. Clearly, $\operatorname{rad}^*_{\mathcal{C}}$ contains no identity morphisms since neither $\operatorname{rad}_{\mathcal{C}}$ does. It is easy to check that $\operatorname{rad}^*_{\mathcal{C}}$ is idempotent [23, Proposition 0.6]. On the other hand, if $\mathfrak{I}$ is idempotent without identity maps, then $\mathfrak{I} = \mathfrak{I}^* \subseteq \operatorname{rad}^*_{\mathcal{C}}$ (since $\mathfrak{I} = \mathfrak{I}^\alpha$ for any ordinal α by idempotency). Hence $\operatorname{rad}_{\mathcal{C}}$ is maximal with respect to those two properties. □

There is also a useful characterization of the morphisms in $\operatorname{rad}^*_{\mathcal{C}}$ "from inside", sheding more light on the concept than a little cryptic definition as the intersection of a series of transfinite powers. The following statement has been proved in [23] for $\mathcal{C} = \operatorname{mod}\Lambda$ using standard means similar to those when one deals with Krull dimension of a poset, and the proof reads equally well for any skeletally small Krull-Schmidt category:

Lemma 2. [23, Proposition 0.6] *Let $\mathcal{C}$ be a Krull-Schmidt category and f be a morphism in $\mathcal{C}$. Then $f \in \operatorname{rad}^*_{\mathcal{C}}$ if and only if there exists a collection of morphisms $f_{pr} : X_r \to X_p$ in $\operatorname{rad}_{\mathcal{C}}$, one for each pair of rational numbers p, r such that $0 \le p < r \le 1$, such that*

1. *$f_{ps} = f_{pr} f_{rs}$ whenever $p < r < s$;*
2. *$f_{01} = f$.*

Note that the collection $(f_{pr})_{0 \le p < r \le 1}$ is nothing else than an inverse system indexed by $[0, 1] \cap \mathbb{Q}$. Using the two lemmas above, we can give some examples of what the transfinite radical can be:

- If Λ is an artin algebra of finite representation type, then $\operatorname{rad}_\Lambda$ is nilpotent. Hence $\operatorname{rad}^*_\Lambda = 0$.

- If Λ is a tame hereditary artin algebra, then $\mathrm{rad}_\Lambda^{\omega+2} = (\mathrm{rad}_\Lambda^\omega)^3 = 0$. Hence $\mathrm{rad}_\Lambda^* = 0$.
- If Λ is a standard (that is, having a simply connected Galois covering) selfinjective algebra of domestic representation type, then $\mathrm{rad}_\Lambda^\omega$ is nilpotent [15]. Hence $\mathrm{rad}_\Lambda^* = 0$.
- If Λ is a special biserial algebra, then $\mathrm{rad}_\Lambda^* = 0$ if and only if $\mathrm{rad}_\Lambda^{\omega^2} = 0$ if and only if Λ is of domestic representation type. If Λ is not domestic, then there exists an indecomposable Λ-module X such that $0 \neq \mathrm{rad}_\Lambda^*(X, X) \subseteq \mathrm{End}_\Lambda(X)$ (see [27, Theorem 2 and Prop. 6.2]).
- As special case of the previous point, one may consider "Gelfand-Ponomarev" algebras $\Lambda_{m,n} = k[x, y]/(xy, yx, x^m, y^n)$, see [11]. The algebra $\Lambda_{2,3}$ is not of domestic represetation type and provides a very illustrative example of non-zero maps in the transfinite radical, see [23].
- If Λ is a wild hereditary artin algebra, it is conjectured that $\mathrm{rad}_\Lambda^\omega$ is idempotent. In view of Lemma 1, this cojecture can be rephrased as $\mathrm{rad}_\Lambda^* = \mathrm{rad}_\Lambda^\omega$.
- It is an unpublished result due to Dieter Vossieck that for the category $\mathcal{C} = \mathrm{mod}\, k\langle x, y\rangle$ of finite dimensional modules over the free algebra $k\langle x, y\rangle$, the radical $\mathrm{rad}_\mathcal{C}$ is idempotent. In particular $\mathrm{rad}_\mathcal{C}^* = \mathrm{rad}_\mathcal{C}$.

There is an important consequence of some of the examples above for wild artin algebras over an algebraically closed field. Namely, they *always* have the transfinite radical non-zero. Let us state this precisely.

Definition 3. Let Λ and Γ be finite dimensional algebras over a field k and let $F : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$ be an additive functor. Then F is called a *representation embedding* if F is faithful, exact, preserves indecomposability (i.e. if X is indecomposable, so is FX) and reflects isomorphism classes (i.e. if $FX \cong FY$ then also $X \cong Y$).

A finite dimensional k-algebra is called *wild* if for any other finite dimensional algebra Γ over k, there is a representation embedding $\mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$.

The following statement immediately follows from [27, Proposition 6.2] and [23, Lemma 0.2] (the same idea is also presented in [19, 8.15]):

Proposition 4. *Let Λ be a wild algebra over an algebraically closed field. Then* $\mathrm{rad}_\Lambda^* \neq 0$. *Moreover, there exists an indecomposable Λ-module X such that* $0 \neq \mathrm{rad}_\Lambda^*(X, X) \subseteq \mathrm{End}_\Lambda(X)$.

3. Idempotent ideals in Krull-Schmidt categories

Let $\mathfrak{I}$ be an ideal of a Krull-Schmidt category. Then clearly, if $\mathfrak{I}$ is generated by a collection of identity morphisms, it is necessarily an idempotent ideal. In the sequel we will show that in "nice" categories, any idempotent ideal is generated by a collection of identity morphisms together with some morphisms from the transfinite radical. To make the word *nice* precise, we need the following definition:

Definition 5. A skeletally small additive category $\mathcal{C}$ is said to have *local descending chain condition on ideals* if for any decreasing series

$$\mathfrak{I}_0 \supseteq \mathfrak{I}_1 \supseteq \mathfrak{I}_2 \supseteq \dots$$

of ideals of $\mathcal{C}$ and any pair of objects X, Y in $\mathcal{C}$, the decreasing chain

$$\mathfrak{I}_0(X,Y) \supseteq \mathfrak{I}_1(X,Y) \supseteq \mathfrak{I}_2(X,Y) \supseteq \dots$$

stabilizes.

Now, our category is "nice" if it is Krull-Schmidt with local d.c.c. on ideals. In fact, this setting is very common in representation theory. Assume that k is a commutative artinian ring and $\mathcal{C}$ is a skeletally small k-category (Hom-spaces are k-modules and composition is k-linear) and satisfies the following conditions:

(C1) $\mathcal{C}$ has splitting idempotents (that is, idempotent morphisms have kernels in $\mathcal{C}$);

(C2) $\mathcal{C}$ is Hom-finite (that is, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finitely generated k-module for any objects $X, Y \in \mathcal{C}$).

Then $\mathcal{C}$ is "nice":

Lemma 6. *Let k be a commutative artinian ring and $\mathcal{C}$ be a skeletally small* Hom*-finite k-category with splitting idempotents. Then $\mathcal{C}$ is Krull-Schmidt with local d.c.c. on ideals.*

Proof. It is a well known fact that $\mathcal{C}$ is Krull-Schmidt under the assumption. It is straightforward to show that $\mathfrak{I}(X,Y)$ is a k-submodule of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any ideal $\mathfrak{I}$ and any pair of objects $X, Y \in \mathcal{C}$. Hence $\mathcal{C}$ has clearly local d.c.c. on ideals thanks to (C2). □

As a consequence, we can give plenty of examples of "nice" categories:

- $\mathrm{mod}\,\Lambda$ for an artin algebra Λ;
- $D^b(\Lambda)$, the derived bounded category for an artin algebra Λ;
- The category of finite dimensional modules over any algebra over a field;

and many others.

Let us start with the proof of the aforementioned statement. First we need a technical lemma.

Lemma 7. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Let $X, Y \in \mathcal{C}$ and α be a limit ordinal. Then there is $\beta < \alpha$ such that $\mathrm{rad}^{\beta}_{\mathcal{C}}(X,Y) = \mathrm{rad}^{\alpha}_{\mathcal{C}}(X,Y)$.*

Proof. Since $\mathcal{C}$ has local d.c.c. on ideals, the decreasing chain $(\mathrm{rad}^{\gamma}_{\mathcal{C}}(X,Y))_{\gamma<\alpha}$ is stationary. Therefore, there is $\beta < \alpha$ such that

$$\mathrm{rad}_{\mathcal{C}}^{\beta}(X,Y) = \bigcap_{\gamma<\alpha} \mathrm{rad}_{\mathcal{C}}^{\gamma}(X,Y) = \mathrm{rad}_{\mathcal{C}}^{\alpha}(X,Y).$$

□

Now, we are in a position to give the structure theorem for idempotent ideals:

Theorem 8. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Let $\mathfrak{I}$ be an idempotent ideal of $\mathcal{C}$ and $f \in \mathfrak{I}$. Then there are $f_1, f_2 \in \mathfrak{I}$ such that $f = f_1 + f_2$, the morphism f_1 is generated by identity morphisms from $\mathfrak{I}$, and $f_2 \in \mathrm{rad}_{\mathcal{C}}^{*}$.*

Proof. We will prove the following statement for all ordinal numbers α by induction:

(∗): For every $f \in \mathfrak{I}$ there are $f_{\alpha,1}, f_{\alpha,2} \in \mathfrak{I}$ such that $f = f_{\alpha,1} + f_{\alpha,2}$, the morphism $f_{\alpha,1}$ is generated by identity morphisms from $\mathfrak{I}$, and $f_{\alpha,2} \in \mathrm{rad}_{\mathcal{C}}^{\alpha}$.

Then the theorem will follow if we take α sufficiently big. Let $f : X \to Y$ be a morphism from $\mathfrak{I}$—we can without loss of generality assume that X and Y are indecomposable.

For $\alpha = 0$, we can simply take $f_{0,1} = 0$ and $f_{0,2} = f$. If α is non-zero finite, we can construct by induction morphisms $g^1, g^2, \dots, g^{\alpha} \in \mathfrak{I}$ such that $f = g^1 g^2 \dots g^{\alpha}$. The morphisms g^i, $1 \le i \le \alpha$, are not necessarily morphisms between indecomposable objects of $\mathcal{C}$, but we can write f as a finite sum of compositions of morphisms between indecomposables. That is:

$$f = \sum_j g^{1j} g^{2j} \dots g^{\alpha j},$$

where we take g^{ij} as components of g^i, so that all g^{ij} are in $\mathfrak{I}$. Finally, we can take $f_{\alpha,1}$ as the sum of those compositions $g^{1j} g^{2j} \dots g^{\alpha j}$ where at least one of the morphisms in the composition is invertible, and $f_{\alpha,2}$ the sum of the remaining compositions. Then clearly $f_{\alpha,1}$ is generated by identities from $\mathfrak{I}$ and $f_{\alpha,2} \in \mathrm{rad}_{\mathcal{C}}^{\alpha}$.

If α is a limit ordinal, there is an ordinal $\beta < \alpha$ such that $\mathrm{rad}_{\mathcal{C}}^{\beta}(X,Y) = \mathrm{rad}_{\mathcal{C}}^{\alpha}(X,Y)$ by Lemma 7. Of course, β depends on X and Y. Hence we can set $f_{\alpha,1} = f_{\beta,1}$ and $f_{\alpha,2} = f_{\beta,2}$, where the existence of $f_{\beta,1}, f_{\beta,2}$ is given by inductive hypothesis.

Assume now that α is an infinite non-limit ordinal and $g_{\beta,1}, g_{\beta,2}$ have been already constructed for all $g \in \mathfrak{I}$ and $\beta < \alpha$. We can write $\alpha = \beta + n$ where β is a limit ordinal and $n \ge 1$ is a natural number. Since $\mathfrak{I}$ is idempotent, we can as in the finite case construct $g^1, g^2, \dots, g^{n+1} \in \mathfrak{I}$ such that $f = g^1 g^2 \dots g^{n+1}$. By inductive hypothesis, we can for each $1 \le i \le n+1$ write $g^i = g^i_{\beta,1} + g^i_{\beta,2}$ where $g^i_{\beta,1}$ is generated by identity morphisms from $\mathfrak{I}$ and $g^i_{\beta,2} \in \mathfrak{I} \cap \operatorname{rad}^\beta_{\mathcal{C}}$. Now,

$$f = \sum g^1_{\beta,k_1} g^2_{\beta,k_2} \dots g^{n+1}_{\beta,k_{n+1}}$$

where the sum is running through all tuples $(k_1, k_2, \dots, k_{n+1}) \in \{1,2\}^{n+1}$. Put $f_{\alpha,2} = g^1_{\beta,2} g^2_{\beta,2} \dots g^{n+1}_{\beta,2}$ and $f_{\alpha,1} = f - f_{\alpha,2}$. Then it immediately follows by the choice of $g^i_{\beta,1}$ and $g^i_{\beta,2}$ that $f_{\alpha,1}$ is generated by identity morphisms from $\mathfrak{I}$ and $f_{\alpha,2} \in (\operatorname{rad}^\beta_{\mathcal{C}})^{n+1} = \operatorname{rad}^\alpha_{\mathcal{C}}$. □

Just by reformulating Theorem 8, we get the following corollary:

Corollary 9. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Let $\mathfrak{I}$ be an idempotent ideal of $\mathcal{C}$, $\mathfrak{L}$ be a representative set of identity maps contained in $\mathfrak{I}$, and let $\mathfrak{R} = \mathfrak{I} \cap \operatorname{rad}^*_{\mathcal{C}}$. Then $\mathfrak{I}$ is generated, as an ideal of $\mathcal{C}$, by $\mathfrak{L} \cup \mathfrak{R}$.*

By combining the above statements, we can also characterize the situation when ideals are idempotent exactly when they are generated by a set of identity maps.

Corollary 10. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Then the following are equivalent:*

(1) *Every idempotent ideal of $\mathcal{C}$ is generated by a set of identity maps.*
(2) $\operatorname{rad}^*_{\mathcal{C}} = 0$.

Proof. (1) $\implies$ (2). If $\operatorname{rad}^*_{\mathcal{C}} \neq 0$, then by Lemma 1 it is a non-zero idempotent ideal without identity maps, hence (1) does not hold.

(2) $\implies$ (1). This is immediate by Corollary 9 since, assuming (2), we always get $\mathfrak{R} = 0$. □

4. Telescope conjecture for module categories

The aim of this section is to prove TCMC for algebras with vanishing transfinite radicals. First, we need to collect some general results about TCMC from [25]. Even though the results are often proved under weaker assumptions and work almost unchanged for left coherent rings, we specialize them to artin algebras since this is our main concern here.

Proposition 11. [25, Theorems 3.5, 4.8 and 4.9] *Let Λ be an artin algebra, $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* $\operatorname{Mod}\Lambda$ *such that $\mathcal{B}$ is closed under unions of well ordered chains, and $\mathfrak{I}$ be the ideal of all morphisms in* $\operatorname{mod}\Lambda$ *which factor through some (infinitely generated) module from $\mathcal{A}$. Then:*

(1) *$(\mathcal{A}, \mathcal{B})$ is of countable type.*
(2) $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathfrak{I}, -) = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^1(f, X) = 0 \quad (\forall f \in \mathfrak{I})\}$.

(3) *Every countably generated module in $\mathcal{A}$ is the direct limit of a countable chain*

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} \dots$$

of finitely generated modules such that $f_i \in \mathfrak{I}$ for each $i \geq 1$.

We also need a technical lemma about filtrations which has been studied in [8, 26, 31], and whose origins can be traced back to an ingenious idea of Paul Hill. Let us recall definitions.

Definition 12. Given a class of modules $\mathcal{S}$, an $\mathcal{S}$-*filtration* of a module M is a well-ordered chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M such that $M_0 = 0$, $M_\sigma = M$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to a module from $\mathcal{S}$ for each $\alpha < \sigma$. A module is called $\mathcal{S}$-*filtered* if it possesses (at least one) $\mathcal{S}$-filtration.

We will use the following specializations of a general statement from [31] for finitely and for countably presented modules:

Lemma 13. [31, Theorem 6]. *Let $\mathcal{S}$ be a set of finitely (countably, resp.) presented modules over an arbitrary ring and M be a module possessing an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{F}$ of submodules of M such that:*

(1) *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.*
(2) *$\mathcal{F}$ is closed under arbitrary sums and intersections.*
(3) *For each $N, P \in \mathcal{F}$ such that $N \subseteq P$, the module P/N is $\mathcal{S}$-filtered.*
(4) *For each $N \in \mathcal{F}$ and a finite (countable, resp.) subset $X \subseteq M$, there is $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is finitely (countably, resp.) presented.*

Most of what we need to do now before proving the main results is to observe that the ideal $\mathfrak{I}$ from Proposition 11 is always idempotent. We state this statement for artin algebras, but it again admits an almost verbatim generalization to left coherent rings.

Lemma 14. *Let Λ, $(\mathcal{A}, \mathcal{B})$ and $\mathfrak{I}$ be as in Proposition 11. Then $\mathfrak{I}$ is an idempotent ideal of* $\operatorname{mod}\Lambda$.

Proof. Let $f : X \to Y$ be a morphism from $\mathfrak{I}$. By definition, f factors as $X \xrightarrow{g} A \xrightarrow{h} Z$ for some $A \in \mathcal{A}$. Since $(\mathcal{A}, \mathcal{B})$ is of countable type, A must be filtered by countably generated modules from $\mathcal{A}$ [31, Theorem 10]. By Lemma 13, we can find a countably generated submodule $A' \subseteq A$ such that $\operatorname{Im} g \subseteq A'$ and $A' \in \mathcal{A}$. More precisely, we use part (4) of the countable version of Lemma 13 for $N = 0$ and X a finite set of generators of $\operatorname{Im} g$. Hence, f factors as $X \xrightarrow{g'} A' \xrightarrow{h'} Z$, and, by Proposition 11, we can express A' as the direct limit of a system

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} \dots$$

of finitely generated modules such that $f_i \in \mathfrak{I}$ for each $i \geq 1$. Finally, since X is finitely generated, g' factors through C_i for some $i \geq 1$. But then we can write $f = h'vf_{i+1}f_iu$ for some morphisms u and v, and clearly both f_iu and $h'vf_{i+1}$ are in $\mathfrak{I}$. Hence $f \in \mathfrak{I}^2$ and $\mathfrak{I}$ is idempotent. □

Now, we can equivalently rephrase Conjecture (B) in the language of ideals:

Proposition 15. *Let Λ, $(\mathcal{A}, \mathcal{B})$ and $\mathfrak{I}$ be as in Proposition 11. Then the following are equivalent:*

(1) *$(\mathcal{A}, \mathcal{B})$ is of finite type.*
(2) *$\mathfrak{I}$ is generated by a set of identity morphisms from* $\operatorname{mod}\Lambda$.

Proof. (1) $\implies$ (2). Assume that $(\mathcal{A}, \mathcal{B})$ is of finite type, that is, $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathcal{S}, -)$ for some set $\mathcal{S}$ of finitely generated modules. We can without loss of generality assume that $\mathcal{S}$ is a representative set of all finitely generated modules in $\mathcal{A}$.

We claim that $\mathfrak{I}$ is then generated by the set $\{1_X \mid X \in \mathcal{S}\}$. To this end we recall that under our assumption, $\mathcal{A}$ consists precisely of direct summands of $\mathcal{S}$-filtered modules (see [32, Theorem 2.2] or [12, Corollary 3.2.3]). Hence, if $f : X \to Y$ is a morphism from $\mathfrak{I}$, then it factors as $X \xrightarrow{g} A \xrightarrow{h} Z$ for some $\mathcal{S}$-filtered module A. Using part (4) of the finite version of Lemma 13 for $N = 0$ and a finite set X of generators of $\operatorname{Im} g$, we can find a module $A' \subseteq A$ such that A' is isomorphic to some module in $X \in \mathcal{S}$ and $\operatorname{Im} g \subseteq A'$. Thus, f factors through 1_X and since f was chosen arbitrarily, the claim is proved.

(2) $\implies$ (1). Suppose that $\mathcal{S}$ is a set of finitely generated modules such that $\{1_X \mid X \in \mathcal{S}\}$ generates $\mathfrak{I}$. It is straightforward by Proposition 11 (2) that $\mathcal{B} = \bigcap_{X \in \mathcal{S}} \operatorname{Ker}\operatorname{Ext}^1_\Lambda(1_X, -)$. But this is exactly the same as saying that $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathcal{S}, -)$. Hence, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type.

□

Finally, we can prove TCMC formulated as Conjecture (B) for those artin algebras Λ for which $\operatorname{rad}^*_\Lambda = 0$. Note that all what we need to do in view of Lemma 14 and Proposition 15 is to show that certain idempotent ideals are generated by identities, and this is always the case when $\operatorname{rad}^*_\Lambda = 0$. As mentioned above, $\operatorname{rad}^*_\Lambda = 0$ whenever Λ is a domestic standard selfinjective algebra [15] or a domestic special biserial algebra [27] over an algebraically closed field.

