Test sets for factorization properties of modules

Jan Trlifaj, Univerzita Karlova, Praha

Algebra Day
*****
In honor of Riccardo Colpi's 60th birthday
****
Department of Statistical Sciences, University of Padova

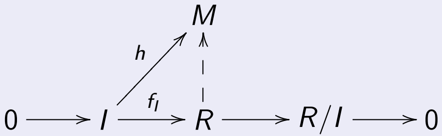
I. Baer Criterion, and testing for (generalized) injectivity

Testing for injectivity

Baer's Criterion '1940

Injectivity coincides with R-injectivity for any ring R and any module M.

M is R-injective, if for each right ideal I, all $h \in \mathrm{Hom}_R(I, M)$ extend to R:



Equivalently: $\mathrm{Ext}^1_R(R/I, M) = 0$ for each right ideal I of R.

So $\{f_I \mid I \in \mathfrak{I}\}$ is a test set for $\mathcal{I}_0$ consisting of monomorphisms:
M is injective, iff $\mathrm{Hom}_R(f_I, M)$ is surjective for each $I \in \mathfrak{I}$.
One morphism suffices: M is injective, iff $\mathrm{Hom}_R(\bigoplus_{I \in \mathfrak{I}} f_I, M)$ is surjective.

Testing for generalized injectivity

Right-hand classes of cotorsion pairs

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathrm{Mod}\text{–}R$ generated by a set $\mathcal{S}$, i.e., $\mathcal{B} = \mathcal{S}^{\perp}$ for a set $\mathcal{S} \subseteq \mathcal{A}$.

For each $A \in \mathcal{S}$, let $0 \to K_A \stackrel{f_A}{\to} P_A \to A \to 0$ be a projective presentation of A.

Then $M \in \mathcal{B}$, iff $\mathrm{Hom}_R(\bigoplus_{A \in \mathcal{S}} f_A, M)$ is surjective. That is, $\{f_A \mid A \in \mathcal{S}\}$ is a test set for $\mathcal{B}$ consisting of monomorphisms.

Corollary

There are test sets of monomorphisms for the properties of being

- a module of injective dimension $\leq n$ (for each fixed $n < \omega$),
- a cotorsion module (in the sense of Enochs, Matlis, and Warfield).

There is a test set of pure monomorphisms for the property of being a pure-injective module.

R-projectivity

M is R-projective, if for each right ideal I, all $h \in \mathrm{Hom}_R(M, R/I)$ factorize through π_I:

$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & & \downarrow & \searrow^{h} & & & \\ 0 & \longrightarrow & I & \xrightarrow{\subseteq} & R & \xrightarrow{\pi_I} & R/I & \longrightarrow & 0 \end{array}$$

If $\mathrm{Ext}^1_R(M, I) = 0$ for each right ideal I of R, then M is R-projective. The converse holds when R is right self-injective, but not in general.

The Dual Baer Criterion (DBC for short) holds for a ring R, in case projectivity coincides with R-projectivity for any module M.

Faith' Problem '1976

For what kind of rings R does DBC hold?

Partial positive answers

- For any ring R, projectivity $=$ R-projectivity for any finitely generated module M, i.e., **DBC holds for all finitely generated modules over any ring**.
- Sandomierski'1964, Ketkar-Vanaja'1981:
  DBC holds for all modules over any right perfect ring.
- Let K be a skew-field, κ an infinite cardinal, and R the endomorphism ring of a κ-dimensional left vector space over K. Then DBC holds for all $\leq \kappa$-generated modules.

In particular, if R is right perfect, then there is always a test set of epimorphisms for $\mathcal{P}_0$ $\{g_j \mid j \in J\}$: M is projective, iff $\mathrm{Hom}_R(M, g_j)$ is surjective for each $j \in J$.
Again, one morphism suffices: M is projective, iff $\mathrm{Hom}_R(M, \prod_{j \in J} g_j)$ is surjective.

The dual setting: testing for generalized projectivity

Left-hand classes of cotorsion pairs

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathrm{Mod}\text{–}R$ cogenerated by a set $\mathcal{S}$, i.e., $\mathcal{A} = {}^{\perp}\mathcal{S}$ for a set $\mathcal{S} \subseteq \mathcal{B}$.
For each $B \in \mathcal{S}$, let $0 \to B \to I_B \stackrel{g_B}{\to} F_B \to 0$ be an injective copresentation of B.
Then $M \in \mathcal{A}$, iff $\mathrm{Hom}_R(M, \prod_{B \in \mathcal{S}} g_B)$ is surjective. That is, $\{g_B \mid B \in \mathcal{S}\}$ is a test set for $\mathcal{A}$ consisting of epimorphisms.

Corollary

There is a test set of epimorphisms for the property of being a module of weak dimension $\leq n$ (for each fixed $n < \omega$).

Example: Dedekind domains

Let R be a Dedekind domain and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathrm{Mod}\text{–}R$.

- Assume $\mathcal{F}_0 \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a cotilting class, and there is a test set of epimorphisms for $\mathcal{A}$.
- Eklof-Shelah-T.'2004: Assume $\mathcal{F}_0 \nsubseteq \mathcal{A}$, Assume moreover that R has countable spectrum, and $\mathfrak{C}$ is generated by a set. Then it is consistent with ZFC + GCH that there is no test set of epimorphisms for $\mathcal{A}$ of the form above. That is, $\mathfrak{C}$ is not cogenerated by a set.

Partial negative solutions to Faith's Problem in ZFC

- Hamsher'1967: If R is a commutative noetherian, but not artinian, ring then there exists a countably generated R-projective module which is not projective.
  So DBC fails for countably generated modules.
- Puninski et al.'2017: If R is a semilocal right noetherian ring. Then DBC holds, iff R is right artinian.

II. Existence/non-existence of sets of epimorphisms testing for projectivity

A set-theoretic barrier

Assume Shelah's Uniformization Principle.
Let κ be an uncountable cardinal of cofinality ω. Then for each non-right perfect ring R of cardinality $\leq \kappa$ there exists a κ^+-generated module M of projective dimension 1 such that $\mathrm{Ext}^1_R(M, N