Theorem 16. *Let Λ be an artin algebra such that $\operatorname{rad}^*_\Lambda = 0$. Then every hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\operatorname{Mod}\Lambda$ such that $\mathcal{B}$ is closed under unions of well ordered chains is of finite type.*

Proof. Let $\mathfrak{I}$ be the ideal of all morphisms in $\operatorname{mod}\Lambda$ which factor through some module from $\mathcal{A}$. Then $\mathfrak{I}$ is an idempotent ideal by

Lemma 14 and, therefore, generated by a set of identity maps by Corollary 10. The latter is equivalent to saying that $(\mathcal{A}, \mathcal{B})$ is of finite type by Proposition 15. □

Another condition on an artin algebra Λ which seems to be closely related to vanishing of the transfinite radical and the domestic representation type is that of the Krull-Gabriel dimension of Λ being an ordinal number. Let us recall first that the category $\mathcal{C}(\Lambda) = \mathrm{fp}(\mathrm{mod}\,\Lambda, \mathrm{Ab})$ of finitely presented covariant additive functors $\mathrm{mod}\,\Lambda \to \mathrm{Ab}$ is an abelian category, and we can inductively define a filtration

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \dots \subseteq \mathcal{S}_\alpha \subseteq \mathcal{S}_{\alpha+1} \subseteq \dots$$

of Serre subcategories of $\mathcal{C}(\Lambda)$ as follows: Let $\mathcal{S}_0$ be the full subcateory of $\mathcal{C}(\Lambda)$ formed by functors of finite length, and for each ordinal number α, let $\mathcal{S}_{\alpha+1}$ be the full subcategory of all functors whose image under the localization functor $\mathcal{C}(\Lambda) \to \mathcal{C}(\Lambda)/\mathcal{S}_\alpha$ is of finite length. At limit ordinals α, we take just the unions $\mathcal{S}_\beta = \bigcup_{\beta<\alpha} \mathcal{S}_\alpha$. We refer to [19, §7] for more details and further references. The construction leads to the following definition:

Definition 17. The *Krull-Gabriel dimension* of an artin algebra Λ is defined as $\mathrm{KGdim}\,\Lambda = \alpha$ where α is the least ordinal number such that $\mathcal{S}_\alpha = \mathcal{C}(\Lambda)$. If no such α exists, one puts $\mathrm{KGdim}\,\Lambda = \infty$.

As a consequence of a deeper and more refined theorem, [19, Corollary 8.14] shows that $\mathrm{rad}^*_\Lambda = 0$ whenever $\mathrm{KGdim}\,\Lambda < \infty$. In particular, we get as a corollary of Theorem 16 that TCMC holds for any artin algebra with ordinal Krull-Gabriel dimension:

Corollary 18. *Let Λ be an artin algebra such that* $\mathrm{KGdim}\,\Lambda < \infty$. *Then every hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in* $\mathrm{Mod}\,\Lambda$ *such that $\mathcal{B}$ is closed under unions of well ordered chains is of finite type.*

Remark. The concept of the Krull-Gabriel dimension has been nicely illustrated by Geigle for tame hereditary algebras Λ in [9], where he explicitly computed that $\mathrm{KGdim}\,\Lambda = 2$ and described the localization categories $\mathcal{S}_1/\mathcal{S}_0$ and $\mathcal{S}_2/\mathcal{S}_1$.

The proof of the fact that $\mathrm{KGdim}\,\Lambda < \infty$ implies $\mathrm{rad}^*_\Lambda = 0$ in [19] goes through a stronger statement and involves many technical arguments. There is, however, a more elementary way to see this. Namely, one can define a so called m-dimension of a modular lattice following [22, §10.2]. Then $\mathrm{KGdim}\,\Lambda$ is equal to the m-dimension of the lattice of subobjects in $\mathrm{fp}(\mathrm{mod}\,\Lambda, \mathrm{Ab})$ of the forgetful functor $\mathrm{Hom}_\Lambda(\Lambda, -)$, [19, 7.2]. Such subobjects precisely correspond to pairs (M, m) where $M \in \mathrm{mod}\,\Lambda$ and $m \in M$, and (M', m') corresponds to a subobject of (M, m) if and only if there is a homomorphism $f : M \to M'$ in $\mathrm{mod}\,\Lambda$ such that $f(m) = m'$, [19, 7.1]. Now, $\mathrm{KGdim}\,\Lambda = \infty$ if and only if there is a factorizable system in $\mathrm{mod}\,\Lambda$ in the sense of [23]. Existence of such a factorizable system is easily implied by Lemma 2 or [23, Proposition 0.6] if $\mathrm{rad}^*_\Lambda \neq 0$.

The Krull-Gabriel dimension of Λ gives also a strong link to model theory of modules, as it is equal to the m-dimension of the lattice of primitive positive formulas in the first order theory of Λ-modules. We refer to [23, Proposition 0.3] and [22, §12] for more details.

5. Telescope conjecture for triangulated categories

We also shortly recall the application on the telescope conjecture for triangulated categories. If Λ is a selfinjective artin algebra, then the stable module category $\underline{\mathrm{Mod}}\,\Lambda$ modulo injective modules is *triangulated* in the sense of [10, IV] or [13, I]. The triangles are, up to isomorphism, of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

where $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathrm{Mod}\,\Lambda$, and the suspension functor $\Sigma : \underline{\mathrm{Mod}}\,\Lambda \to \underline{\mathrm{Mod}}\,\Lambda$ corresponds to taking cosyzygies in $\mathrm{Mod}\,\Lambda$. Clearly, Σ is an auto-equivalence of $\underline{\mathrm{Mod}}\,\Lambda$ and the corresponding inverse Σ^{-1} is given by taking syzygies in $\mathrm{Mod}\,\Lambda$.

An object X in a triangulated category with (set-indexed) coproducts is called *compact* if the representable functor $\mathrm{Hom}(X, -)$ commutes with coproducts. In particular, an object $X \in \underline{\mathrm{Mod}}\,\Lambda$ is compact if and only if it is isomorphic to a finitely generated Λ-module in $\underline{\mathrm{Mod}}\,\Lambda$ (see [18, §1.5] or [17, §6.5]).

A full triangulated subcategory $\mathcal{X}$ of $\underline{\mathrm{Mod}}\,\Lambda$ is called *localizing* if it is closed under forming coproducts in $\underline{\mathrm{Mod}}\,\Lambda$. A localizing subcategory $\mathcal{X}$ is called *smashing* if the inclusion $\mathcal{X} \hookrightarrow \underline{\mathrm{Mod}}\,\Lambda$ has a right adjoint which preserves coproducts. We say that a localizing subcategory $\mathcal{X}$ is *generated* by a class $\mathcal{C}$ of objects if there is no proper localizing subclass of $\mathcal{X}'$ of $\mathcal{X}$ such that $\mathcal{C} \subseteq \mathcal{X}'$. We refer to [18, 16] for a thorough discussion of these concepts. It follows that $\underline{\mathrm{Mod}}\,\Lambda$ is a *compactly generated* triangulated category, that is, $\underline{\mathrm{Mod}}\,\Lambda$ is generated, as a localizing class, by a set of compact objects.

The telescope conjecture studied in [18, 16] asserts that every smashing localizing subcategory of a compactly generated triangulated category is generated by a set of compact objects. Even though it is generally false as mentioned in the introduction, we can give an affirmative answer in a special case. Namely Theorem 16 together with results from [20] imply that the conjecture holds for $\underline{\mathrm{Mod}}\,\Lambda$ where Λ is a selfinjective artin algebra with vanishing transfinite radical.

Theorem 19. *Let Λ be a selfinjective artin algebra such that* $\mathrm{rad}^*_\Lambda = 0$. *Let $\mathcal{X}$ be a smashing localizing subcategory of* $\underline{\mathrm{Mod}}\,\Lambda$. *Then $\mathcal{X}$ is generated by a set of finitely generated Λ-modules.*

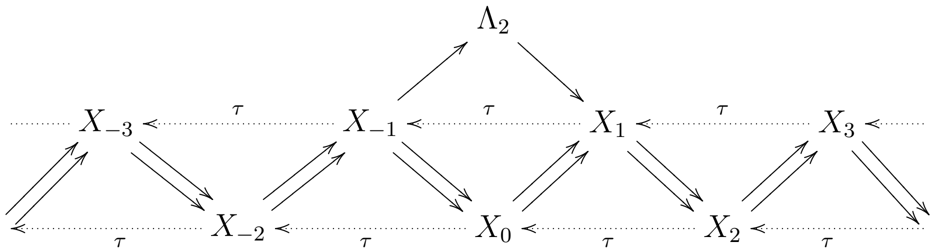
Proof. We know that Conjecture (B) (see page 72) holds for Λ by Theorem 16. Hence also Conjecture (A) holds by the discussion in Section 1. The rest follows immediately from [20, Corollary 7.7]. □

6. Examples

We conclude with some examples of particular representation-infinite selfinjective algebras with vanishing transfinite radical.

Example 20. The simplest example is probably the exterior algebra of a 2-dimensional vector space over an algebraically closed field. That is, $\Lambda_2 = k\langle x, y\rangle/(x^2, y^2, xy + yx)$. It is a special biserial algebra in the sense of [30] and it has, up to rotation equivalence and inverse, only one band xy^{-1}. In particular, Λ_2 is domestic and we have exactly one one-parametric family of indecomposable modules in each even dimension. For example, we have $M_{(a:b)} = \Lambda_2/\Lambda_2(ax + by)$ for each $(a:b) \in \mathbb{P}^1(k)$ in dimension 2. Thus, $\operatorname{rad}^*_{\Lambda_2} = 0$ by [27, Theorem 2].

With a little more effort, we can classify all smashing localizations and all hereditary cotorsion pairs with the right hand class closed under unions of chains. Using the representation theory of special biserial algeras, one can readily compute the Auslander-Reiten quiver of Λ_2. It consists of a family $(\mathcal{T}_{(a:b)} \mid (a:b) \in \mathbb{P}^1(k))$ of homogeneous tubes, the corresponding quasi-simples being precisely the modules $M_{(a:b)}$ above. In addition, there is one more component, which we denote by $\mathcal{C}$, of the form



where X_0 is the unique simple module, and X_n and X_{-n} are the string modules corresponding to the strings $(yx^{-1})^n$ and $(x^{-1}y)^n$, respectively. In particular, $\dim_k X_n = 2\cdot|n|+1$. It is easy to compute that $\Omega^-(X_n) \cong X_{n+1}$ and $\Omega^-(M) = M$ for each indecomposable finite dimensional module in a tube. This describes the restriction of the suspension functor $\Sigma : \underline{\operatorname{Mod}}\,\Lambda_2 \to \underline{\operatorname{Mod}}\,\Lambda_2$ to $\underline{\operatorname{mod}}\,\Lambda_2$.

We recall that a full triangulated subcategory $\mathcal{X}_0$ of $\underline{\operatorname{mod}}\,\Lambda_2$ is called *thick* if it is closed under direct summands. There is a bijective correspondence between thick subcategories $\mathcal{X}_0$ of $\underline{\operatorname{mod}}\,\Lambda_2$ and localizing subcategories $\mathcal{X}$ of $\underline{\operatorname{Mod}}\,\Lambda_2$ generated by a set of compact objects. More precisely, if $\mathcal{X}$ is generated by $\mathcal{X}_0 \subseteq \underline{\operatorname{mod}}\,\Lambda_2$ and $\mathcal{X}_0$ is thick, then $\mathcal{X} \cap \underline{\operatorname{mod}}\,\Lambda_2 = \mathcal{X}_0$, [21, 2.2]. It is clear that each thick subcategory is uniquely determined by its indecomposable objects.

We will now describe thick subcategories of $\underline{\mathrm{mod}}\,\Lambda_2$. It is straightforward to check that if an indecomposable non-injective module $M \in \mathrm{mod}\,\Lambda_2$ is contained in a thick subcategory $\mathcal{X}_0$, then all modules in the same component of the Auslander-Reiten quiver are in $\mathcal{X}_0$, too. On the other hand, if $\mathcal{T}_p$ is a tube for some $p \in \mathbb{P}^1(k)$, then one can check that in $\mathrm{mod}\,\Lambda_2$, the additive closure of $\mathcal{T}_p \cup \{\Lambda_2\}$ equals to

$$\{X \in \mathrm{mod}\,\Lambda_2 \mid \underline{\mathrm{Hom}}_{\Lambda_2}(X, \mathcal{T}_q) = 0 = \underline{\mathrm{Hom}}_{\Lambda_2}(\mathcal{T}_q, X) \ \ (\forall q \in \mathbb{P}^1(k) \backslash \{p\})\}$$

Therefore, $\mathrm{add}(\mathcal{T}_p \cup \{\Lambda_2\})$ is closed under extensions, syzygies and cosyzygies in $\mathrm{mod}\,\Lambda_2$, and consequently $\mathrm{add}\mathcal{T}_p$ is thick in $\underline{\mathrm{mod}}\,\Lambda_2$. It is easy to see that $\underline{\mathrm{Hom}}_{\Lambda_2}(\mathcal{T}_p, \mathcal{T}_q) = 0$ for $p \neq q$, so the additive closure of any set of tubes is thick in $\underline{\mathrm{mod}}\,\Lambda_2$. Finally, there is an exact sequence $0 \to M \to X_m \to X_{m+1} \to 0$ for each $m < 0$ and each quasi-simple module M in a tube; hence a thick subcategory containing the component $\mathcal{C}$ contains all the tubes, too. When summarizing all the facts (and using Theorem 19), we obtain the following classification:

Proposition 21. *Let k be an algebraically closed field, $\Lambda_2 = k\langle x, y\rangle/(x^2, y^2, xy + yx)$, and $\mathcal{C}$ and $\mathcal{T}_p$, $p \in \mathbb{P}^1(k)$, be the components of the Auslander-Reiten quiver of Λ_2 as above. Then each smashing localizing class $\mathcal{X}$ in $\underline{\mathrm{Mod}}\,\Lambda_2$ is generated by $\mathcal{X}_0 = \mathcal{X} \cap \underline{\mathrm{mod}}\,\Lambda_2$, and the possible intersections $\mathcal{X}_0$ are classified as follows:*

(1) $\mathcal{X}_0 = 0$*; or*
(2) $\mathcal{X}_0$ *is the additive closure of* $\bigcup_{p \in P} \mathcal{T}_p$ *for some* $P \subseteq \mathbb{P}^1(k)$*; or*
(3) $\mathcal{X}_0 = \underline{\mathrm{mod}}\,\Lambda_2$.

In the same spirit, we can classify the hereditary cotorsion pairs $(\mathcal{A}, \mathcal{B})$ in $\mathrm{Mod}\,\Lambda_2$ such that $\mathcal{B}$ is closed under unions of chains. Recall that a subcategory $\mathcal{A}_0$ of $\mathrm{mod}\,\Lambda_2$ is called *resolving* if it contains Λ_2 and it is closed under extensions, kernels of epimorphisms and direct summands. There is a bijective correspondence between resolving subcategories $\mathcal{A}_0$ in $\mathrm{mod}\,\Lambda_2$ and hereditary cotorsion pairs $(\mathcal{A}, \mathcal{B})$ of finite type in $\mathrm{Mod}\,\Lambda_2$, [3, 2.5]. Note that if $\mathcal{A}_0$ is resolving and contains a module $X_m \in \mathcal{C}$, it must contain all X_z, $z \leq m$, and all tubes. On the other hand, it is not difficult to see that there is an exact sequence $0 \to X_n \to U \to X_{-k} \to 0$ with an indecomposable (string) module U from a tube for each $n, k > 0$. Hence $\mathcal{A}_0$ must contain all of $\mathcal{C}$, too. We will leave details of the following statement (using Theorem 16) for the reader:

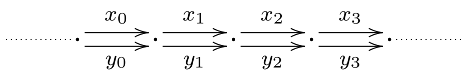
Proposition 22. *Let k be an algebraically closed field, $\Lambda_2 = k\langle x, y\rangle/(x^2, y^2, xy + yx)$, and $\mathcal{C}$ and $\mathcal{T}_p$, $p \in \mathbb{P}^1(k)$, be the components of the Auslander-Reiten quiver of Λ_2 as above. Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in $\mathrm{Mod}\,\Lambda_2$ such that $\mathcal{B}$ is closed under unions of chains, and let $\mathcal{A}_0 = \mathcal{A} \cap \mathrm{mod}\,\Lambda_2$. Then $\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_{\Lambda_2}(\mathcal{A}_0, -)$, and the possible classes $\mathcal{A}_0$ are classified as follows:*

(1) $\mathcal{A}_0 = \mathrm{add}\{\Lambda_2\}$*; or*

(2) $\mathcal{A}_0$ *is the additive closure of* $\{\Lambda_2\} \cup \bigcup_{p\in P} \mathcal{T}_p$ *for* $P \subseteq \mathbb{P}^1(k)$*; or*
(3) $\mathcal{A}_0 = \operatorname{mod} \Lambda_2$.

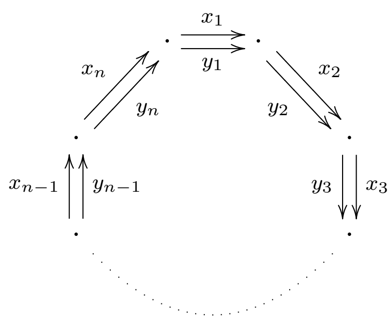
Example 23. A recipe for construction of more complicated examples is given in [15]. Let B be a representation-infinite tilted algebra of Euclidean type over an algebraically closed field and $\hat{B}$ be its repetitive algebra. Put $\Lambda = \hat{B}/G$ where G is an admissible infinite cyclic group of k-linear automorphisms of $\hat{B}$ (see [29, §1] for unexplained terminology). Then Λ is selfinjective and $\operatorname{rad}^*_\Lambda = 0$ by the main result of [15].

We illustrate the construction on $B = k(\cdot \rightrightarrows \cdot)$, the Kronecker algebra. The repetitive algebra $\hat{B}$ is then given by the following infinite quiver with relations:



$$x_{i+1}x_i - y_{i+1}y_i = 0, \quad x_{i+1}y_i = 0, \quad y_{i+1}x_i = 0 \qquad \text{for each } i \in \mathbb{Z}.$$

Let $n \geq 1$ and $\bar{q} = (q_1, \dots, q_n)$ be an n-tuple of non-zero elements of k. It is not difficult to see that we get the algebra $\Lambda_{n,\bar{q}}$ described by the quiver and relations below as $\hat{B}/G$ for a suitable G:



$$x_{i+1}y_i + q_i y_{i+1}x_i = 0, \; x_{i+1}x_i = 0, \; y_{i+1}y_i = 0 \quad \text{for each } i \in \{1, 2, \dots, n\}.$$

The addition in indicies of arrows above is considered modulo n. It is easy to see that $\Lambda_{n,\bar{q}}$ is special biserial and there are exactly n one-parametric families of indecomposable $\Lambda_{n,\bar{q}}$-modules in each even dimension. They correspond to the bands $x_i y_i^{-1}$. In fact, if $n = 1$ and $q_1 = 1$, we get precisely the exterior algebra on a 2-dimensional space.

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III.

THE TELESCOPE CONJECTURE FOR HEREDITARY RINGS VIA EXT-ORTHOGONAL PAIRS

(JOINT WITH HENNING KRAUSE)

Abstract

For the module category of a hereditary ring, the Ext-orthogonal pairs of subcategories are studied. For each Ext-orthogonal pair that is generated by a single module, a 5-term exact sequence is constructed. The pairs of finite type are characterized and two consequences for the class of hereditary rings are established: homological epimorphisms and universal localizations coincide, and the telescope conjecture for the derived category holds true. However, we present examples showing that neither of these two statements is true in general for rings of global dimension 2.

This paper is a preprint available as arXiv:0810.1401.

THE TELESCOPE CONJECTURE FOR HEREDITARY RINGS VIA EXT-ORTHOGONAL PAIRS

HENNING KRAUSE AND JAN ŠŤOVÍČEK

Dedicated to Helmut Lenzing on the occasion of his 70th birthday.

Abstract. For the module category of a hereditary ring, the Ext-orthogonal pairs of subcategories are studied. For each Ext-orthogonal pair that is generated by a single module, a 5-term exact sequence is constructed. The pairs of finite type are characterized and two consequences for the class of hereditary rings are established: homological epimorphisms and universal localizations coincide, and the telescope conjecture for the derived category holds true. However, we present examples showing that neither of these two statements is true in general for rings of global dimension 2.

1. Introduction

In this paper, we prove the telescope conjecture for the derived category of any hereditary ring. To achieve this, we study Ext-orthogonal pairs of subcategories for hereditary module categories.

The telescope conjecture for the derived category of a module category is also called smashing conjecture. It is the analogue of the telescope conjecture from stable homotopy theory which is due to Bousfield and Ravenel [6, 28]. In each case one deals with a compactly generated triangulated category. The conjecture then claims that a localizing subcategory is generated by compact objects provided it is smashing, that is, the localizing subcategory arises as the kernel of a localization functor that preserves arbitrary coproducts [24]. In this general form, the telescope conjecture seems to be wide open. For the stable homotopy category, we refer to the work of Mahowald, Ravenel, and Shick [22] for more details. In our case, the conjecture takes the following form and is proved in §7:

Theorem A. *Let A be a hereditary ring. For a localizing subcategory $\mathcal{C}$ of $\mathbf{D}(\operatorname{Mod} A)$ the following conditions are equivalent:*

1. *There exists a localization functor $L\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ that preserves coproducts and such that $\mathcal{C} = \operatorname{Ker} L$.*
2. *The localizing subcategory $\mathcal{C}$ is generated by perfect complexes.*

Version from August 13, 2009.
The second author is supported by the Research Council of Norway through the Storforsk-project "Homological and geometric methods in algebra".

For the derived category of a module category, only two results seem to be known so far. Neeman proved the conjecture for the derived category of a commutative noetherian ring [25], essentially by classifying all localizing subcategories; see [16] for a treatment of this approach in the context of axiomatic stable homotopy theory. On the other hand, Keller gave an explicit example of a commutative ring where the conjecture does not hold [17]. In fact, an analysis of Keller's argument [18] shows that there are such examples having global dimension 2; see Example 7.8.

The approach for hereditary rings presented here is completely different from Neeman's. In particular, we are working in a non-commutative setting and without using any noetherianess assumption. The main idea here is to exploit the very close connection between the module category and the derived category in the hereditary case. Unfortunately, this approach cannot be extended directly even to global dimension 2, as mentioned above.

At a first glance, the telescope conjecture seems to be a rather abstract statement about unbounded derived categories. However in the context of a fixed hereditary ring, it turns out that smashing localizing subcategories are in bijective correspondence to various natural structures; see §8:

Theorem B. *For a hereditary ring A there are bijections between the following sets:*

1. *Extension closed abelian subcategories of* $\operatorname{Mod} A$ *that are closed under products and coproducts.*
2. *Extension closed abelian subcategories of* $\operatorname{mod} A$.
3. *Homological epimorphisms $A \to B$ (up to isomorphism).*
4. *Universal localizations $A \to B$ (up to isomorphism).*
5. *Localizing subcategories of* $\mathbf{D}(\operatorname{Mod} A)$ *that are closed under products.*
6. *Localization functors* $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ *preserving coproducts (up to natural isomorphism).*
7. *Thick subcategories of* $\mathbf{D}^b(\operatorname{mod} A)$.

This reveals that the telescope conjecture and its proof are related to interesting recent work by some other authors. In [34], Schofield describes for any hereditary ring its universal localizations in terms of appropriate subcategories of finitely presented modules. This is a consequence of the present work since we show that homological epimorphisms and universal localizations coincide for any hereditary ring; see §6. However, as we mention at the end of §6, the identification between homological epimorphisms and universal localizations also fails already for rings of global dimension 2.

In [27], Nicolás and Saorín establish for a differential graded algebra a correspondence between recollements for its derived category and differential graded homological epimorphisms. This correspondence specializes for a hereditary ring to the above mentioned bijection between smashing localizing subcategories and homological epimorphisms.

The link between the structures mentioned in Theorem B is provided by so-called Ext-orthogonal pairs. This concept seems to be new, but it is based on the notion of a perpendicular category which is one of the fundamental tools for studying hereditary categories arising in representation theory [32, 13].

Given any abelian category $\mathcal{A}$, we call a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories *Ext-orthogonal* if $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal to each other with respect to the bifunctor $\coprod_{n\geq 0} \operatorname{Ext}^n_{\mathcal{A}}(-,-)$. This concept is the analogue of a *torsion pair* and a *cotorsion pair* where one considers instead the bifunctors $\operatorname{Hom}_{\mathcal{A}}(-,-)$ and $\coprod_{n>0} \operatorname{Ext}^n_{\mathcal{A}}(-,-)$, respectively [9, 30].

Torsion and cotorsion pairs are most interesting when they are *complete*. For a torsion pair this means that each object M in $\mathcal{A}$ admits a short exact sequence $0 \to X_M \to M \to Y^M \to 0$ with $X_M \in \mathcal{X}$ and $Y^M \in \mathcal{Y}$. In the second case this means that each object M admits short exact sequences $0 \to Y_M \to X_M \to M \to 0$ and $0 \to M \to Y^M \to X^M \to 0$ with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

It turns out that there is also a reasonable notion of completeness for Ext-orthogonal pairs. In that case each object M in $\mathcal{A}$ admits a 5-term exact sequence

$$0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. This notion of a complete Ext-orthogonal pair is meaningful also for non-hereditary module categories, see Example 4.5.

In this work, however, we study Ext-orthogonal pairs mainly for the module category of a hereditary ring. As already mentioned, this assumption implies a close connection between the module category and its derived category, which we exploit in both directions. We use Bousfield localization functors which exist for the derived category to establish the completeness of certain Ext-orthogonal pairs for the module category; see §2. On the other hand, we are able to prove the telescope conjecture for the derived category by showing first a similar result for Ext-orthogonal pairs; see §5 and §7.

Specific examples of Ext-orthogonal pairs arise in the representation theory of finite dimensional algebras via perpendicular categories; see §4. Note that a perpendicular category is always a part of an Ext-othogonal pair. Schofield introduced perpendicular categories for representations of quivers [32] and this fits into our set-up because the path algebra of any quiver is hereditary. In fact, the concept of a perpendicular category is fundamental for studying hereditary categories arising in representation theory [13]. It is therefore somewhat surprising that the 5-term exact sequence for a complete Ext-orthogonal pair seems to appear for the first time in this work.

Acknowledgements. The authors would like to thank Lidia Angeleri Hügel and Manolo Saorín for helpful discussions concerning this work.

2. Ext-orthogonal pairs

Let $\mathcal{A}$ be an abelian category. Given a pair of objects $X, Y \in \mathcal{A}$, set

$$\operatorname{Ext}^*_{\mathcal{A}}(X,Y) = \coprod_{n\in\mathbb{Z}} \operatorname{Ext}^n_{\mathcal{A}}(X,Y).$$

For a subcategory $\mathcal{C}$ of $\mathcal{A}$ we consider its full Ext-orthogonal subcategories

$$\begin{aligned} {}^{\perp}\mathcal{C} &= \{X \in \mathcal{A} \mid \operatorname{Ext}^*_{\mathcal{A}}(X,C) = 0 \text{ for all } C \in \mathcal{C}\}, \\ \mathcal{C}^{\perp} &= \{Y \in \mathcal{A} \mid \operatorname{Ext}^*_{\mathcal{A}}(C,Y) = 0 \text{ for all } C \in \mathcal{C}\}. \end{aligned}$$

If $\mathcal{C} = \{X\}$ is a singleton, we write ${}^{\perp}X$ instead of ${}^{\perp}\{X\}$, and similarly with $X^{\perp}$.

Definition 2.1. An *Ext-orthogonal pair* for $\mathcal{A}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories such that $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. An Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if there exists for each object $M \in \mathcal{A}$ an exact sequence

$$\varepsilon_M\colon\ 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. The pair $(\mathcal{X}, \mathcal{Y})$ is *generated* by a subcategory $\mathcal{C}$ of $\mathcal{A}$ if $\mathcal{Y} = \mathcal{C}^{\perp}$.

The definition can be extended to the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ if we put for each pair of complexes $X, Y \in \mathbf{D}(\mathcal{A})$ and $n \in \mathbb{Z}$

$$\operatorname{Ext}^n_{\mathcal{A}}(X,Y) = \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y[n]).$$

Thus an *Ext-orthogonal pair* for $\mathbf{D}(\mathcal{A})$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathbf{D}(\mathcal{A})$ such that $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$.

Recall that an *abelian subcategory* of $\mathcal{A}$ is a full subcategory $\mathcal{C}$ such that the category $\mathcal{C}$ is abelian and the inclusion functor $\mathcal{C} \to \mathcal{A}$ is exact. Suppose $\mathcal{A}$ is *hereditary*, that is, $\operatorname{Ext}^n_{\mathcal{A}}(-,-)$ vanishes for all $n > 1$. Then a simple calculation shows that for any subcategory $\mathcal{C}$ of $\mathcal{A}$, the subcategories $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ are extension closed abelian subcategories; see [13, Proposition 1.1].

The following result establishes the completeness for certain Ext-orthogonal pairs. Recall that an abelian category is a *Grothendieck category* if it has a set of generators and admits colimits that are exact when taken over filtered categories.

Theorem 2.2. *Let $\mathcal{A}$ be a hereditary abelian Grothendieck category and X an object in $\mathcal{A}$. Set $\mathcal{Y} = X^{\perp}$ and let $\mathcal{X}$ denote the smallest extension closed abelian subcategory of $\mathcal{A}$ that is closed under taking coproducts and contains X. Then $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair for $\mathcal{A}$. Thus there exists for each object $M \in \mathcal{A}$ an exact sequence*

$$0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. This sequence is natural and induces bijections $\operatorname{Hom}_{\mathcal{A}}(X, X_M) \to \operatorname{Hom}_{\mathcal{A}}(X, M)$ and $\operatorname{Hom}_{\mathcal{A}}(Y^M, Y) \to \operatorname{Hom}_{\mathcal{A}}(M, Y)$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

The proof uses derived categories and Bousfield localization functors. Thus we need to collect some basic facts about hereditary abelian categories and their derived categories.

The derived category of a hereditary abelian category. Let $\mathcal{A}$ be a hereditary abelian category and let $\mathbf{D}(\mathcal{A})$ denote its derived category. We assume that $\mathcal{A}$ admits coproducts and that the coproduct of any set of exact sequences is again exact. Thus the category $\mathbf{D}(\mathcal{A})$ admits coproducts, and for each integer n these coproducts are preserved by the functor $H^n \colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ which takes a complex to its cohomology in degree n.

It is well-known that each complex is quasi-isomorphic to its cohomology. That is:

Lemma 2.3. *Given a complex X in $\mathbf{D}(\mathcal{A})$, there are (non-canonical) isomorphisms*

$$\coprod_{n \in \mathbb{Z}} (H^n X)[-n] \cong X \cong \prod_{n \in \mathbb{Z}} (H^n X)[-n].$$

Proof. See for instance [19, §1.6]. □

A full subcategory $\mathcal{C}$ of $\mathbf{D}(\mathcal{A})$ is called *thick* if it is a triangulated subcategory which is, in addition, closed under taking direct summands. A thick subcategory is *localizing* if it is closed under taking coproducts. Note that for each subcategory $\mathcal{C}$ the subcategories $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ are thick.

To a subcategory $\mathcal{C}$ of $\mathbf{D}(\mathcal{A})$ we assign the full subcategory

$$H^0\mathcal{C} = \{M \in \mathcal{A} \mid M = H^0 X \text{ for some } X \in \mathcal{C}\},$$

and given a subcategory $\mathcal{X}$ of $\mathcal{A}$, we define the full subcategory

$$\mathbf{D}_{\mathcal{X}}(\mathcal{A}) = \{X \in \mathbf{D}(\mathcal{A}) \mid H^n X \in \mathcal{X} \text{ for all } n \in \mathbb{Z}\}.$$

Both assignments induce mutually inverse bijections between appropriate subcategories. This is a useful fact which we recall from [7, Theorem 6.1].

Proposition 2.4. *The functor $H^0\colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ induces a bijection between the localizing subcategories of $\mathbf{D}(\mathcal{A})$ and the extension closed abelian subcategories of $\mathcal{A}$ that are closed under coproducts. The inverse map sends a subcategory $\mathcal{X}$ of $\mathcal{A}$ to $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$.* □

Remark 2.5. The bijection in Proposition 2.4 has an analogue for thick subcategories. Given any hereditary abelian category $\mathcal{B}$, the functor $H^0\colon \mathbf{D}^b(\mathcal{B}) \to \mathcal{B}$ induces a bijection between the thick subcategories of $\mathbf{D}^b(\mathcal{B})$ and the extension closed abelian subcategories of $\mathcal{B}$; see [7, Theorem 5.1].

Next we extend these maps to bijections between Ext-orthogonal pairs.

Proposition 2.6. *The functor $H^0\colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ induces a bijection between the Ext-orthogonal pairs for $\mathbf{D}(\mathcal{A})$ and the Ext-orthogonal pairs for $\mathcal{A}$. The inverse map sends a pair $(\mathcal{X}, \mathcal{Y})$ for $\mathcal{A}$ to $(\mathbf{D}_{\mathcal{X}}(\mathcal{A}), \mathbf{D}_{\mathcal{Y}}(\mathcal{A}))$.*

Proof. First observe that for each pair of complexes $X, Y \in \mathbf{D}(\mathcal{A})$, we have $\operatorname{Ext}^*_{\mathcal{A}}(X, Y) = 0$ if and only if $\operatorname{Ext}^*_{\mathcal{A}}(H^pX, H^qY) = 0$ for all $p, q \in \mathbb{Z}$. This is a consequence of Lemma 2.3. It follows that H^0 and its inverse send Ext-orthogonal pairs to Ext-orthogonal pairs. Each Ext-orthogonal pair is determined by its first half, and therefore an application of Proposition 2.4 shows that both maps are mutually inverse. □

Localization functors. Let $\mathcal{T}$ be a triangulated category. A *localization functor* $L\colon \mathcal{T} \to \mathcal{T}$ is an exact functor that admits a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{T}} \to L$ such that $L\eta_X$ is an isomorphism and $L\eta_X = \eta_{LX}$ for all objects $X \in \mathcal{T}$. Basic facts about localization functors one finds, for example, in [4, §3].

Proposition 2.7. *Let $\mathcal{A}$ be a hereditary abelian category. For a full subcategory $\mathcal{X}$ of $\mathcal{A}$ the following are equivalent.*

1. *There exists a localization functor $L\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ such that $\operatorname{Ker} L = \mathbf{D}_{\mathcal{X}}(\mathcal{A})$.*
2. *There exists a complete Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for $\mathcal{A}$.*

Proof. (1) ⇒ (2): The kernel $\operatorname{Ker} L$ and the essential image $\operatorname{Im} L$ of a localization functor L form an Ext-orthogonal pair for $\mathbf{D}(\mathcal{A})$; see for instance [4, Lemma 3.3]. Then it follows from Proposition 2.6 that the pair $(\mathcal{X}, \mathcal{Y}) = (H^0 \operatorname{Ker} L, H^0 \operatorname{Im} L)$ is Ext-orthogonal for $\mathcal{A}$.

The localization functor L comes equipped with a natural transformation $\eta\colon \mathrm{Id}_{\mathbf{D}(\mathcal{A})} \to L$, and for each complex M we complete the morphism $\eta_M\colon M \to LM$ to an exact triangle

$$\varGamma M \to M \to LM \to \varGamma M[1].$$

Note that $\varGamma M \in \operatorname{Ker} L$ and $LM \in \operatorname{Im} L$ since $L\eta_M$ is an isomorphism and L is exact. Now suppose that M is concentrated in degree zero.

Applying H^0 to this triangle yields an exact sequence

$$0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

(2) $\Rightarrow$ (1): Let $(\mathcal{X}, \mathcal{Y})$ be an Ext-orthogonal pair for $\mathcal{A}$. This pair induces an Ext-orthogonal pair $(\mathbf{D}_{\mathcal{X}}(\mathcal{A}), \mathbf{D}_{\mathcal{Y}}(\mathcal{A}))$ for $\mathbf{D}(\mathcal{A})$ by Proposition 2.6. In order to construct a localization functor $L\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ such that $\operatorname{Ker} L = \mathbf{D}_{\mathcal{X}}(\mathcal{A})$, it is sufficient to construct for each object M in $\mathbf{D}(\mathcal{A})$ an exact triangle $X \to M \to Y \to X[1]$ with $X \in \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ and $Y \in \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$. Then one defines $LM = Y$ and the morphism $M \to Y$ induces a natural transformation $\eta\colon \mathrm{Id}_{\mathbf{D}(\mathcal{A})} \to L$ having the required properties. In view of Lemma 2.3 it is sufficient to assume that M is a complex concentrated in degree zero.

Suppose that M admits an approximation sequence

$$\varepsilon_M\colon\ 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. Let M' denote the image of $X_M \to M$ and M'' the image of $M \to Y^M$. Then ε_M induces the following three exact sequences

$$\begin{aligned}
\alpha_M\colon&\ 0 \to M' \to M \to M'' \to 0,\\
\beta_M\colon&\ 0 \to Y_M \to X_M \to M' \to 0,\\
\gamma_M\colon&\ 0 \to M'' \to Y^M \to X^M \to 0.
\end{aligned}$$

In $\mathbf{D}(\mathcal{A})$ these three exact sequence give rise to the following commuting square

$$\begin{array}{ccc}
X^M[-2] & \xrightarrow{\gamma_M} & M''[-1] \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha_M} \\
X_M & \xrightarrow{\bar\beta_M} & M'
\end{array}$$

where $\bar\beta_M$ is the second morphism in β_M. Commutativity of the diagram is clear since $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(U[-2], V) = 0$ for any $U, V \in \mathcal{A}$. An application of the octahedral axiom shows that this square can be extended as follows to a diagram where each row and each column is an exact triangle.

$$\begin{array}{ccccccc}
X^M[-2] & \longrightarrow & M''[-1] & \longrightarrow & Y^M[-1] & \longrightarrow & X^M[-1] \\
\downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
X_M & \longrightarrow & M' & \longrightarrow & Y_M[1] & \longrightarrow & X_M[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_M \oplus X^M[-1] & \longrightarrow & M & \longrightarrow & Y_M[1] \oplus Y^M & \longrightarrow & X_M[1] \oplus X^M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^M[-1] & \longrightarrow & M'' & \longrightarrow & Y^M & \longrightarrow & X^M
\end{array}$$

The first and third column are split exact triangles, and this explains the objects appearing in the third row. In particular, this yields the desired exact triangle $X \to M \to Y \to X[1]$ with $X \in \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ and $Y \in \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$. $\square$

Remark 2.8. The proof of the implication (2) ⇒ (1) comes as a special case of a more general result on the existence of exact triangles with a specified long exact sequence of cohomology objects. We refer to work of Neeman [23] for more details.

Next we formulate the functorial properties of the 5-term exact sequence constructed in Proposition 2.7.

Lemma 2.9. *Let $\mathcal{A}$ be an abelian category and $(\mathcal{X},\mathcal{Y})$ an Ext-orthogonal pair for $\mathcal{A}$. Suppose there is an exact sequence*

$$\varepsilon_M\colon\ 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

in $\mathcal{A}$ with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

(1) *The sequence ε_M induces for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ bijections* $\mathrm{Hom}_{\mathcal{A}}(X, X_M) \to \mathrm{Hom}_{\mathcal{A}}(X, M)$ *and* $\mathrm{Hom}_{\mathcal{A}}(Y^M, Y) \to \mathrm{Hom}_{\mathcal{A}}(M, Y)$.
(2) *Let $\varepsilon_N\colon\ 0 \to Y_N \to X_N \to N \to Y^N \to X^N \to 0$ be an exact sequence in $\mathcal{A}$ with $X_N, X^N \in \mathcal{X}$ and $Y_N, Y^N \in \mathcal{Y}$. Then each morphism $M \to N$ extends uniquely to a morphism $\varepsilon_M \to \varepsilon_N$ of exact sequences.*
(3) *Any exact sequence $0 \to Y' \to X' \to M \to Y'' \to X'' \to 0$ in $\mathcal{A}$ with $X', X'' \in \mathcal{X}$ and $Y', Y'' \in \mathcal{Y}$ is uniquely isomorphic to ε_M.*

Proof. We prove part (1). Then parts (2) and (3) are immediate consequences.

Fix an object $X \in \mathcal{X}$. The map $\mu\colon \mathrm{Hom}_{\mathcal{A}}(X, X_M) \to \mathrm{Hom}_{\mathcal{A}}(X, M)$ is injective because $\mathrm{Hom}_{\mathcal{A}}(X, Y_M) = 0$. Any morphism $X \to M$ factors through the kernel M' of $M \to Y^M$ since $\mathrm{Hom}_{\mathcal{A}}(X, Y^M) = 0$. The induced morphism $X \to M'$ factors through $X_M \to M'$ since

$\operatorname{Ext}^1_{\mathcal{A}}(X, Y_M) = 0$. Thus μ is surjective. The argument for the other map $\operatorname{Hom}_{\mathcal{A}}(Y^M, Y) \to \operatorname{Hom}_{\mathcal{A}}(M, Y)$ is dual. □

Ext-orthogonal pairs for Grothendieck categories. Now we give the proof of Theorem 2.2. The basic idea is to establish a localization functor for $\mathbf{D}(\mathcal{A})$ and to derive the exact approximation sequence in $\mathcal{A}$ by taking the cohomology of some appropriate exact triangle as in Proposition 2.7.

Proof of Theorem 2.2. Let $\mathcal{X}$ denote the smallest extension closed abelian subcategory of $\mathcal{A}$ that contains X and is closed under coproducts. Then Proposition 2.4 implies that $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ is the smallest localizing subcategory of $\mathbf{D}(\mathcal{A})$ containing X. Thus there exists a localization functor $L\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ with $\operatorname{Ker} L = \mathbf{D}_{\mathcal{X}}(\mathcal{A})$. This is a result which goes back to Bousfield's work in algebraic topology, [6]. In the context of derived categories we refer to [2, Theorem 5.7]. Now apply Proposition 2.7 to get the 5-term exact sequence for each object M in $\mathcal{A}$. The properties of this sequence follow from Lemma 2.9. □

Remark 2.10. We do not know an example of an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for a hereditary abelian Grothendieck category such that the pair $(\mathcal{X}, \mathcal{Y})$ is not complete.

Ext-orthogonal pairs naturally arise also for non-hereditary abelian categories. Here we mention one such class of examples, but we do not know whether or when exactly they are complete:

Example 2.11. Let $\mathcal{A}$ be any abelian Grothendieck category and $\mathcal{X}$ a *localizing subcategory*. That is, $\mathcal{X}$ is a full subcategory closed under taking coproducts and such that for any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$ we have $M \in \mathcal{X}$ if and only if $M', M'' \in \mathcal{X}$. Set $\mathcal{Y} = \mathcal{X}^{\perp}$ and let $\mathcal{Y}_{\mathrm{inj}}$ denote the full subcategory of injective objects of $\mathcal{A}$ contained in $\mathcal{Y}$. Then $\mathcal{X} = {}^{\perp}\mathcal{Y}_{\mathrm{inj}}$ and therefore $(\mathcal{X}, \mathcal{Y})$ is an Ext-orthogonal pair for $\mathcal{A}$; see [11, III.4] for details.

Torsion and cotorsion pairs. We also sketch an interpretation of an Ext-orthogonal pair in terms of torsion and cotorsion pairs. Here, a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{A}$ is called a *torsion pair* if $\mathcal{U}$ and $\mathcal{V}$ are orthogonal to each other with respect to $\operatorname{Hom}_{\mathcal{A}}(-,-)$. Analogously, a pair of full subcategories is a *cotorsion pair* if both categories are orthogonal to each other with respect to $\coprod_{n>0} \operatorname{Ext}^n_{\mathcal{A}}(-,-)$.

Let $\mathcal{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair. The subcategory $\mathcal{X}$ generates a torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ and a cotorsion pair $(\mathcal{X}_1, \mathcal{Y}_1)$ for $\mathcal{A}$, if one defines the corresponding full subcategories of $\mathcal{A}$ as follows:

$$\begin{aligned}
\mathcal{Y}_0 &= \{Y \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(X,Y) = 0 \text{ for all } X \in \mathcal{X}\},\\
\mathcal{X}_0 &= \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(X,Y) = 0 \text{ for all } Y \in \mathcal{Y}_0\},\\
\mathcal{Y}_1 &= \{Y \in \mathcal{A} \mid \operatorname{Ext}^n_{\mathcal{A}}(X,Y) = 0 \text{ for all } X \in \mathcal{X},\ n > 0\},\\
\mathcal{X}_1 &= \{X \in \mathcal{A} \mid \operatorname{Ext}^n_{\mathcal{A}}(X,Y) = 0 \text{ for all } Y \in \mathcal{Y}_1,\ n > 0\}.
\end{aligned}$$

Note that $\mathcal{X} = \mathcal{X}_0 \cap \mathcal{X}_1$ and $\mathcal{Y} = \mathcal{Y}_0 \cap \mathcal{Y}_1$. In particular, one recovers the pair $(\mathcal{X}, \mathcal{Y})$ from $(\mathcal{X}_0, \mathcal{Y}_0)$ and $(\mathcal{X}_1, \mathcal{Y}_1)$.

Suppose an object $M \in \mathcal{A}$ admits an approximation sequence

$$\varepsilon_M \colon\ 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. We give the following interpretation of this sequence. Let M' denote the image of $X_M \to M$ and M'' the image of $M \to Y^M$. Then there are three short exact sequences:

$$\begin{aligned}
\alpha_M &\colon\ 0 \to M' \to M \to M'' \to 0,\\
\beta_M &\colon\ 0 \to Y_M \to X_M \to M' \to 0,\\
\gamma_M &\colon\ 0 \to M'' \to Y^M \to X^M \to 0.
\end{aligned}$$

The sequence α_M is the approximation sequence of M with respect to the torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$, that is, $M' \in \mathcal{X}_0$ and $M'' \in \mathcal{Y}_0$. On the other hand, β_M and γ_M are approximation sequences of M' and M'' respectively, with respect to the cotorsion pair $(\mathcal{X}_1, \mathcal{Y}_1)$, that is, $X_M, X^M \in \mathcal{X}_1$ and $Y_M, Y^M \in \mathcal{Y}_1$. Thus the 5-term exact sequence ε_M is obtained by splicing together three short exact approximation sequences.

Suppose finally that the Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ is complete. It is not hard to see that then the associated torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ has an explicit description: we have $\mathcal{X}_0 = \operatorname{Fac}\mathcal{X}$ and $\mathcal{Y}_0 = \operatorname{Sub}\mathcal{Y}$, where

$$\operatorname{Fac}\mathcal{X} = \{X/U \mid U \subseteq X,\ X \in \mathcal{X}\} \quad\text{and}\quad \operatorname{Sub}\mathcal{Y} = \{U \mid U \subseteq Y,\ Y \in \mathcal{Y}\}.$$

3. Homological epimorphisms

From now on we will study Ext-orthogonal pairs only for module categories. Thus we fix a ring A and denote by $\operatorname{Mod} A$ the category of (right) A-modules. The full subcategory formed by all finitely presented A-modules is denoted by $\operatorname{mod} A$.

Most of our results require the ring A to be (right) *hereditary.* This means the category of A-modules is hereditary, that is, $\operatorname{Ext}^n_A(-,-)$ vanishes for all $n > 1$.

We are going to show that Ext-orthogonal pairs for module categories over hereditary rings are closely related to homological epimorphisms. Recall that a ring homomorphism $A \to B$ is a *homological epimorphism* if

$$B \otimes_A B \cong B \quad\text{and}\quad \operatorname{Tor}^A_n(B,B) = 0 \quad\text{for all}\quad n > 0,$$

or equivalently, if restriction induces isomorphisms

$$\operatorname{Ext}^*_B(X,Y) \xrightarrow{\sim} \operatorname{Ext}^*_A(X,Y)$$

for all B-modules X, Y; see [13] for details. The first observation is that every homological epimorphism naturally induces two complete Ext-orthogonal pairs:

Proposition 3.1. *Let A be a hereditary ring and $f\colon A \to B$ a homological epimorphism. Denote by $\mathcal{Y}$ the category of A-modules which are restrictions of modules over B. Set $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y}^{\perp} = \mathcal{Z}$. Then $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{Y},\mathcal{Z})$ are complete Ext-orthogonal pairs for* $\operatorname{Mod} A$ *with $\mathcal{Y} = (\operatorname{Ker} f \oplus \operatorname{Coker} f)^{\perp}$ and $\mathcal{Z} = B^{\perp}$.*

Proof. We wish to apply Theorem 2.2 which provides a construction for complete Ext-orthogonal pairs.

First observe that $\mathcal{Y}$ is the smallest extension closed abelian subcategory of $\operatorname{Mod} A$ closed under coproducts and containing B. This yields $\mathcal{Z} = B^{\perp}$.

Next we show that $\mathcal{Y} = (\operatorname{Ker} f \oplus \operatorname{Coker} f)^{\perp}$. In fact, an A-module Y is the restriction of a B-module if and only if f induces an isomorphism $\operatorname{Hom}_A(B,Y) \to \operatorname{Hom}_A(A,Y)$. Using the assumptions on A and f, a simple calculation shows that this implies $\mathcal{Y} = (\operatorname{Ker} f \oplus \operatorname{Coker} f)^{\perp}$.

It remains to apply Theorem 2.2. Thus $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{Y},\mathcal{Z})$ are complete Ext-orthogonal pairs. □

Now we use a crucial theorem of Gabriel and de la Peña. It identifies, only by their closure properties, the full subcategories of a module category $\operatorname{Mod} A$ that arise as the images of the restriction functors $\operatorname{Mod} B \to \operatorname{Mod} A$ for ring epimorphisms $A \to B$. In our version, we identify in a similar way the essential images of the restriction functors of homological epimorphisms, provided A is hereditary.

Proposition 3.2. *Let A be a hereditary ring and $\mathcal{Y}$ an extension closed abelian subcategory of* $\operatorname{Mod} A$ *that is closed under taking products and coproducts. Then there exists a homological epimorphism $f\colon A \to B$ such that the restriction functor* $\operatorname{Mod} B \to \operatorname{Mod} A$ *induces an equivalence* $\operatorname{Mod} B \xrightarrow{\sim} \mathcal{Y}$.

Proof. It follows from [12, Theorem 1.2] that there exists a ring epimorphism $f\colon A \to B$ such that the restriction functor $\operatorname{Mod} B \to \operatorname{Mod} A$ induces an equivalence $\operatorname{Mod} B \xrightarrow{\sim} \mathcal{Y}$. To be more specific, one constructs a left adjoint $F\colon \operatorname{Mod} A \to \mathcal{Y}$ for the inclusion $\mathcal{Y} \to \operatorname{Mod} A$. Then FA is a small projective generator for $\mathcal{Y}$, because A has this property for $\operatorname{Mod} A$ and the inclusion of $\mathcal{Y}$ is an exact functor that preserves coproducts. Thus one takes for f the induced map $A \cong \operatorname{End}_A(A) \to \operatorname{End}_A(FA)$.

We claim that restriction via f induces an isomorphism

$$\operatorname{Ext}^n_B(X,Y) \xrightarrow{\sim} \operatorname{Ext}^n_A(X,Y)$$

for all B-modules X, Y and all $n \geq 0$. This is clear for $n = 0, 1$ since $\mathcal{Y}$ is extension closed. On the other hand, the isomorphism for $n = 1$ implies that $\operatorname{Ext}^1_B(X, -)$ is right exact since A is hereditary. It follows that B is hereditary and $\operatorname{Ext}^n_B(-, -)$ vanishes for all $n > 1$. □

We get as an immediate consequence that any class $\mathcal{Y}$ satisfying the assumptions of Proposition 3.2 belongs to two complete cotorsion pairs. In order to obtain more information about the corresponding 5-term approximation sequences, we prefer, however, to postpone this corollary after the following lemma:

Lemma 3.3. *Let $A \to B$ be a homological epimorphism and denote by $\mathcal{Y}$ the category of A-modules which are restrictions of modules over B.*

1. *The functor $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex X to $X \otimes^{\mathbf{L}}_A B$ is a localization functor with essential image equal to $\mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A)$.*
2. *The functor $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex X to the cone (which is in this case functorial) of the natural morphism $\mathbf{R}\operatorname{Hom}_A(B, X) \to X$ is a localization functor with kernel equal to $\mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A)$.*

Proof. Restriction along $f \colon A \to B$ identifies $\operatorname{Mod} B$ with $\mathcal{Y}$. The functor induces an isomorphism

$$\operatorname{Ext}^n_B(X, Y) \xrightarrow{\sim} \operatorname{Ext}^n_A(X, Y)$$

for all B-modules X, Y and all $n \geq 0$, because f is a homological epimorphism. This isomorphism implies that the induced functor $f_* \colon \mathbf{D}(\operatorname{Mod} B) \to \mathbf{D}(\operatorname{Mod} A)$ is fully faithful with essential image $\mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A)$. Moreover, f_* is naturally isomorphic to both $\mathbf{R}\operatorname{Hom}_B({}_AB, -)$ and $- \otimes^{\mathbf{L}}_B B_A$. It follows that:

(1) The functor f_* admits a left adjoint $f^* = - \otimes^{\mathbf{L}}_A B$ and we therefore have a localization functor $L \colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex X to $f_* f^*(X)$; see [4, Lemma 3.1]. It remains to note that the essential images of L and f_* coincide.

(2) The functor f_* admits a right adjoint $f^! = \mathbf{R}\operatorname{Hom}_A(B, -)$ and we therefore have a colocalization functor $\varGamma \colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex X to $f_* f^!(X)$. Note that the adjunction morphism $\varGamma X \to X$ is an isomorphism if and only if X belongs to $\mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A)$. Completing $\varGamma X \to X$ to a triangle yields a well defined localization functor $\mathbf{D}(\operatorname{Mod} B) \to \mathbf{D}(\operatorname{Mod} A)$ with kernel $\mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A)$; see [4, Lemma 3.3]. □

Now we state the above mentioned immediate consequence of Propositions 3.1 and 3.2, but with an alternative and more explicit proof.

Corollary 3.4. *Let A be a hereditary ring and $\mathcal{Y}$ an extension closed abelian subcategory of* $\operatorname{Mod} A$ *that is closed under taking products and* *coproducts. Set $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Z} = \mathcal{Y}^{\perp}$. Then $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are both complete Ext-orthogonal pairs.*

Proof. There exists a homological epimorphism $f\colon A \to B$ such that restriction identifies $\operatorname{Mod} B$ with $\mathcal{Y}$; see Proposition 3.2. Then Lemma 3.3 produces two localization functors $L_1, L_2\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ with $\operatorname{Im} L_1 = \mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A) = \operatorname{Ker} L_2$. Thus

$$\begin{aligned} \operatorname{Ker} L_1 &= {}^{\perp}(\operatorname{Im} L_1) &&= \mathbf{D}_{\mathcal{X}}(\operatorname{Mod} A), \text{ and} \\ \operatorname{Im} L_2 &= (\operatorname{Ker} L_2)^{\perp} &&= \mathbf{D}_{\mathcal{Z}}(\operatorname{Mod} A), \end{aligned}$$

where in both cases the first equality follows from [4, Lemma 3.3] and the second from Proposition 2.6. It remains to apply Proposition 2.7 which yields in both cases for each A-module the desired 5-term exact sequence. □

Remark 3.5. The proof of Lemma 3.3 and Corollary 3.4 yields for any A-module M an explicit description of some terms of the 5-term exact sequence ε_M, using the homological epimorphism $A \to B$. In the first case, we have

$$\varepsilon_M\colon\ 0 \to \operatorname{Tor}_1^A(M, B) \to X_M \to M \to M \otimes_A B \to X^M \to 0,$$

and in the second case, we have

$$\varepsilon_M\colon\ 0 \to Z_M \to \operatorname{Hom}_A(B, M) \to M \to Z^M \to \operatorname{Ext}_A^1(B, M) \to 0.$$

We also mention another consequence of the above discussion, which is immediately implied by Corollary 3.4. It reflects the fact that given a homological epimorphism $A \to B$ and the fully faithful functor $f_*\colon \mathbf{D}(\operatorname{Mod} B) \to \mathbf{D}(\operatorname{Mod} A)$ having both a left and a right adjoint, there exists a corresponding recollement of the derived category $\mathbf{D}(\operatorname{Mod} A)$; see [20, §4.13].

Corollary 3.6. *Let A be a hereditary ring and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair for the category of A-modules.*

1. *There is an Ext-orthogonal pair $(\mathcal{W}, \mathcal{X})$ if and only if $\mathcal{X}$ is closed under products.*
2. *There is an Ext-orthogonal pair $(\mathcal{Y}, \mathcal{Z})$ if and only if $\mathcal{Y}$ is closed under coproducts.*

4. Examples

We present a number of examples of Ext-orthogonal pairs which illustrate the results of this work. The first example is classical and provides one of the motivations for studying perpendicular categories in representation theory of finite dimensional algebras. We refer to Schofield's work [33, 32] which contains some explicit calculations; see also [13, 14].

Example 4.1. Let A be a finite dimensional hereditary algebra over a field k and X a finite dimensional A-module. Then $X^\perp = \mathcal{Y}$ identifies via a homological epimorphism $A \to B$ with the category of modules over a k-algebra B and this yields a complete Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$. If X is *exceptional*, that is, $\mathrm{Ext}^1_A(X,X) = 0$, then B is finite dimensional (see the proposition below) and often can be constructed explicitly. We refer to [33] for particular examples. Note that in this case for each finite dimensional A-module M the corresponding 5-term exact sequence ε_M consists of finite dimensional modules. Moreover, the category $\mathcal{X}$ is equivalent to the module category of another finite dimensional algebra. We do not know of a criterion on X that characterizes the fact that B is finite dimensional; see however the following proposition.

Proposition 4.2. *Let A be a finite dimensional hereditary algebra over a field k and $(\mathcal{X}, \mathcal{Y})$ a complete Ext-orthogonal pair such that $\mathcal{Y}$ is closed under coproducts. Fix a homological epimorphism $A \to B$ inducing an equivalence $\mathrm{Mod}\, B \xrightarrow{\sim} \mathcal{Y}$. Then the following are equivalent.*

1. *There exists an exceptional module $X \in \mathrm{mod}\, A$ such that $\mathcal{Y} = X^\perp$.*
2. *The algebra B is finite dimensional over k.*
3. *For each $M \in \mathrm{mod}\, A$, the 5-term exact sequence ε_M belongs to $\mathrm{mod}\, A$.*

Proof. (1) $\Rightarrow$ (2): This follows, for example, from [13, Proposition 3.2].

(2) $\Rightarrow$ (3): This follows from Remark 3.5.

(3) $\Rightarrow$ (1): Let $\mathcal{X}_{\mathrm{fp}} = \mathcal{X} \cap \mathrm{mod}\, A$ and $\mathcal{Y}_{\mathrm{fp}} = \mathcal{Y} \cap \mathrm{mod}\, A$. The assumption on $(\mathcal{X}, \mathcal{Y})$ implies that $(\mathcal{X}_{\mathrm{fp}}, \mathcal{Y}_{\mathrm{fp}})$ is a complete Ext-orthogonal pair for $\mathrm{mod}\, A$. Moreover, every object in $\mathcal{X}$ is a filtered colimit of objects in $\mathcal{X}_{\mathrm{fp}}$. To see this, we first express X as a filtered colimit $\varinjlim M_i$ of finitely presented modules. Then, using the forthcoming Lemma 5.3(2), we see that $\varepsilon_X = \varinjlim \varepsilon_{M_i}$, from which it easily follows that $X \cong \varinjlim X_{M_i}$. Now choose an injective cogenerator Q in $\mathrm{mod}\, A$ and let $X = X_Q$ be the module from the 5-term exact sequence ε_Q. This module is the image of Q under a right adjoint of the inclusion $\mathcal{X}_{\mathrm{fp}} \to \mathrm{mod}\, A$. Note that a right adjoint of an exact functor preserves injectivity. It follows that X is an exceptional object and that $\mathcal{X}_{\mathrm{fp}}$ is the smallest extension closed abelian subcategory of $\mathrm{mod}\, A$ containing X. Thus $X^\perp = \mathcal{X}_{\mathrm{fp}}^\perp = \mathcal{X}^\perp = \mathcal{Y}$, using the fact that $\mathcal{X} = \varinjlim \mathcal{X}_{\mathrm{fp}}$. □

As a special case, any finitely generated projective module generates an Ext-orthogonal pair that can be described explicitly; see [13, §5]. For cyclic projective modules, this is discussed in more generality in the following example.

Example 4.3. Let A be a hereditary ring and $e^2 = e \in A$ an idempotent. Let $\mathcal{X}$ denote the category of A-modules M such that the natural map $Me \otimes_{eAe} eA \to M$ is an isomorphism, and let $\mathcal{Y} = eA^{\perp} = \{M \in \operatorname{Mod} A \mid Me = 0\}$. Thus $- \otimes_{eAe} eA$ identifies $\operatorname{Mod} eAe$ with $\mathcal{X}$ and restriction via $A \to A/AeA$ identifies $\operatorname{Mod} A/AeA$ with $\mathcal{Y}$. Then $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair for $\operatorname{Mod} A$, and for each A-module M the 5-term exact sequence ε_M is of the form

$$0 \to \operatorname{Tor}_1^A(M, A/AeA) \to Me \otimes_{eAe} eA \to M \to M \otimes_A A/AeA \to 0 \to 0.$$

The next example[1] arises from the work of Reiten and Ringel on infinite dimensional representations of canonical algebras; see [29] which is our reference for all concepts and results in the following discussion. Note that these algebras are not necessarily hereditary. The example shows the interplay between Ext-orthogonal pairs and (co)torsion pairs.

Example 4.4. Let A be a finite dimensional canonical algebra over a field k. Take for example a tame hereditary algebra, or, more specifically, the Kronecker algebra $\left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$. For such algebras, there is the concept of a *separating tubular family.* We fix such a family and denote by $\mathcal{T}$ the category of finite dimensional modules belonging to this family. There is also a particular *generic module* over A which depends in some cases on the choice of the tubular family; it is denoted by G. Then the full subcategory $\mathcal{X} = \varinjlim \mathcal{T}$ consisting of all filtered colimits of modules in $\mathcal{T}$ and the full subcategory $\mathcal{Y} = \operatorname{Add} G$ consisting of all coproducts of copies of G form an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for $\operatorname{Mod} A$. Note that the endomorphism ring $D = \operatorname{End}_A(G)$ of G is a division ring and that the canonical map $A \to B$ with $B = \operatorname{End}_D(G)$ is a homological epimorphism which induces an equivalence $\operatorname{Mod} B \xrightarrow{\sim} \mathcal{Y}$. In the particular case of the Kronecker algebra $A = \left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$, a direct computation shows that $B = M_2\big(k(x)\big)$.

The category of A-modules which are generated by $\mathcal{T}$ and the category of A-modules which are cogenerated by G form a torsion pair $(\operatorname{Fac} \mathcal{X}, \operatorname{Sub} \mathcal{Y})$ for $\operatorname{Mod} A$ which equals the torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ generated by $\mathcal{X}$. On the other hand, let $\mathcal{C}$ denote the category of A-modules which are cogenerated by $\mathcal{X}$, and let $\mathcal{D}$ denote the category of A-modules M satisfying $\operatorname{Hom}_A(M, \mathcal{T}) = 0$. Then the pair $(\mathcal{C}, \mathcal{D})$ forms a cotorsion pair for $\operatorname{Mod} A$ which identifies with the cotorsion pair $(\mathcal{X}_1, \mathcal{Y}_1)$ generated by $\mathcal{X}$.

If A is hereditary, then the Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ is complete by Corollary 3.4; see also Remark 3.5 for an explicit description of the 5-term approximation sequence ε_M for each A-module M. Alternatively, one obtains the sequence ε_M by splicing together appropriate approximation sequences which arise from $(\mathcal{X}_0, \mathcal{Y}_0)$ and $(\mathcal{X}_1, \mathcal{Y}_1)$.

[1]The first author is grateful to Lidia Angeleri Hügel for suggesting this example.

The following example of an Ext-orthogonal pair arises from a localizing subcategory; it is a specialization of Example 2.11 and provides a simple (and not necessarily hereditary) model for the previous example.

Example 4.5. Let A be an integral domain with quotient field Q. Let $\mathcal{X}$ denote the category of torsion modules and $\mathcal{Y}$ the category of torsion free divisible modules. Note that the modules in $\mathcal{Y}$ are precisely the coproducts of copies of Q. Then $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair for $\operatorname{Mod} A$, and for each A-module M the 5-term exact sequence ε_M is of the form

$$0 \to 0 \to tM \to M \to M \otimes_A Q \to \bar{M} \to 0.$$

We conclude the section by showing that there are examples of abelian categories that admit only trivial Ext-orthogonal pairs.

Example 4.6. Let A be a local artinian ring and set $\mathcal{A} = \operatorname{Mod} A$. Then $\operatorname{Hom}_A(X, Y) \neq 0$ for any pair X, Y of non-zero A-modules. This is because the unique (up to isomorphism) simple module S is a submodule of Y and a factor of X. Thus if $(\mathcal{X}, \mathcal{Y})$ is an Ext-orthogonal pair for $\mathcal{A}$, then $\mathcal{X} = \mathcal{A}$ or $\mathcal{Y} = \mathcal{A}$.

5. Ext-orthogonal pairs of finite type

At this point, we use the results from §3 to characterize for hereditary rings the Ext-orthogonal pairs of *finite type*. Those are, by definition, the Ext-orthogonal pairs generated by a set of finitely presented modules.

Theorem 5.1. *Let A be a hereditary ring and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair for the module category of A. Then the following are equivalent.*

(1) *The subcategory $\mathcal{Y}$ is closed under taking coproducts.*
(2) *Every module in $\mathcal{X}$ is a filtered colimit of finitely presented modules from $\mathcal{X}$.*
(3) *There exists a category $\mathcal{C}$ of finitely presented modules such that $\mathcal{C}^\perp = \mathcal{Y}$.*

We need some preparations for the proof of this result. The first lemma is a slight modification of [3, Proposition 2.1].

Lemma 5.2. *Let A be a ring and $\mathcal{Y}$ a subcategory of its module category. Denote by $\mathcal{X}$ the category of A-modules X of projective dimension at most 1 satisfying $\operatorname{Ext}^1_A(X, Y) = 0$ for all $Y \in \mathcal{Y}$. Then any module in $\mathcal{X}$ is a filtered colimit of finitely presented modules from $\mathcal{X}$.*

Proof. Let $X \in \mathcal{X}$. Choose an exact sequence $0 \to P \xrightarrow{\phi} Q \to X \to 0$ such that P is free and Q is projective. Note that $\operatorname{Ext}^1_A(X, Y) = 0$ implies that every morphism $P \to Y$ factors through ϕ. The commuting diagrams of A-module morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_i & \xrightarrow{\phi_i} & Q_i & \longrightarrow & X_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P & \xrightarrow{\phi} & Q & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

with P_i and Q_i finitely generated projective form a filtered system of exact sequences such that $\varinjlim \phi_i = \phi$. Note that P is a filtered colimit of its finitely generated direct summands since P is free. Thus there is a cofinal subsystem such that each morphism $P_i \to P$ is a split monomorphism. Therefore we may without loss of generality assume that each morphism $P_i \to P$ is a split monomorphism.

Clearly $\varinjlim X_i = X$, and it remains to prove that $\operatorname{Ext}^1_A(X_i, \mathcal{Y}) = 0$ for all i. This is equivalent to showing that each morphism $\mu\colon P_i \to Y$ with $Y \in \mathcal{Y}$ factors through ϕ_i. For this, we first factor each such μ through the split monomorphism $P_i \to P$, then through ϕ, and finally compose the morphism $Q \to Y$ which we have obtained with the morphism $Q_i \to Q$. The result is a morphism $\nu\colon Q_i \to Y$ such that $\nu\phi_i = \mu$, as desired. □

The second lemma establishes some necessary properties of the 5-term sequences.

Lemma 5.3. *Let A be a hereditary ring and $(\mathcal{X}, \mathcal{Y})$ a complete Ext-orthogonal pair for* $\operatorname{Mod} A$. *Let M be an A-module and ε_M the corresponding 5-term exact sequence.*

(1) *If $\operatorname{Ext}^1_A(M, \mathcal{Y}) = 0$, then $Y_M = 0$.*
(2) *Suppose that $\mathcal{Y}$ is closed under coproducts and let $M = \varinjlim M_i$ be a filtered colimit of A-modules M_i. Then $\varepsilon_M = \varinjlim \varepsilon_{M_i}$.*

Proof. We use the uniqueness of the 5-term exact sequences guaranteed by Lemma 2.9. If $\operatorname{Ext}^1_A(M, \mathcal{Y}) = 0$, then the image of the morphism $X_M \to M$ belongs to $\mathcal{X}$. Thus $X_M \to M$ is a monomorphism since ε_M is unique, and this yields (1).

To prove (2), one uses that $\mathcal{X}$ and $\mathcal{Y}$ are closed under taking colimits and that taking filtered colimits is exact. Thus $\varinjlim \varepsilon_{M_i}$ is an exact sequence with middle term M and all other terms in $\mathcal{X}$ or $\mathcal{Y}$. Now the uniqueness of ε_M implies that $\varepsilon_M = \varinjlim \varepsilon_{M_i}$. □

Finally, the following lemma is needed for hereditary rings which are not noetherian.

Lemma 5.4. *Let M be a finitely presented module over a hereditary ring and $N \subseteq M$ any submodule. Then N is a direct sum of finitely presented modules.*

Proof. We combine two results. Over a hereditary ring, any submodule of a finitely presented module is a direct sum of a finitely presented module and a projective module; see [8, Theorem 5.1.6]. In addition, one uses that any projective module is a direct sum of finitely generated projective modules; see [1]. □

Proof of Theorem 5.1. (1) ⇒ (2): Suppose that $\mathcal{Y}$ is closed under taking coproducts. We apply Corollary 3.4 to obtain for each module M the natural exact sequence ε_M. Here note that we a priori did not assume completeness of $(\mathcal{X}, \mathcal{Y})$. Now suppose that M belongs to $\mathcal{X}$. Then one can write $M = \varinjlim M_i$ as a filtered colimit of finitely presented modules with $\operatorname{Ext}^1_A(M_i, \mathcal{Y}) = 0$ for all i; see Lemma 5.2. Next we apply Lemma 5.3. Thus

$$\varinjlim X_{M_i} \xrightarrow{\sim} X_M \xrightarrow{\sim} M,$$

and each X_{M_i} is a submodule of the finitely presented module M_i. Finally, each X_{M_i} is a filtered colimit of finitely presented direct summands by Lemma 5.4. Thus M is a filtered colimit of finitely presented modules from $\mathcal{X}$.

(2) ⇒ (3): Let $\mathcal{X}_{\mathrm{fp}}$ denote the full subcategory that is formed by all finitely presented modules in $\mathcal{X}$. Observe that ${}^\perp Y$ is closed under taking colimits for each module Y, because ${}^\perp Y$ is closed under taking coproducts and cokernels. Thus $\mathcal{X}_{\mathrm{fp}}^\perp = \mathcal{X}^\perp = \mathcal{Y}$ provided that $\mathcal{X} = \varinjlim \mathcal{X}_{\mathrm{fp}}$.

(3) ⇒ (1): Use that for each finitely presented A-module X, the functor $\operatorname{Ext}^*_A(X, -)$ preserves all coproducts. □

Note that Theorem 5.1 gives rise to a bijection between extension closed abelian subcategories of finitely presented modules and Ext-orthogonal pairs of finite type. We will state this explicitly in §8, but we in fact prove it here by the following proposition.

Proposition 5.5. *Let A be a hereditary ring and $\mathcal{C}$ a category of finitely presented A-modules. Then ${}^\perp(\mathcal{C}^\perp) \cap \operatorname{mod} A$ equals the smallest extension closed abelian subcategory of* $\operatorname{mod} A$ *containing $\mathcal{C}$.*

Proof. Let $\mathcal{D}$ denote the smallest extension closed abelian subcategory of $\operatorname{mod} A$ containing $\mathcal{C}$. We claim that the category $\varinjlim \mathcal{D}$ which is formed by all filtered colimits of modules in $\mathcal{D}$ is an extension closed abelian subcategory of $\operatorname{Mod} A$.

Assume for the moment that the claim holds. Then Theorem 2.2 implies that $\mathcal{X} = {}^\perp(\mathcal{C}^\perp)$ equals the smallest extension closed abelian subcategory of $\operatorname{Mod} A$ closed under coproducts and containing $\mathcal{C}$. Our claim then implies $\mathcal{X} = \varinjlim \mathcal{D}$, so $\mathcal{X} \cap \operatorname{mod} A = \mathcal{D}$ and we are finished.

Therefore, it only remains to prove the claim. First observe that every morphism in $\varinjlim \mathcal{D}$ can be written as a filtered colimit of morphisms in $\mathcal{D}$. Using that taking filtered colimits is exact, it follows immediately that $\varinjlim \mathcal{D}$ is closed under kernels and cokernels in $\operatorname{Mod} A$.

It remains to show that $\varinjlim \mathcal{D}$ is closed under extensions. To this end let $\eta\colon 0 \to L \to M \to N \to 0$ be an exact sequence with L and N in $\varinjlim \mathcal{D}$. We can without loss of generality assume that N belongs to $\mathcal{D}$, because otherwise the sequence η is a filtered colimit of the pull-back exact sequences with the last terms in $\mathcal{D}$. Next we choose a morphism $\phi\colon M' \to M$ with M' finitely presented. All we need to do now is to show that ϕ factors through an object in $\mathcal{D}$; see [21]. We may, moreover, assume that the composite of ϕ with $M \to N$ is an epimorphism. This is because otherwise we can take an epimorphism $P \to N$ with P finitely generated projective, factor it through $M \to N$, and replace ϕ by $\phi'\colon M' \oplus P \to M$. Finally, denote by L' the kernel of ϕ, which is necessarily a finitely presented module. The induced map $L' \to L$ then factors through an object L'' in $\mathcal{D}$ since L belongs to $\varinjlim \mathcal{D}$. Forming the push-out exact sequence of $0 \to L' \to M' \to N \to 0$ along the morphism $L' \to L''$ gives an exact sequence $0 \to L'' \to M'' \to N \to 0$. Now ϕ factors through M'' which belongs to $\mathcal{D}$. □

6. Universal localizations

A ring homomorphism $A \to B$ is called a *universal localization* if there exists a set Σ of morphisms between finitely generated projective A-modules such that

(1) $\sigma \otimes_A B$ is an isomorphism of B-modules for all $\sigma \in \Sigma$, and
(2) every ring homomorphism $A \to B'$ such that $\sigma \otimes_A B'$ is an isomorphism of B-modules for all $\sigma \in \Sigma$ factors uniquely through $A \to B$.

Let A be a ring and Σ a set of morphisms between finitely generated projective A-modules. Then there exists a universal localization inverting Σ and this is unique up to a unique isomorphism; see [31] for details. The universal localization is denoted by $A \to A_\Sigma$ and restriction identifies $\operatorname{Mod} A_\Sigma$ with the full subcategory consisting of all A-modules M such that $\operatorname{Hom}_A(\sigma, M)$ is an isomorphism for all $\sigma \in \Sigma$. Note that $\operatorname{Hom}_A(\sigma, M)$ is an isomorphism if and only if M belongs to $\{\operatorname{Ker}\sigma, \operatorname{Coker}\sigma\}^\perp$, provided that A is hereditary. The main result of this section is then the following theorem.

Theorem 6.1. *Let A be a hereditary ring. A ring homomorphism $f\colon A \to B$ is a homological epimorphism if and only if f is a universal localization.*

Proof. Suppose first that $f\colon A \to B$ is a homological epimorphism. This gives rise to an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for $\operatorname{Mod} A$, if we identify $\operatorname{Mod} B$ with a full subcategory $\mathcal{Y}$ of $\operatorname{Mod} A$; see Proposition 3.1. Let $\mathcal{X}_{\mathrm{fp}}$ denote the subcategory that is formed by all finitely presented modules in $\mathcal{X}$. It follows from Theorem 5.1 that $\mathcal{X}_{\mathrm{fp}}^\perp = \mathcal{Y}$. Now fix for each $X \in \mathcal{X}_{\mathrm{fp}}$ an exact sequence

$$0 \to P_X \xrightarrow{\sigma_X} Q_X \to X \to 0$$

such that P_X and Q_X are finitely generated projective, and let $\Sigma = \{\sigma_X \mid X \in \mathcal{X}_{\mathrm{fp}}\}$. Then

$$\operatorname{Mod} B = \mathcal{X}_{\mathrm{fp}}^{\perp} = \operatorname{Mod} A_\Sigma.$$

Therefore, $f\colon A \to B$ is a universal localization, since $\mathcal{X}_{\mathrm{fp}}^{\perp}$ determines the corresponding ring epimorphism uniquely up to isomorphism, see the proof of Proposition 3.2.

Now suppose $f\colon A \to B$ is a universal localization. Then restriction identifies the category of B-modules with an extension closed subcategory of $\operatorname{Mod} A$. Thus we have induced isomorphisms

$$\operatorname{Ext}^*_B(X,Y) \xrightarrow{\sim} \operatorname{Ext}^*_A(X,Y)$$

for all B-modules X, Y, since A is hereditary. It follows that f is a homological epimorphism. □

Remark 6.2. Neither implication in Theorem 6.1 is true if one drops the assumption on the ring A to be hereditary, not even if the global dimension is 2. In [17], Keller gives an example of a Bézout domain A and a non-zero ideal I such that the canonical map $A \to A/I$ is a homological epimorphism, but any map σ between finitely generated projective A-modules needs to be invertible if $\sigma \otimes_A A/I$ is invertible. We refine the constrcution so that $\operatorname{gldim} A = 2$, see Example 7.8. On the other hand, Neeman, Ranicki, and Schofield use finite dimensional algebras to construct in [26] examples of universal localizations that are not homological epimorphisms. They are also able to construct such examples of global dimension 2, see [26, Remark 2.13].

7. The telescope conjecture

Now we are ready to state and prove an extended version of Theorem A after recalling the necessary notions.

Let A be a ring. A complex of A-modules is called *perfect* if it is isomorphic to a bounded complex of finitely generated projective modules. Note that a complex X is perfect if and only if the functor $\operatorname{Hom}_{\mathbf{D}(\operatorname{Mod} A)}(X, -)$ preserves coproducts. One direction of this statement is easy to prove since $\operatorname{Hom}_{\mathbf{D}(\operatorname{Mod} A)}(A, -)$ preserves coproducts and every perfect complex is finitely built from A. The converse follows from [24, Lemma 2.2] and [5, Proposition 3.4]. Recall also that a localizing subcategory $\mathcal{C}$ of $\mathbf{D}(\operatorname{Mod} A)$ is *generated by perfect complexes* if $\mathcal{C}$ admits no proper localizing subcategory containing all perfect complexes from $\mathcal{C}$.

Theorem 7.1. *Let A be a hereditary ring. For a localizing subcategory $\mathcal{C}$ of $\mathbf{D}(\operatorname{Mod} A)$ the following conditions are equivalent:*

(1) *There exists a localization functor* $L\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ *that preserves coproducts and such that* $\mathcal{C} = \operatorname{Ker} L$.
(2) *The localizing subcategory* $\mathcal{C}$ *is generated by perfect complexes.*
(3) *There exists a localizing subcategory* $\mathcal{D}$ *of* $\mathbf{D}(\operatorname{Mod} A)$ *that is closed under products such that* $\mathcal{C} = {}^{\perp}\mathcal{D}$.

Proof. (1) ⇒ (2): The kernel $\operatorname{Ker} L$ and the essential image $\operatorname{Im} L$ of a localization functor L form an Ext-orthogonal pair for $\mathbf{D}(\operatorname{Mod} A)$; see [4, Lemma 3.3]. We obtain an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for $\operatorname{Mod} A$ by taking $\mathcal{X} = H^0 \operatorname{Ker} L$ and $\mathcal{Y} = H^0 \operatorname{Im} L$; see Proposition 2.6. The fact that L preserves coproducts implies that $\mathcal{Y}$ is closed under taking coproducts. It follows from Theorem 5.1 that $\mathcal{X}$ is generated by finitely presented modules. Each finitely presented module is isomorphic in $\mathbf{D}(\operatorname{Mod} A)$ to a perfect complex, and therefore $\operatorname{Ker} L$ is generated by perfect complexes.

(2) ⇒ (3): Suppose that $\mathcal{C}$ is generated by perfect complexes. Then there exists a localization functor $L\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ such that $\operatorname{Ker} L = \mathcal{C}$. Thus we have an Ext-orthogonal pair $(\mathcal{C}, \mathcal{D})$ for $\mathbf{D}(\operatorname{Mod} A)$ with $\mathcal{D} = \operatorname{Im} L$; see [4, Lemma 3.3]. Now observe that $\mathcal{D} = \mathcal{C}^{\perp}$ is closed under coproducts, since for any perfect complex X the functor $\operatorname{Hom}_{\mathbf{D}(\operatorname{Mod} A)}(X, -)$ preserves coproducts. It follows that $\mathcal{D}$ is a localizing subcategory.

(3) ⇒ (1): Let $\mathcal{D}$ be a localizing subcategory that is closed under products such that $\mathcal{C} = {}^{\perp}\mathcal{D}$. Then $\mathcal{Y} = H^0\mathcal{D}$ is an extension closed abelian subcategory of $\operatorname{Mod} A$ that is closed under products and coproducts; see Proposition 2.4. In the proof of Corollary 3.4 we have constructed a localization functor $L\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ such that $\mathcal{C} = \operatorname{Ker} L$. More precisely, there exists a homological epimorphism $A \to B$ such that $L = - \otimes_A^{\mathbf{L}} B$. It remains to notice that this functor preserves coproducts. □

Remark 7.2. The implication (1) ⇒ (2) is known as the *telescope conjecture*. Let us sketch the essential ingredients of the proof of this implication. In fact, the proof is not as involved as one might expect from the references to preceding results of this work.

We need the 5-term exact sequence ε_M for each module M which one gets immediately from the the localization functor L; see Proposition 2.7. The perfect complexes generating $\mathcal{C}$ are constructed in the proof of Theorem 5.1, where the relevant implication is (1) ⇒ (2). For this proof, one uses Lemmas 5.2 – 5.4, but this is all.

Remark 7.3. Let A be a herditary ring and B a ring that is derived equivalent to A, that is, there is an equivalence of triangulated categories $\mathbf{D}(\operatorname{Mod} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} B)$. Then the statement of Theorem 7.1 carries over from A to B. In particular, the statement of Theorem 7.1 holds for every tilted algebra in the sense of Happel and Ringel [15].

Given the proof of the telescope conjecture for the derived categories of hereditary rings, one may be tempted to think that perhaps it is possible to get a similar result for rings of higher global dimension. Here we show that this is not the case. Namely, we construct a class of rings for which the conjecture fails for the derived category, and we will see that some of them have global dimension 2. To achieve this, we use the following result due to Keller [17].

Lemma 7.4. *Let A be a ring and I a non-zero two-sided ideal of A such that*

(1) $\operatorname{Tor}_i^A(A/I, A/I) = 0$ *for all $i \geq 1$ (that is, the surjection $A \to A/I$ is a homological epimorphism), and*
(2) *I is contained in the Jacobson radical of A.*

Then $L = - \otimes_A^{\mathbf{L}} A/I \colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ is a coproduct preserving localization functor but $\operatorname{Ker} L$, which is the smallest localizing subcategory containing I, contains no non-zero perfect complexes. In particular, the telescope conjecture fails for $\mathbf{D}(\operatorname{Mod} A)$.

In order to find such A and I with (right) global dimension of A equal to 2, we restrict ourself to the case when A is a valuation domain. That is, A is a commutative domain with the property that for each pair $a, b \in A$, either a divides b or b divides a. We refer to [10, Chapter II] for a discussion of such domains. Here, we mention only the properties which we need for our example:

Lemma 7.5. *The following holds for a valuation domain A which is not a field.*

(1) *The ring A is local and its weak global dimension equals* 1*.*
(2) *The maximal ideal P of A is either principal or idempotent.*
(3) *For any ideal I of A we have the isomorphism* $\operatorname{Tor}_1^A(A/I, A/I) \cong I/I^2$*.*

Proof. (1) The ring A is local since the ideals of A are totally ordered by inclusion. The second part of (1) follows from [10, VI.10.4].

(2) This is a direct consequence of results in [10, Section II.4]. For an ideal I, one defines

$$I' = \{a \in A \mid aI \subsetneq I\}.$$

It turns out that I' is always a prime ideal and I is naturally an $R_{I'}$-module. Moreover, $I = I'$ if I itself is a prime ideal, [10, II.4.3 (iv)]. In particular we have $P' = P$. On the other hand, [10, p. 69, item (d)] says that $I' \cdot I \subsetneq I$ if and only if I is a principal ideal of $R_{I'}$. Specialized to P, this precisely says that $P^2 = P' \cdot P \subsetneq P$ if and only if P is a principal ideal of R.

(3) Tensoring the exact sequence $0 \to I \to A \to A/I \to 0$ with A/I gives the exact sequence

$$A/I \otimes_A I \xrightarrow{0} A/I \xrightarrow{\sim} A/I \otimes_A A/I \to 0.$$

It follows that $\operatorname{Tor}_1^A(A/I, A/I) \cong A/I \otimes_A I$, and the right exactness of the tensor product yields $A/I \otimes_A I \cong I/I^2$. □

The following result is a straightforward consequence.

Proposition 7.6. *Let A be a valuation domain whose maximal ideal P is non-principal. Then the telescope conjecture fails for $\mathbf{D}(\operatorname{Mod} A)$. More precisely, $L = - \otimes_A^{\mathbf{L}} A/P$ is a coproduct preserving localization functor on $\mathbf{D}(\operatorname{Mod} A)$ whose kernel is non-trivial (it contains P) but not generated by perfect complexes.*

Proof. It is enough to prove that the maximal ideal P meets the conditions of Lemma 7.4. As P is the Jacobson radical of A, condition (2) is fulfilled. Condition (1) follows easily from Lemma 7.5. □

What we are left with now is to construct a valuation domain whose maximal ideal is non-principal and whose global dimension is 2. To this end, we recall the basic tool to construct valuation domains with given properties: the value group. If A is a valuation domain, denote by Q its quotient field and by U the group of units of A. Then U is clearly a subgroup of the multiplicative group $Q^* = Q \setminus \{0\}$ and

$$G = Q^*/U$$

is a totally ordered abelian group. More precisely, G is an abelian group, the relation $\leq$ on G defined by $aU \leq bU$ if $ba^{-1} \in A$ gives a total order on G, and we have the compatibility condition

$$\alpha \leq \beta \quad \text{implies} \quad \alpha \cdot \gamma \leq \beta \cdot \gamma \qquad \text{for all } \alpha, \beta, \gamma \in G.$$

The pair $(G, \leq)$ is called the *value group* of A. We will use the following fundamental result [10, Theorem 3.8].

Proposition 7.7. *Let k be a field and $(G, \leq)$ a totally ordered abelian group. Then there is a valuation domain A whose residue field A/P is isomorphic to k, and whose value group is isomorphic to G as an ordered group.*

Now, we can give the promised example.

Example 7.8. Let G be a free abelian group of countable rank. If we view G as the group $\mathbb{Z}^{(\mathbb{N})}$ (with additive notation), then G is naturally equipped with the lexicographic ordering which makes it to a totally ordered group. Let A be a valuation domain whose value group is isomorphic to G. In fact, looking closer at the particular construction in [10, Section II.3], we can construct A such that it is countable.

We claim that the maximal ideal P of A is non-principal and that $\operatorname{gldim} A = 2$. Indeed, each ideal of A is flat and countably generated since the value group is countable. Thus, each ideal is of projective dimension at most 1 and $\operatorname{gldim} A \leq 2$. On the other hand, it is easy to see that A has non-principal, hence non-projective, ideals and so is not hereditary. One of them is P, which is generated by elements of A whose cosets in the value group Q^*/U correspond, under the isomorphism $Q^*/U \cong \mathbb{Z}^{(\mathbb{N})}$, to the canonical basis elements $e_1, e_2, e_3, \ldots \in \mathbb{Z}^{(\mathbb{N})}$.

This way, we obtain a countable valuation domain A of global dimension 2 such that the telescope conjecture fails for $\mathbf{D}(\operatorname{Mod} A)$ by Proposition 7.6.

8. A bijective correspondence

In this final section we summarize our findings by stating explicitly the correspondence between various structures arising from Ext-orthogonal pairs for hereditary rings. In particular, this completes the proof of an extended version of Theorem B:

Theorem 8.1. *For a hereditary ring A there are bijections between the following sets:*

(1) *Ext-orthogonal pairs $(\mathcal{X}, \mathcal{Y})$ for $\operatorname{Mod} A$ such that $\mathcal{Y}$ is closed under coproducts.*
(2) *Ext-orthogonal pairs $(\mathcal{Y}, \mathcal{Z})$ for $\operatorname{Mod} A$ such that $\mathcal{Y}$ is closed under products.*
(3) *Extension closed abelian subcategories of $\operatorname{Mod} A$ that are closed under products and coproducts.*
(4) *Extension closed abelian subcategories of $\operatorname{mod} A$.*
(5) *Homological epimorphisms $A \to B$ (up to isomorphism).*
(6) *Universal localizations $A \to B$ (up to isomorphism).*
(7) *Localizing subcategories of $\mathbf{D}(\operatorname{Mod} A)$ that are closed under products.*
(8) *Localization functors $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ preserving coproducts (up to natural isomorphism).*
(9) *Thick subcategories of $\mathbf{D}^b(\operatorname{mod} A)$.*

Proof. We state the bijections explicitly in the following table and give the references to the places where these bijections are established.

Direction	Map	Reference
(1) $\leftrightarrow$ (3)	$(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{Y}$	Corollary 3.4
(2) $\leftrightarrow$ (3)	$(\mathcal{Y}, \mathcal{Z}) \mapsto \mathcal{Y}$	Corollary 3.4
(3) $\to$ (4)	$\mathcal{Y} \mapsto ({}^{\perp}\mathcal{Y}) \cap \operatorname{mod} A$	Thm. 5.1 & Prop. 5.5
(4) $\to$ (3)	$\mathcal{C} \mapsto \mathcal{C}^{\perp}$	Thm. 5.1 & Prop. 5.5
(3) $\to$ (5)	$\mathcal{Y} \mapsto (A \to \operatorname{End}_A(FA))$	Proposition 3.2
(5) $\to$ (3)	$f \mapsto (\operatorname{Ker} f \oplus \operatorname{Coker} f)^{\perp}$	Proposition 3.1
(5) $\leftrightarrow$ (6)	$f \mapsto f$	Theorem 6.1
(3) $\to$ (7)	$\mathcal{Y} \mapsto \mathbf{D}_{\mathcal{Y}}(\operatorname{Mod} A)$	Proposition 2.4
(7) $\to$ (3)	$\mathcal{C} \mapsto H^0\mathcal{C}$	Proposition 2.4
(7) $\to$ (8)	$\mathcal{C} \mapsto (X \mapsto GX)$	Theorem 7.1

Direction	Map	Reference
(8) → (7)	$L \mapsto \operatorname{Im} L$	Theorem 7.1
(4) → (9)	$\mathcal{X} \mapsto \mathbf{D}^b_{\mathcal{X}}(\operatorname{mod} A)$	Remark 2.5
(9) → (4)	$\mathcal{C} \mapsto H^0\mathcal{C}$	Remark 2.5

For (3) → (5), the functor F denotes a left adjoint of the inclusion $\mathcal{Y} \to \operatorname{Mod} A$. For (7) → (8), the functor G denotes a left adjoint of the inclusion $\mathcal{C} \to \mathbf{D}(\operatorname{Mod} A)$, □

Let us mention that this correspondence is related to recent work of some other authors. In [34], Schofield establishes for any hereditary ring the bijection (4) ↔ (6). In [27], Nicolás and Saorín establish for a differential graded algebra A a correspondence between recollements for the derived category $\mathbf{D}(A)$ and differential graded homological epimorphisms $A \to B$. This correspondence specializes for a hereditary ring to the bijection (5) ↔ (8).[2]

A finiteness condition. Given an Ext-orthogonal pair for the category of A-modules as in Theorem 8.1, it is a natural question to ask when its restriction to the category of finitely presented modules yields a complete Ext-orthogonal pair for $\operatorname{mod} A$. This is very important especially when considering relations of results from this paper to representation theory of finite dimensional algebras. For that setting, we characterize this finiteness condition in terms of finitely presented modules; see also Proposition 4.2.

Proposition 8.2. *Let A be a finite dimensional hereditary algebra over a field and $\mathcal{C}$ an extension closed abelian subcategory of* $\operatorname{mod} A$*. Then the following are equivalent.*

(1) *There exists a complete Ext-orthogonal pair $(\mathcal{C}, \mathcal{D})$ for* $\operatorname{mod} A$*.*
(2) *The inclusion $\mathcal{C} \to \operatorname{mod} A$ admits a right adjoint.*
(3) *There exists an exceptional object $X \in \mathcal{C}$ such that $\mathcal{C}$ is the smallest extension closed abelian subcategory of* $\operatorname{mod} A$ *containing X.*
(4) *Let $(\mathcal{X}, \mathcal{Y})$ be the Ext-orthogonal pair for* $\operatorname{Mod} A$ *generated by $\mathcal{C}$. Then for each $M \in \operatorname{mod} A$ the 5-term exact sequence ε_M belongs to* $\operatorname{mod} A$*.*

Proof. (1) ⇒ (2): For $M \in \operatorname{mod} A$ let $0 \to D_M \to C_M \to M \to D^M \to C^M \to 0$ be its 5-term exact sequence. Sending a module M to C_M induces a right adjoint for the inclusion $\mathcal{C} \to \operatorname{mod} A$; see Lemma 2.9.

(2) ⇒ (3): Choose an injective cogenerator Q in $\operatorname{mod} A$ and let X denote its image under the right adjoint of the inclusion of $\mathcal{C}$. A right adjoint of an exact functor preserves injectivity. It follows that X is an

[2] The first author is grateful to Manolo Saorín for pointing out this bijection.

exceptional object and that $\mathcal{C}$ is the smallest extension closed abelian subcategory of $\operatorname{mod} A$ containing X.

(3) $\Rightarrow$ (4): See Proposition 4.2.

(4) $\Rightarrow$ (1): The property of the pair $(\mathcal{X}, \mathcal{Y})$ implies that $(\mathcal{X} \cap \operatorname{mod} A, \mathcal{Y} \cap \operatorname{mod} A)$ is a complete Ext-orthogonal pair for $\operatorname{mod} A$. An application of Proposition 5.5 yields the equality $\mathcal{X} \cap \operatorname{mod} A = \mathcal{C}$. Thus there exists a complete Ext-orthogonal pair $(\mathcal{C}, \mathcal{D})$ for $\operatorname{mod} A$. □

Remark 8.3. There is a dual result which is obtained by applying the duality between modules over the algebra A and its opposite A^{op}. Note that condition (3) is self-dual.

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IV.

LOCALLY WELL GENERATED HOMOTOPY CATEGORIES OF COMPLEXES

Abstract

We show that the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ over any finitely accessible additive category $\mathcal{B}$ is locally well generated. That is, any localizing subcategory $\mathcal{L}$ in $\mathbf{K}(\mathcal{B})$ which is generated by a set is well generated in the sense of Neeman. We also show that $\mathbf{K}(\mathcal{B})$ itself being well generated is equivalent to $\mathcal{B}$ being pure semisimple, a concept which naturally generalizes right pure semisimplicity of a ring R for $\mathcal{B} = \text{Mod-}R$.

This paper is a preprint available as arXiv:0810.5684.

LOCALLY WELL GENERATED HOMOTOPY CATEGORIES OF COMPLEXES

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Abstract. We show that the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ over any finitely accessible additive category $\mathcal{B}$ is locally well generated. That is, any localizing subcategory $\mathcal{L}$ in $\mathbf{K}(\mathcal{B})$ which is generated by a set is well generated in the sense of Neeman. We also show that $\mathbf{K}(\mathcal{B})$ itself being well generated is equivalent to $\mathcal{B}$ being pure semisimple, a concept which naturally generalizes right pure semisimplicity of a ring R for $\mathcal{B} = \text{Mod-}R$.

Introduction

The main motivation for this paper is to study when the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ over an additive category $\mathcal{B}$ is compactly generated or, more generally, well generated.

In the last few decades, the theory of compactly generated triangulated categories has become an important tool unifying concepts from various fields of mathematics. Standard examples are the unbounded derived category of a ring or the stable homotopy category of spectra. The key property of such a category $\mathcal{T}$ is the Brown Representability Theorem, cf. [28, 23], originally due to Brown [8]:

> Any contravariant cohomological functor $F : \mathcal{T} \to \text{Ab}$ which sends coproducts to products is representable.

This theorem is an important tool and has been used in several places. We mention Neeman's proof of the Grothendieck Duality Theorem [28], Krause's work on the Telescope Conjecture [26, 22], or Keller's representation theorem for algebraic compactly generated triangulated categories [21].

Recently, there has been a growing interest in giving criteria for certain homotopy categories $\mathbf{K}(\mathcal{B})$ to be compactly generated, [14, 19, 27, 29]. Here, $\mathcal{B}$ typically was a suitable subcategory of a module category. The main reason for studying such homotopy categories were results

Date: March 19, 2009.

2000 *Mathematics Subject Classification.* 18G35 (primary), 18E30, 18E35, 16D90 (secondary).

Key words and phrases. compactly and well generated triangulated categories, complexes, pure semisimplicity.

The author was supported by the Research Council of Norway through the Storforsk-project "Homological and geometric methods in algebra".

concerning the Grothendieck Duality Theorem [16, 29] and relative homological algebra [18]. There is, however, a conceptual reason, too. Namely, every algebraic triangulated category is triangle equivalent to a full subcategory of some homotopy category, [23, §7.5].

It turned out when studying the homotopy category of complexes of projective modules in [29] that it is useful to consider well generated triangulated categories in this context. More precisely, $\mathbf{K}(\text{Proj-}R)$ is always well generated, but may not be compactly generated. Well generated categories have been defined by Neeman [30] in a natural attempt to extend results such as the Brown Representability from compactly generated triangulated categories to a wider class of triangulated categories.

Although one has already known for some time that there exist rather natural triangulated categories, such as the homotopy category of complexes of abelian groups, which are not even well generated, one has typically viewed those as rare and exceptional cases.

We will give some arguments to show that this interpretation is not very accurate. First, the categories of the shape $\mathbf{K}(\text{Mod-}R)$ for a ring R are rarely well generated. It happens if and only if R is right pure semisimple, which establishes the converse of [14, §4 (3), p. 17]. Moreover, we generalize this result to the homotopy categories $\mathbf{K}(\mathcal{B})$ with $\mathcal{B}$ additive finitely accessible. This way, we obtain a fairly complete answer regarding when $\mathbf{K}(\text{Flat-}R)$ is compactly or well generated, see [14, Question 4.2].

We also give a partial remedy for the typical failure of $\mathbf{K}(\mathcal{B})$ to be well generated. Roughly said, the main problem with $\mathbf{K}(\mathcal{B})$, where $\mathcal{B}$ is finitely accessible, is that it may not have any set of generators at all. But if we take a localizing subcategory $\mathcal{L}$ generated by any set of objects, it will automatically be well generated. We will call a triangulated category with this property locally well generated.

We will as well give basic properties of locally well generated categories and see that some usual results regarding localization work fine in the new setting. For example, any localizing subcategory generated by a set of objects is realized as the kernel of a localization endofunctor. This version of a Bousfield localization theorem generalizes [24, §7.2] and [2, 5.7]. However, one has to be more careful. The Brown Representability theorem as stated above does not work for locally well generated categories in general, and there are localizing subcategories which are not associated to any localization endofunctor. We illustrate this in Example 3.7.

Acknowledgements. The author would like to thank Henning Krause for several helpful discussions and suggestions, as well as for his hospitality during the author's visits in Paderborn.

1. Preliminaries

Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called *thick* if, whenever $X \amalg Y \in \mathcal{S}$, then also $X \in \mathcal{S}$. From now on, we will assume that $\mathcal{T}$ has arbitrary (set-indexed) coproducts. A full triangulated subcategory $\mathcal{L} \subseteq \mathcal{T}$ is called *localizing* if it is closed under forming coproducts. Note that by [30, 1.6.8], $\mathcal{T}$ has splitting idempotents and any localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ is thick.

If $\mathcal{S}$ is any class of objects of $\mathcal{T}$, we denote by $\operatorname{Loc} \mathcal{S}$ the smallest localizing subcategory of $\mathcal{T}$ which contains $\mathcal{S}$. In other words, $\operatorname{Loc} \mathcal{S}$ is the closure of $\mathcal{S}$ under shifts, coproducts and triangle completions.

Given $\mathcal{T}$ and a localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$, one can construct the so-called *Verdier quotient* $\mathcal{T}/\mathcal{L}$ by formally inverting in $\mathcal{T}$ all morphisms in the class $\Sigma(\mathcal{L})$ defined as

$$\Sigma(\mathcal{L}) = \{f \mid \exists \text{ triangle } X \xrightarrow{f} Y \to Z \to X[1] \text{ in } \mathcal{T} \text{ such that } Z \in \mathcal{L}\}.$$

It is a well known fact that the Verdier quotient always has coproducts, admits a natural triangulated structure, and the canonical localization functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{L}$ is exact and preserves coproducts, [30, Chapter 2]. However, one has to be careful, since $\mathcal{T}/\mathcal{L}$ might not be a usual category in the sense that the homomorphism spaces might be proper classes rather than sets. This fact, although often inessential and neglected, as $\mathcal{T}/\mathcal{L}$ has a very straightforward and constructive description, may nevertheless have important consequences in some cases; see eg. [5].

Let $L : \mathcal{T} \to \mathcal{T}$ be an exact endofunctor of $\mathcal{T}$. Then L is called a *localization functor* if there exists a natural transformation $\eta : \mathrm{Id}_{\mathcal{T}} \to L$ such that $L\eta_X = \eta_{LX}$ and $\eta_{LX} : LX \to L^2X$ is an isomorphism for each $X \in \mathcal{T}$.

It is easy to check that the full subcategory $\operatorname{Ker} L$ of $\mathcal{T}$ given by

$$\operatorname{Ker} L = \{X \in \mathcal{T} \mid LX = 0\}$$

is always localizing [2, 1.2]. Moreover, there is a canonical triangle equivalence between $\mathcal{T}/\operatorname{Ker} L$ and $\operatorname{Im} L$, the essential image of L; see [30, 9.1.16] or [24, 4.9.1]. This among others implies that all morphism spaces in $\mathcal{T}/\operatorname{Ker} L$ are sets. Note that although $\operatorname{Im} L$ has coproducts as a category, it might *not* be closed under coproducts in $\mathcal{T}$. This type of localization, coming from a localization functor, is often refered to as *Bousfield localization.* However, not every localizing subcategory $\mathcal{L}$ is realized as the kernel of a localization functor, [5, 1.3]. Namely, $\mathcal{L}$ is of the form $\operatorname{Ker} L$ for some localization functor if and only if the inclusion $\mathcal{L} \to \mathcal{T}$ has a right adjoint, [2, 1.6].

A central concept in this paper is that of a well generated triangulated category. Let κ be a regular cardinal number. An object Y in a category with arbitrary coproducts is called *κ–small* provided that every morphism of the form

$$Y \longrightarrow \coprod_{i \in I} X_i$$

factorizes through a subcoproduct $\coprod_{i \in J} X_i$ with $|J| < \kappa$.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and κ be a regular cardinal. Then $\mathcal{T}$ is called *κ–well generated* provided there is a set $\mathcal{S}$ of objects of $\mathcal{T}$ satisfying the following conditions:

(1) If $X \in \mathcal{T}$ such that $\mathcal{T}(Y, X) = 0$ for each $Y \in \mathcal{S}$, then $X = 0$;
(2) Each object $Y \in \mathcal{S}$ is κ–small;
(3) For any morphism in $\mathcal{T}$ of the form $f : Y \to \coprod_{i \in I} X_i$ with $Y \in \mathcal{S}$, there exists a family of morphisms $f_i : Y_i \to X_i$ such that $Y_i \in \mathcal{S}$ for each $i \in I$ and f factorizes as

$$Y \longrightarrow \coprod_{i \in I} Y_i \xrightarrow{\coprod f_i} \coprod_{i \in I} X_i.$$

The category $\mathcal{T}$ is called *well generated* if it is κ–well generated for some regular cardinal κ.

This definition differs to some extent from Neeman's original definition in [30, 8.1.7]. The equivalence between the two follows from [25, Theorem A] and [25, Lemmas 4 and 5]. Note that if $\kappa = \aleph_0$, then condition (3) is vacuous and $\aleph_0$–well generated triangulated categories are precisely the *compactly generated* triangulated categories in the usual sense.

The key property of well generated categories is that the Brown Representability Theorem holds:

Proposition 1.2. [30, 8.3.3] *Let $\mathcal{T}$ be a well generated triangulated category. Then:*

(1) *Any contravariant cohomological functor $F : \mathcal{T} \to \mathrm{Ab}$ which takes coproducts to products is, up to isomorphism, of the form $\mathcal{T}(-, X)$ for some $X \in \mathcal{T}$.*
(2) *If $\mathcal{S}$ is a set of objects of $\mathcal{T}$ which meets assumptions* (1), (2) *and* (3) *of Definition 1.1 for some cardinal κ, then $\mathcal{T} = \operatorname{Loc} \mathcal{S}$.*

Next we turn our attention to categories of complexes. Let $\mathcal{B}$ be an additive category. Using a standard notation, we denote by $\mathbf{C}(\mathcal{B})$ the category of chain complexes

$$X : \qquad \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \dots,$$

of objects of $\mathcal{B}$. By $\mathbf{K}(\mathcal{B})$, we denote the factor-category of $\mathbf{C}(\mathcal{B})$ modulo the ideal of null-homotopic chain complex morphisms. It is well known that $\mathbf{K}(\mathcal{B})$ has a triangulated structure where triangle completions are constructed using mapping cones (see for example [13, Chapter I]). Moreover, if $\mathcal{B}$ has arbitrary coproducts, so have them both

$\mathbf{C}(\mathcal{B})$ and $\mathbf{K}(\mathcal{B})$, and the canonical functor $\mathbf{C}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$ preserves coproducts.

We will often take for $\mathcal{B}$ module categories or their subcategories. In this case, R will denote an associative unital ring and Mod-R the category of all (unital) right R–modules. By Proj-R and Flat-R we denote, respectively, the full subcategories of projective and flat R–modules.

In fact, our considerations will usually work in a more general setting. Let $\mathcal{A}$ be a skeletally small additive category and Mod-$\mathcal{A}$ be the category of all contravariant additive functors $\mathcal{A} \to \mathrm{Ab}$. We will call such functors *right $\mathcal{A}$–modules*. Then Mod-$\mathcal{A}$ shares many formal properties with usual module categories. We refer to [17, Appendix B] for more details. Correspondingly, we denote by Proj-$\mathcal{A}$ the full subcategory of projective functors and by Flat-$\mathcal{A}$ the category of flat functors. We discuss the categories of the form Flat-$\mathcal{A}$ more in detail in Section 4 since those are, up to equivalence, precisely the so called additive finitely accessible categories. Many natural abelian categories are of this form.

Finally, we spend a few words on set-theoretic considerations. All our proofs work in ZFC with an extra technical assumption: the axiom of choice for proper classes. The latter assumption has no algebraic significance, it is only used to keep arguments simple in the following case:

Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant additive functor. If we know, for example by the Brown Representability Theorem, that the composition of functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\mathcal{D}(-,X)} \mathrm{Ab}$$

is representable for each $X \in \mathcal{D}$, we would like to conclude that F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$. In order to do that, we must for each $Y \in \mathcal{C}$ *choose* one particular value for GY from a class of mutually isomorphic candidates.

2. Pure semisiplicity

A relatively straightforward but crucial obstacle causing a homotopy category of complexes $\mathbf{K}(\mathcal{B})$ not to be well generated is that the additive base category $\mathcal{B}$ is not pure semisimple. Here, we use the following very general definition:

Definition 2.1. An additive category $\mathcal{B}$ with arbitrary coproducts is called *pure semisimple* if it has an additive generator. That is, there is an object $X \in \mathcal{B}$ such that $\mathcal{B} = \mathrm{Add}X$, where $\mathrm{Add}X$ stands for the full subcategory formed by all objects which are summands in (possibly infinite) coproducts of copies of X.

The term is inspired by the case $\mathcal{B} = \text{Mod-}R$, where we have the following proposition:

Proposition 2.2. *A ring R is right pure semisimple (that is, each pure monomorphism between right R–modules splits) if and only if* Mod-R *is pure semisimple in the sense of Definition 2.1.*

Proof. If every pure monomorphism in Mod-R splits, then also every pure epimorphism splits. That is, every module is pure projective, or equivalently a summand in a direct sum of finitely presented modules. By a theorem of Kaplansky, [20, Theorem 1], it follows that every module is a direct sum of countably generated modules. Hence, Mod-R is pure semisimple according to our definition. In fact, one can show more in this case: Every module is even a direct sum of finitely presented modules; see for example [15] or [17, App. B].

Let us conversely assume that Mod-R is a pure semisimple additive category. First, one can use a straightforward variation of [20, Theorem 1] for higher cardinalities to see, provided Mod-$R = \mathrm{Add}X$ for some κ–generated module X, that each module in Mod-R is a direct sum of λ–generated modules where $\lambda = \max(\kappa, \aleph_0)$. This fact implies that every module is Σ–pure injective, [11]. In particular, each pure monomorphism in Mod-R splits and R is right pure semisimple. □

If R is an artin algebra, then the conditions of Proposition 2.2 are well-known to be further equivalent to R being of finite representation type; see [3, Theorem A]. For more details and references on this topic, we also refer to [15]. It turns out that the pure semisimplicity condition has a nice interpretation for finitely accessible additive categories as well. We will discuss this more in detail in Section 4.

For giving a connection between pure semisimplicity of $\mathcal{B}$ and properties of $\mathbf{K}(\mathcal{B})$, we recall a structure result for the so-called contractible complexes in $\mathbf{C}(\mathcal{B})$. A complex $Y \in \mathbf{C}(\mathcal{B})$ is *contractible* if it is mapped to a zero object under $\mathbf{C}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$. It is clear that the complexes of the form

$$I_{X,n}: \qquad \cdots \to 0 \to 0 \to X = X \to 0 \to 0 \to \dots,$$

such that the first X is in degree n, are contractible. Moreover, all other contractible complexes are obtained in the following way:

Lemma 2.3. *Let $\mathcal{B}$ be an additive category with splitting idempotents and $Y \in \mathbf{C}(\mathcal{B})$. Then the following are equivalent:*

(1) *Y is contractible;*
(2) *Y is isomorphic in $\mathbf{C}(\mathcal{B})$ to a complex of the form $\coprod_{n\in\mathbb{Z}} I_{X_n,n}$.*

Proof. (2) $\implies$ (1). This is trivial given the fact that the functor $\mathbf{C}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$ preserves those componentwise coproducts of complexes which exist in $\mathbf{C}(\mathcal{B})$.

(1) $\implies$ (2). Let us fix a contractible complex in $\mathbf{K}(\mathcal{B})$:

$$Y: \qquad \dots \xrightarrow{d^{n-2}} Y^{n-1} \xrightarrow{d^{n-1}} Y^n \xrightarrow{d^n} Y^{n+1} \xrightarrow{d^{n+1}} \dots.$$

By definition, the identity morphism of Y is homotopy equivalent to the zero morphism in $\mathbf{C}(\mathcal{B})$, so there are morphisms $s^n : Y^n \to Y^{n-1}$ in $\mathcal{B}$ such that

$$1_{Y^n} = d^{n-1}s^n + s^{n+1}d^n.$$

When composing with d^n, we get $d^n = d^n s^{n+1} d^n$, so $s^{n+1}d^n : Y^n \to Y^n$ is idempotent in $\mathcal{B}$ for each $n \in \mathbb{Z}$. Hence there are morphisms $p^n : Y^n \to X_n$ and $j^n : X_n \to Y^n$ in $\mathcal{B}$ such that $p^n j^n = 1_{X_n}$ and $j^n p^n = s^{n+1}d^n$. Let us denote by $f^n : X_{n-1} \amalg X_n \to Y^n$ and $g^n : Y^n \to X_{n-1} \amalg X_n$ the morphisms defined as follows:

$$f^n = (d^{n-1}j^{n-1}, j^n), \quad \text{and} \quad g^n = \begin{pmatrix} p^{n-1}s^n \\ p^n \end{pmatrix}.$$

Using the identities above, it is easy to check that $f^n g^n = 1_{Y^n}$ and $g^n f^n$ is an isomorphism in $\mathcal{B}$ for each n. Therefore, both f^n and g^n are isomorphisms and $g^n f^n$ is the identity morphism. Finally, it is straightforward to check that the family of morphisms $(f_n \mid n \in \mathbb{Z})$ induces an (iso)morphism $f : \coprod_{n \in Z} I_{X_n,n} \to Y$ in $\mathbf{C}(\mathcal{B})$. □

It is not difficult to see that the condition of $\mathcal{B}$ having splitting idempotents is really necessary in Lemma 2.3. However, there is a standard construction which allows us to amend $\mathcal{B}$ with the missing summands if $\mathcal{B}$ does not have splitting idempotents.

Definition 2.4. Let $\mathcal{B}$ be an additive category. Then an additive category $\bar{\mathcal{B}}$ is called an *idempotent completion* of $\mathcal{B}$ if

(1) $\bar{\mathcal{B}}$ has splitting idempotents;
(2) $\mathcal{B}$ is a full subcategory of $\bar{\mathcal{B}}$;
(3) Every object in $\bar{\mathcal{B}}$ is a direct summand of an object in $\mathcal{B}$.

It is a classical result that idempotent completions always exist. We refer for example to [4, §1] for a particular construction. Moreover, it is well-known that if $\mathcal{B}$ has arbitrary coproducts, then also $\bar{\mathcal{B}}$ has them and they are compatible with coproducts in $\mathcal{B}$.

Now we can state the main result of the section showing that for $\mathbf{K}(\mathcal{B})$ being generated by a set (and, in particular, for $\mathbf{K}(\mathcal{B})$ being well generated), the category $\mathcal{B}$ is necessarily pure semisimple.

Theorem 2.5. *Let $\mathcal{B}$ be an additive category with arbitrary coproducts and assume that there is a set of objects $\mathcal{S} \subseteq \mathbf{K}(\mathcal{B})$ such that $\mathbf{K}(\mathcal{B}) = \operatorname{Loc}\mathcal{S}$. Then $\mathcal{B}$ is pure semisimple.*

Proof. Note that we can replace $\mathcal{S}$ by a singleton $\{Y\}$; take for instance $Y = \coprod_{Z \in \mathcal{S}} Z$. Let us denote by $X \in \mathcal{B}$ the coproduct $\coprod_{n \in \mathbb{Z}} Y^n$ of all components of Y. We will show that $\mathcal{B} = \operatorname{Add}X$.

First, we claim that $\mathbf{K}(\operatorname{Add}X)$ is a dense subcategory of $\mathbf{K}(\mathcal{B})$, that is, each object in $\mathbf{K}(\mathcal{B})$ is isomorphic to one in $\mathbf{K}(\operatorname{Add}X)$. Clearly, $Y \in \mathbf{K}(\operatorname{Add}X)$ and any family of objects from $\mathbf{K}(\operatorname{Add}X)$ admits a coproduct in $\mathbf{K}(\mathcal{B})$ which lies again in $\mathbf{K}(\operatorname{Add}X)$. Now, if $f : U \to V$

is a morphism in $\mathbf{C}(\mathrm{Add}X)$, then the mapping cone C_f is in $\mathbf{C}(\mathrm{Add}X)$ and $U \to V \to C_f \to U[1]$ forms a triangle in $\mathbf{K}(\mathcal{B})$. As coproducts and triangle completions are unique up to isomorphism, the closure of $\mathbf{K}(\mathrm{Add}X)$ under taking isomorphic objects in $\mathbf{K}(\mathcal{B})$ is obviously a localizing subcategory of $\mathbf{K}(\mathcal{B})$. Hence it coincides with $\mathbf{K}(\mathcal{B})$ and the claim is proved.

Suppose for the moment that $\mathcal{B}$ has splitting idempotents. If we identify $\mathcal{B}$ with the full subcategory of $\mathbf{K}(\mathcal{B})$ formed by complexes concentrated in degree zero, we have proved that each object $Z \in \mathcal{B}$ is isomorphic to a complex $Q \in \mathbf{K}(\mathrm{Add}X)$. That is, there is a chain complex homomorphism $f : Z \to Q$ such that $Q \in \mathbf{C}(\mathrm{Add}X)$ and f becomes an isomorphism in $\mathbf{K}(\mathcal{B})$. In particular, the mapping cone C_f of f is contractible:

$$C_f : \qquad \dots \longrightarrow Q^{-3} \xrightarrow{d^{-3}} Q^{-2} \xrightarrow{\binom{d^{-2}}{0}} Q^{-1} \amalg Z \xrightarrow{(d^{-1}, f^0)} Q^0 \xrightarrow{d^0} Q^1 \longrightarrow \dots$$

Here, f^0 is the degree 0 component of f. Consequently, Lemma 2.3 yields the following commutatuve diagram in $\mathcal{B}$ with isomorphisms in columns:

$$\begin{array}{ccccc} Q^{-2} & \xrightarrow{\binom{d^{-2}}{0}} & Q^{-1} \amalg Z & \xrightarrow{(d^{-1}, f^0)} & Q^0 \\ \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\ U \amalg V & \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} & V \amalg W & \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} & W \amalg Z \end{array}$$

It follows that V, W and also $Q^{-1} \amalg Z$ and Z are in $\mathrm{Add}X$. Hence $\mathcal{B} = \mathrm{Add}X$.

Finally, let $\mathcal{B}$ be a general additive category with coproducts and $\bar{\mathcal{B}}$ be its idempotent completion. From the fact that $\mathbf{K}(\mathcal{B})$ has splitting idempotents, [30, 1.6.8], one easily sees that the full embedding $\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\bar{\mathcal{B}})$ is dense. We already know that if $\mathbf{K}(\mathcal{B}) = \mathrm{Loc}\,\mathcal{S}$ for a set $\mathcal{S}$, then $\bar{\mathcal{B}} = \mathrm{Add}X$ for some $X \in \bar{\mathcal{B}}$. In fact, we can take $X \in \mathcal{B}$ by the above construction. But then clearly $\mathcal{B} = \mathrm{Add}X$ when the additive closure is taken in $\mathcal{B}$. Hence $\mathcal{B}$ is pure semisimple. □

Remark. When studying well generated triangulated categories, an important role is played by so-called κ–localizing subcategories, see [30, 24]. We recall that given a cardinal number κ, a *κ–coproduct* is a coproduct with fewer than κ summands. If $\mathcal{T}$ is a triangulated category with arbitrary κ–coproducts, a thick subcategory $\mathcal{L} \subseteq \mathcal{T}$ is called *κ–localizing* if it is closed under taking κ–coproducts. In this context, one can state the following "bounded" version of Theorem 2.5:

Let κ be an uncountable regular cardinal and $\mathcal{B}$ be an additive category with κ–coproducts. If $\mathbf{K}(\mathcal{B})$ is generated as a κ–localizing subcategory by a set $\mathcal{S}$ of fewer than κ objects, then there is $X \in \mathcal{B}$ such that every object of $\mathcal{B}$ is a summand in a κ–coproduct of copies of X.

Note that Theorem 2.5 gives immediately a wide range of examples of categories which are not well generated. For instance, $\mathbf{K}(\text{Mod-}R)$ is not well generated for any ring R which is not right pure semisimple. One can take $R = \mathbb{Z}$ or $R = k(\cdot \rightrightarrows \cdot)$, the Kronecker algebra over a field k. The fact that $\mathbf{K}(\text{Ab})$ is not well generated was first observed by Neeman, [30, E.3.2], using different arguments. In fact, we can state the following proposition, which we later generalize in Section 5:

Proposition 2.6. *Let R be a ring. Then the following are equivalent:*

(1) $\mathbf{K}(\text{Mod-}R)$ *is well generated;*
(2) $\mathbf{K}(\text{Mod-}R)$ *is compactly generated;*
(3) R *is right pure semisimple.*

If R is an artin algebra, the conditions are further equivalent to:

(4) R *is of finite representation type.*

Proof. (2) $\implies$ (1) is clear, as compactly generated is the same as $\aleph_0$–well generated. (1) $\implies$ (3) follows by Theorem 2.5 and Proposition 2.2. (3) $\implies$ (2) has been proved by Holm and Jørgensen, [14, §4 (3), p. 17]. Finally, the equivalence between (3) and (4) is due to Auslander, [3, Theorem A]. □

3. Locally well generated triangulated categories

We have seen in the last section that a triangulated category of the form $\mathbf{K}(\text{Mod-}R)$ is often not well generated. One might get an impression that handling such categories is hopeless, but the main problem here actually is that the category is very big in the sense that it is not generated by any set. Otherwise, it has a very reasonable structure. We shall see that it is locally well generated in the following sense:

Definition 3.1. A triangulated category $\mathcal{T}$ with arbitrary coproducts is called *locally well generated* if $\operatorname{Loc} \mathcal{S}$ is well generated for any set $\mathcal{S}$ of objects of $\mathcal{T}$.

In fact, we prove that $\mathbf{K}(\text{Mod-}\mathcal{A})$ is locally well generated for any skeletally small additive category $\mathcal{A}$. To this end, we first need to be able to measure the size of modules and complexes.

Definition 3.2. Let $\mathcal{A}$ be a skeletally small additive category and $M \in \text{Mod-}\mathcal{A}$. Recall that M is a contravariant additive functor $\mathcal{A} \to \text{Ab}$ by definition. Then the *cardinality of M*, denoted by $|M|$, is defined as

$$|M| = \sum_{A \in \mathcal{S}} |M(A)|,$$

where $|M(A)|$ is just the usual cardinality of the group $M(A)$ and $\mathcal{S}$ is a fixed representative set for isomorphism classes of objects from $\mathcal{A}$. The *cardinality of a complex* $Y = (Y^n, d^n) \in \mathbf{K}(\text{Mod-}\mathcal{A})$ is defined as

$$|Y| = \sum_{n \in \mathbb{Z}} |Y^n|.$$

It is not so difficult to see that the category of all complexes whose cardinalities are bounded by a given regular cardinal always gives rise to a well-generated subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$:

Lemma 3.3. *Let $\mathcal{A}$ be a skeletally small additive category and κ be an infinite cardinal. Then the full subcategory $\mathcal{S}_\kappa$ formed by all complexes of cardinality less than κ meets conditions* (2) *and* (3) *of Definition 1.1.*

In particular, $\mathcal{T}_\kappa = \operatorname{Loc}\mathcal{S}_\kappa$ is a κ–well generated subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$ for any regular cardinal κ.

Proof. Let $Y \in \mathbf{K}(\text{Mod-}\mathcal{A})$ such that $|Y| < \kappa$. If $(Z_i \mid i \in I)$ is an arbitrary family of complexes in $\mathbf{K}(\text{Mod-}\mathcal{A})$, we can construct their coproduct as a componentwise coproduct in $\mathbf{C}(\text{Mod-}\mathcal{A})$. Then whenever $f : Y \to \coprod_{i\in I} Z_i$ is a morphisms in $\mathbf{C}(\text{Mod-}\mathcal{A})$, it is straightforward to see that f factorizes through $\coprod_{i\in J} Z_i$ for some $J \subseteq I$ of cardinality less than κ. Hence Y is κ–small in $\mathbf{K}(\text{Mod-}\mathcal{A})$.

Regarding part (3) of Definition 1.1, consider a morphism $f : Y \to \coprod_{i\in I} Z_i$. We have the following factorization in the abelian category of complexes $\mathbf{C}(\text{Mod-}\mathcal{A})$:

$$Y \xrightarrow{(f_i)} \coprod_{i\in I} \operatorname{Im} f_i \xrightarrow{j} \coprod_{i\in I} Z_i.$$

Here, $f_i : Y \to Z_i$ are the compositions of f with the canonical projections $\pi_i : \coprod_{i'\in I} Z_{i'} \to Z_i$, and j stands for the obvious inclusion. It is easy to see that $|\operatorname{Im} f_i| < \kappa$ for each $i \in I$ and that the morphism j is a coproduct of the inclusions $\operatorname{Im} f_i \to Z_i$. Hence (3) is satisfied.

For the second part, let κ be regular and $\mathcal{T}_\kappa = \operatorname{Loc}\mathcal{S}_\kappa$. Let us denote by $\mathcal{S}'$ a representative set of objects in $\mathcal{S}_\kappa$. It only remains to prove that $\mathcal{S}'$ satisfies condition (1) of Definition 1.1, which is rather easy. Namely, let $X \in \mathcal{T}_\kappa$ such that $\mathcal{T}_\kappa(Y, X) = 0$ for each $Y \in \mathcal{S}'$. Then $\mathcal{T}' = \{Y \in \mathcal{T}_\kappa \mid \mathcal{T}_\kappa(Y, X) = 0\}$ defines a localizing subcategory of $\mathcal{T}_\kappa$ containing $\mathcal{S}_\kappa$. Hence, $\mathcal{T}' = \mathcal{T}_\kappa$ and $X = 0$. □

We will also need (a simplified version of) an important result, which is essentially contained already in [30]. It says that the property of being well generated is preserved when passing to any localizing subcategory generated by a set. In particular, every well generated category is locally well generated.

Proposition 3.4. [24, Theorem 7.2.1] *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ be a set of objects. Then $\operatorname{Loc}\mathcal{S}$ is a well generated triangulated category, too.*

Now, we are in a position to state a theorem which gives us a major source of examples of locally well generated triangulated categories.

Theorem 3.5. *Let $\mathcal{A}$ be a skeletally small additive category. Then the triangulated category $\mathbf{K}(\text{Mod-}\mathcal{A})$ is locally well generated.*

Proof. As in Lemma 3.3, we denote by $\mathcal{S}_\kappa$ the full subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$ formed by complexes of cardinality less than κ and put $\mathcal{T}_\kappa = \operatorname{Loc}\mathcal{S}_\kappa$, the localizing class generated by $\mathcal{S}_\kappa$ in $\mathbf{K}(\text{Mod-}\mathcal{A})$. Then $\mathcal{T}_\kappa$ is (κ–)well generated for each regular cardinal κ by Lemma 3.3 and clearly

$$\mathbf{K}(\text{Mod-}\mathcal{A}) = \bigcup_{\kappa \text{ regular}} \mathcal{S}_\kappa = \bigcup_{\kappa \text{ regular}} \mathcal{T}_\kappa.$$

Now, if $\mathcal{S} \subseteq \mathbf{K}(\text{Mod-}\mathcal{A})$ is a set of objects, then $\mathcal{S} \subseteq \mathcal{T}_\kappa$ for some κ. Hence also $\operatorname{Loc}\mathcal{S} \subseteq \mathcal{T}_\kappa$ and $\operatorname{Loc}\mathcal{S}$ is well generated by Proposition 3.4. It follows that $\mathbf{K}(\text{Mod-}\mathcal{A})$ is locally well generated.

□

Having obtained a large class of examples of locally well generated triangulated categories, one might ask for some basic properties of such categories. We will prove a version of the so-called Bousfield Localization Theorem here:

Proposition 3.6. *Let $\mathcal{T}$ be a locally well generated triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ be a set of objects. Then $\mathcal{T}/\operatorname{Loc}\mathcal{S}$ is a Bousfield localization; that is, there is a localization functor $L : \mathcal{T} \to \mathcal{T}$ such that* $\operatorname{Ker} L = \operatorname{Loc}\mathcal{S}$. *In particular, we have*

$$\operatorname{Im} L = \{X \in \mathcal{T} \mid \mathcal{T}(Y, X) = 0 \text{ for each } Y \in \mathcal{S}\},$$

there is a canonical triangle equivalence between $\mathcal{T}/\operatorname{Loc}\mathcal{S}$ and $\operatorname{Im} L$ given by the composition

$$\operatorname{Im} L \xrightarrow{\subseteq} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\operatorname{Loc}\mathcal{S},$$

and all morphism spaces in $\mathcal{T}/\operatorname{Loc}\mathcal{S}$ are sets.

Proof. The proof is rather standard. $\operatorname{Loc}\mathcal{S}$ is well generated, so it satisfies the Brown Representability Theorem (see Proposition 1.2). Hence the inclusion $\mathbf{i} : \operatorname{Loc}\mathcal{S} \to \mathcal{T}$ has a right adjoint by [30, 8.4.4]. The composition of this right adjoint with $\mathbf{i}$ gives a so-called colocalization functor $\Gamma : \mathcal{T} \to \mathcal{T}$ whose essential image is equal to $\operatorname{Loc}\mathcal{S}$. The definition of a colocalization functor is formally dual to the one of a localization functor; see [24, §4.12] for details. A well-known construction then yields a localization functor $L : \mathcal{T} \to \mathcal{T}$ such that $\operatorname{Ker} L = \operatorname{Loc}\mathcal{S}$. We refer to [30, 9.1.14] or [24, 4.12.1] for details. The rest follows from [30, 9.1.16] or [24, 4.9.1]. □

Remark. Proposition 3.6 has been proved before for well generated triangulated categories. This is implicitly contained for example in [24, §7.2]. It also generalizes more classical results, such as a corresponding statement for the derived category $\mathbf{D}(\mathcal{B})$ of a Grothendieck abelian category $\mathcal{B}$, [2, 5.7]. To see this, one only needs to observe that $\mathbf{D}(\mathcal{B})$ is well generated, see [24, Example 7.7].

An obvious question is whether the Brown Representability Theorem also holds for locally well generated categories, as this was the crucial feature of well generated categories. Unfortunately, this is not the case in general, as the following example suggested by Henning Krause shows.

Example 3.7. According to [9, Exercise 1, p. 131], one can construct an abelian category $\mathcal{B}$ with some Ext-spaces being proper classes. Namely, let U be the class of all cardinals, and let $\mathcal{B} = \text{Mod-}\mathbb{Z}\langle U\rangle$, the category of all "modules over the free ring on the proper class of generators U." That is, an object X of $\mathcal{B}$ is an abelian group such that each $\kappa \in U$ has a $\mathbb{Z}$-linear action on X and this action is trivial for all but a set of cardinals. Such a category admits a valid set-theoretical description in ZFC. If we denote by $\mathbb{Z}$ the object of $\mathcal{B}$ whose underlying group is free of rank 1 and $\kappa \cdot \mathbb{Z} = 0$ for each $\kappa \in U$, then $\text{Ext}^1_{\mathcal{B}}(\mathbb{Z}, \mathbb{Z})$ is a proper class (see also [24, 4.15] or [5, 1.1]).

Given the above description of objects of $\mathcal{B}$, one can easily adjust the proof of Theorem 3.5 to see that $\mathbf{K}(\mathcal{B})$ is locally well generated. Let $\mathbf{K}_{\text{ac}}(\mathcal{B})$ stand for the full subcategory of all acyclic complexes in $\mathbf{K}(\mathcal{B})$. Then $\mathbf{K}_{\text{ac}}(\mathcal{B})$ is clearly a localizing subcategory of $\mathbf{K}(\mathcal{B})$, hence locally well-generated.

It has been shown in [5] that $\mathbf{K}_{\text{ac}}(\mathcal{B})$ does not satisfy the Brown Representability Theorem. In fact, one proved even more: $\mathbf{K}_{\text{ac}}(\mathcal{B})$ is localizing in $\mathbf{K}(\mathcal{B})$, but it is not a kernel of any localization functor $L : \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$. More specifically, the composition of functors, the second of which is contravariant,

$$\mathbf{K}_{\text{ac}}(\mathcal{B}) \xrightarrow{\subseteq} \mathbf{K}(\mathcal{B}) \xrightarrow{\mathbf{K}(\mathcal{B})(-,\mathbb{Z})} \text{Ab}$$

is not representable by any object of $\mathbf{K}_{\text{ac}}(\mathcal{B})$.

Yet another natural question is what other triangulated categories are locally well generated. A deeper analysis of this problem is left for future research, but we will see in Section 4 that $\mathbf{K}(\mathcal{B})$ is locally well generated for any finitely accessible additive category $\mathcal{B}$. For now, we will prove that the class of locally well generated triangulated categories is closed under some natural constructions. Let us start with a general lemma, which holds even if morphism spaces in the quotient $\mathcal{T}/\mathcal{L}$ are proper classes:

Lemma 3.8. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{L} \subseteq \mathcal{L}'$ be two localizing subcategories of $\mathcal{T}$. Then $\mathcal{L}'/\mathcal{L}$ is a localizing subcategory of $\mathcal{T}/\mathcal{L}$.*

Proof. Given the construction of the Verdier quotients and their triangulated structures (see [30, §2.1]), proof of the lemma immediately reduces to proof of the following two statements:

(1) $\mathcal{L}'/\mathcal{L}$ is a full subcategory of $\mathcal{T}/\mathcal{L}$;

(2) $\mathcal{L}'/\mathcal{L}$ is closed under taking isomorphic objects in $\mathcal{T}/\mathcal{L}$.

To this end, let $g : X \to Y$ be a homomorphism in $\mathcal{T}/\mathcal{L}$. Then g is represented by a fraction $f\sigma^{-1}$ of morphisms from $\mathcal{T}$. More precisely, there are triangles

$$Z \xrightarrow{\sigma} X \longrightarrow V \longrightarrow Z[1],$$
$$Z \xrightarrow{f} Y \longrightarrow W \longrightarrow Z[1]$$

in $\mathcal{T}$ such that $V \in \mathcal{L}$. Now, if both $X, Y \in \mathcal{L}'$, then clearly $Z \in \mathcal{L}'$ and $g = f\sigma^{-1}$ is a morphism in $\mathcal{L}'/\mathcal{L}$. This proves (1). On the other hand, if g is an isomorphism, then $W \in \mathcal{L}$ by [30, 2.1.35 and 1.6.8]. If, moreover, $X \in \mathcal{L}'$, then immediately $Z, Y \in \mathcal{L}'$. This shows (2). $\square$

Now we can show that taking localizing subcategories and localizing with respect to a set of objects preserves the locally well generated property.

Proposition 3.9. *Let $\mathcal{T}$ be a locally well generated triangulated category.*

1. *Any localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ is itself locally well generated.*
2. *The Verdier quotient $\mathcal{T}/\operatorname{Loc}\mathcal{S}$ is locally well generated for any set $\mathcal{S}$ of objects in $\mathcal{T}$.*

Proof. (1) is trivial. For (2), put $\mathcal{L} = \operatorname{Loc}\mathcal{S}$ and consider a set $\mathcal{C}$ of objects in $\mathcal{T}/\mathcal{L}$. We have to prove that the localizing subcategory generated by $\mathcal{C}$ in $\mathcal{T}/\mathcal{L}$ is well generated. Since the objects of $\mathcal{T}$ and $\mathcal{T}/\mathcal{L}$ coincide by definition, we can consider a localizing subcategory $\mathcal{L}' \subseteq \mathcal{T}$ defined by $\mathcal{L}' = \operatorname{Loc}(\mathcal{S} \cup \mathcal{C})$. One easily sees using Lemma 3.8 that $\mathcal{L}'/\mathcal{L} = \operatorname{Loc}\mathcal{C}$ in $\mathcal{T}/\mathcal{L}$. Since both $\mathcal{L}$ and $\mathcal{L}'$ are well generated by definition, so is $\mathcal{L}'/\mathcal{L}$ by [24, 7.2.1]. Hence $\mathcal{T}/\mathcal{L}$ is locally well generated. $\square$

We conclude this section with an immediate consequence of Theorem 3.5 and Proposition 3.9, which will be useful in the next section:

Corollary 3.10. *Let $\mathcal{A}$ be a small additive category and $\mathcal{B}$ be a full subcategory of $\mathbf{K}(\text{Mod-}\mathcal{A})$ which is closed under arbitrary coproducts. Then $\mathbf{K}(\mathcal{B})$ is locally well generated.*

4. Finitely accessible additive categories

There is a natural generalization of module categories, namely the additive version of finitely accessible categories in the terminology of [1]. As we have seen, there is quite a lot of freedom to choose $\mathcal{B}$ in the above Corollary 3.10. We will use this fact and a standard trick to (seemingly) generalize Theorem 3.5 from module categories to finitely accessible additive categories. We start with a definition.

Definition 4.1. Let $\mathcal{B}$ be an additive category which admits arbitrary filtered colimits. Then:

- An object $X \in \mathcal{B}$ is called *finitely presentable* if the representable functor $\mathcal{B}(X, -) : \mathcal{B} \to \mathrm{Ab}$ preserves filtered colimits.
- The category $\mathcal{B}$ is called *finitely accessible* if there is a set $\mathcal{A}$ of finitely presentable objects from $\mathcal{B}$ such that every object in $\mathcal{B}$ is a filtered colimit of objects from $\mathcal{A}$.

Note that if $\mathcal{B}$ is finitely accessible, the full subcategory $\mathrm{fp}(\mathcal{B})$ of $\mathcal{B}$ formed by all finitely presentable objects in $\mathcal{B}$ is skeletally small, [1, 2.2]. Several other general properties of finitely accessible categories will follow from Proposition 4.2.

Finitely accessible categories occur at many occasions. The simplest and most natural example is the module category Mod-R over an associative unital ring. It is well-known that finitely presentable objects in Mod-R coincide with finitely presented R–modules in the usual sense. The same holds for Mod-$\mathcal{A}$, the category of modules over a small additive category $\mathcal{A}$. Motivated by representation theory, finitely accessible categories were studied by Crawley-Boevey [7] under the name locally finitely presented categories; see [7, §5] for further examples. The term from [7], however, may cause some confusion in the light of other definitions. Namely, Gabriel and Ulmer [10] have defined the concept of a *locally finitely presentable* category which is, in our terminology, a cocomplete finitely accessible category. As the latter concept has been used quite substantially in one of our main references, [24], we stick to the terminology of [1].

The crucial fact about finitely accessible additive categories is the following representation theorem:

Proposition 4.2. *The assignments*

$$\mathcal{A} \mapsto \text{Flat-}\mathcal{A} \quad \textit{and} \quad \mathcal{B} \mapsto \mathrm{fp}(\mathcal{B})$$

form a bijective correspondence between

(1) *equivalence classes of skeletally small additive categories $\mathcal{A}$ with splitting idempotents, and*
(2) *equivalence classes of additive finitely accessible categories $\mathcal{B}$.*

Proof. See [7, §1.4]. □

Remark. The correspondence from Proposition 4.2 restricts, using [7, §2.2], to a bijection between equivalence classes of skeletally small additive categories with finite colimits (equivalently, with cokernels) and equivalence classes of locally finitely presentable categories in the sense of Gabriel and Ulmer [10].

One of the main results of this paper has now become a mere corollary of preceding results:

Theorem 4.3. *Let $\mathcal{B}$ be a finitely accessible additive category. Then* $\mathbf{K}(\mathcal{B})$ *is locally well generated.*

Proof. Let us put $\mathcal{A} = \mathrm{fp}(\mathcal{B})$, the full subcategory of $\mathcal{B}$ formed by all finitely presentable objects. Using Proposition 4.2, we see that $\mathcal{B}$ is equivalent to the category Flat-$\mathcal{A}$. The category $\mathbf{K}(\text{Flat-}\mathcal{A})$ is locally well generated by Corollary 3.10, and so must be $\mathbf{K}(\mathcal{B})$. □

The remaining question when $\mathbf{K}(\mathcal{B})$ is κ–well generated and which cardinals κ can occur will be answered in the next section. For now, we know by Theorem 2.5 that a neccessary condition is that $\mathcal{B}$ be pure semisimple. In fact, we will show that this is also sufficient, but at the moment we will only give a better description of pure semisimple finitely accessible additive categories.

Proposition 4.4. *Let $\mathcal{B}$ be a finitely accessible additive category. Then the following are equivalent:*

(1) *$\mathcal{B}$ is pure semisimple in the sense of Definition 2.1;*
(2) *Each object in $\mathcal{B}$ is a coproduct of (indecomposable) finitely presentable objects;*
(3) *Each flat right $\mathcal{A}$–module is projective, where $\mathcal{A} = \mathrm{fp}(\mathcal{B})$.*

Proof. For the whole argument, we put $\mathcal{A} = \mathrm{fp}(\mathcal{B})$ and without loss of generality assume that $\mathcal{B} = \text{Flat-}\mathcal{A}$.

(1) $\Longrightarrow$ (3). Assume that Flat-$\mathcal{A}$ is pure semisimple. As in the proof for Proposition 2.2, we can use an obvious generalization of Kaplansky's theorem, [20, Theorem 1], to deduce that there is a cardinal number λ such that each flat $\mathcal{A}$–module is a direct sum of at most λ–generated flat $\mathcal{A}$–modules. The key step is then contained in [12, Corollary 3.6] which says that under the latter condition $\mathcal{A}$ is a right perfect category. That is, it satisfies the equivalent conditions of Bass' theorem [17, B.12] (or more precisely, its version for contravariant functors $\mathcal{A} \to \mathrm{Ab}$). One of the equivalent conditions is condition (3).

(3) $\Longrightarrow$ (2). This is a consequence of Bass' theorem; see [17, B.13].

(2) $\Longrightarrow$ (1). Trivial, $\mathcal{B} = \mathrm{Add} X$ where $X = \bigoplus_{Y \in \mathcal{A}} Y$. □

For further reference, we mention one more condition which one might impose on a finitely accessible additive category. Namely, it is well known that for a ring R, the category Flat-R is closed under products if and only if R is left coherent. This generalizes in a natural way for finitely accessible additive categories. Let us recall that an additive category $\mathcal{A}$ is said to have *weak cokernels* if for each morphism $X \to Y$ there is a morphism $Y \to Z$ such that $\mathcal{A}(Z,W) \to \mathcal{A}(Y,W) \to \mathcal{A}(X,W)$ is exact for all $W \in \mathcal{A}$.

Lemma 4.5. *Let $\mathcal{B}$ be a finitely accessible additive category and $\mathcal{A} = \mathrm{fp}(\mathcal{B})$. Then the following are equivalent:*

(1) *$\mathcal{B}$ has products.*
(2) Flat-$\mathcal{A}$ *is closed under products in* Mod-$\mathcal{A}$.
(3) *$\mathcal{A}$ has weak cokernels.*

Proof. See [7, §2.1]. □

Remark. If $\mathcal{B}$ has products, one can give a more classical proof for Proposition 4.4. Namely, one can then replace the argument by Guil Asensio, Izurdiaga and Torrecillas [12] by an older and simpler argument by Chase [6, Theorem 3.1].

5. When is the homotopy category well generated?

In this final section, we have developed enough tools to answer the question when exactly is the homotopy category of complexes $\mathbf{K}(\mathcal{B})$ well generated if $\mathcal{B}$ is a finitely accessible additive category. This way, we will generalize Proposition 2.6 and also give a rather complete answer to [14, Question 4.2] asked by Holm and Jørgensen. Finally, we will give another criterion for a triangulated category to be (or not to be) well generated and this way construct other classes of examples of categories which are not well generated.

First, we recall a crucial result due to Neeman:

Lemma 5.1. *Let $\mathcal{A}$ be a skeletally small additive category. Then the homotopy category $\mathbf{K}(\text{Proj-}\mathcal{A})$ is $\aleph_1$–well generated. If, moreover, $\mathcal{A}$ has weak cokernels, then $\mathbf{K}(\text{Proj-}\mathcal{A})$ is compactly generated.*

Proof. Neeman has proved in [29, Theorem 1.1] that, given a ring R, the category $\mathbf{K}(\text{Proj-}R)$ is $\aleph_1$–well generated, and if R is left coherent then $\mathbf{K}(\text{Proj-}R)$ is even compactly generated. The actual arguments, contained in [29, §§4–7], immediately generalize to the setting of projective modules over small categories. The role of finitely generated free modules over R is taken by representable functors, and instead of the duality between the categories of left and right projective finitely generated modules we consider the duality between the idempotent completions of the categories of covariant and contravariant representable functors. □

We already know that $\mathbf{K}(\mathcal{B})$ is always locally well generated. When employing Lemma 5.1, we can show the following statement, which is one of the main results of this paper:

Theorem 5.2. *Let $\mathcal{B}$ be a finitely accessible additive category. Then the following are equivalent:*

(1) $\mathbf{K}(\mathcal{B})$ *is well generated;*
(2) $\mathbf{K}(\mathcal{B})$ *is $\aleph_1$–well generated;*
(3) $\mathcal{B}$ *is pure semisimple.*

If, moreover, $\mathcal{B}$ has products, then the conditions are further equivalent to

(4) $\mathbf{K}(\mathcal{B})$ *is compactly generated.*

Proof. (1) $\implies$ (3). If $\mathbf{K}(\mathcal{B})$ is well generated, it is in particular generated by a set of objects as a localizing subcategory of itself; see Proposition 1.2. Hence $\mathcal{B}$ is pure semisimple by Theorem 2.5.

(3) $\implies$ (2) and (4). If $\mathcal{B}$ is pure semisimple and $\mathcal{A} = \mathrm{fp}(\mathcal{B})$, then $\mathcal{B}$ is equivalent to Flat-$\mathcal{A}$ by Proposition 4.2, and Flat-$\mathcal{A}$ = Proj-$\mathcal{A}$ by Proposition 4.4. The conlusion follows by Lemmas 5.1 and 4.5.

(2) or (4) $\implies$ (1). This is obvious. □

Remark. (1) Neeman proved in [29] more than stated in Lemma 5.1. He described a particular set of generators for $\mathbf{K}(\text{Proj-}\mathcal{A})$ satisfying conditions of Definition 1.1. Namely, $\mathbf{K}(\text{Proj-}\mathcal{A})$ is always $\aleph_1$–well generated by a representative set of bounded below complexes of finitely generated projectives. Moreover, he gave an explicit description of compact objects in $\mathbf{K}(\text{Proj-}\mathcal{A})$ in [29, 7.12].

(2) An exact characterization of when $\mathbf{K}(\mathcal{B})$ is compactly generated and thereby a complete answer to [14, Question 4.2] does not seem to be known. We have shown that this reduces to the problem when $\mathbf{K}(\text{Proj-}\mathcal{A})$ is compactly generated. A sufficient condition is given in Lemma 5.1, but it is probably not necessary. On the other hand, if $R = k[x_1, x_2, x_3, \dots]/(x_i x_j;\ i, j \in \mathbb{N})$ where k is a field, then $\mathbf{K}(\text{Flat-}R)$ coincides with $\mathbf{K}(\text{Proj-}R)$, but the latter is not a compactly generated triangulated category; see [29, 7.16] for details.

Example 5.3. The above theorem adds other locally well generated but not well generated triangulated categories to our repertoire. For example $\mathbf{K}(\mathcal{TF})$, where $\mathcal{TF}$ stands for the category of all torsion-free abelian groups, has this property.

We finish the paper with some examples of triangulated categories where the fact that they are not generated by a set is less obvious. For this purpose, we will use the following criterion:

Proposition 5.4. *Let $\mathcal{T}$ be a locally well generated triangulated category and $\mathcal{L}$ be a localizing subcategory. Consider the diagram*

$$\mathcal{L} \xrightarrow{\subseteq} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{L}.$$

If two of the categories $\mathcal{L}$, $\mathcal{T}$ and $\mathcal{T}/\mathcal{L}$ are well generated, so is the third.

Proof. If $\mathcal{L} = \operatorname{Loc}\mathcal{S}$ and $\mathcal{T}/\mathcal{L} = \operatorname{Loc}\mathcal{C}$ for some sets $\mathcal{S}, \mathcal{C}$, let $\mathcal{L}'$ be the localizing subcateogry of $\mathcal{T}$ generated by the set of objects $\mathcal{S} \cup \mathcal{C}$. Lemma 3.8 yields the equality $\mathcal{T}/\mathcal{L} = \mathcal{L}'/\mathcal{L}$. Hence also $\mathcal{T} = \mathcal{L}'$, so $\mathcal{T}$ is generated by a set, and consequently $\mathcal{T}$ is well generated.

If $\mathcal{L}$ and $\mathcal{T}$ are well generated, so is $\mathcal{T}/\mathcal{L}$ by [24, 7.2.1]. Finally, one knows that $X \in \mathcal{T}$ belongs to $\mathcal{L}$ if and only if $QX = 0$; see [30, 2.1.33 and 1.6.8]. Therefore, if $\mathcal{T}$ and $\mathcal{T}/\mathcal{L}$ are well generated, so is $\mathcal{L}$ by [24, 7.4.1]. □

Remark. We stress here that by saying that $\mathcal{T}/\mathcal{L}$ is well generated, we in particular mean that $\mathcal{T}/\mathcal{L}$ is a usual category in the sense that all morphism spaces are sets and *not* proper classes.

Now we can conclude by showing that some homotopy categories of acyclic complexes are not well generated.

Example 5.5. Let R be a ring, $\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R)$ be the full subcategory of $\mathbf{K}(\text{Mod-}R)$ formed by all acyclic complexes, and $\mathcal{L} = \operatorname{Loc}\{R\}$. It is well-known but also an easy consequence of Proposition 3.6 that the composition

$$\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R) \xrightarrow{\subseteq} \mathbf{K}(\text{Mod-}R) \xrightarrow{Q} \mathbf{K}(\text{Mod-}R)/\mathcal{L}$$

is a triangle equivalence between $\mathbf{K}(\text{Mod-}R)/\mathcal{L}$ and $\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R)$.

By Proposition 2.6, $\mathbf{K}(\text{Mod-}R)$ is well generated if and only if R is right pure semisimple. Therefore, $\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R)$ is well generated if and only if R is right pure semisimple by Proposition 5.4. In fact, $\mathbf{K}_{\mathrm{ac}}(\text{Mod-}R)$ is not generated by any set of objects if R is not right pure semisimple. As particular examples, we may take $R = \mathbb{Z}$ or $R = k(\cdot \rightrightarrows \cdot)$ for any field k.

Example 5.6. Let $\mathcal{B}$ be a finitely accessible category. Recall that $\mathcal{B}$ is equivalent to Flat-$\mathcal{A}$ for $\mathcal{A} = \operatorname{fp}(\mathcal{B})$. Then the natural exact structure on Flat-$\mathcal{A}$ coming from Mod-$\mathcal{A}$ is nothing else than the well-known exact structure given by pure exact short sequences in $\mathcal{B}$ (see eg. [7]).

We denote by $\mathbf{K}_{\mathrm{pac}}(\text{Flat-}\mathcal{A})$ the full subcategory of $\mathbf{K}(\text{Flat-}\mathcal{A})$ formed by all complexes exact with respect to this exact structure, and call such complexes *pure acyclic*. More explicitly, $X \in \mathbf{K}(\text{Flat-}\mathcal{A})$ is pure acyclic if and only if X is acyclic in Mod-$\mathcal{A}$ and all the cycles $Z^i(X)$ are flat. Note that $\mathbf{K}_{\mathrm{pac}}(\text{Flat-}\mathcal{A})$ is closed under taking coproducts in $\mathbf{K}(\text{Flat-}\mathcal{A})$.

Neeman proved in [29, Theorem 8.6] that $X \in \mathbf{K}(\text{Flat-}\mathcal{A})$ is pure acyclic if and only if there are no non-zero homomorphisms from any $Y \in \mathbf{K}(\text{Proj-}\mathcal{A})$ to X. Then either by combining Proposition 3.6 with Lemma 5.1 or by using [29, 8.1 and 8.2], one shows that the composition

$$\mathbf{K}_{\mathrm{pac}}(\text{Flat-}\mathcal{A}) \xrightarrow{\subseteq} \mathbf{K}(\text{Flat-}\mathcal{A}) \xrightarrow{Q} \mathbf{K}(\text{Flat-}\mathcal{A})/\mathbf{K}(\text{Proj-}\mathcal{A})$$

is a triangle equivalence. Now again, Proposition 5.4 implies that $\mathbf{K}_{\mathrm{pac}}(\text{Flat-}\mathcal{A})$ is well generated if and only if $\mathcal{B}$ is pure semisimple. If $\mathcal{B}$ is of the form Flat-R for a ring R, this precisely means that R is right perfect.

As a particular example, $\mathbf{K}_{\mathrm{pac}}(\mathcal{TF})$ is locally well generated but not well generated, where $\mathcal{TF}$ stands for the class of all torsion-free abelian groups.

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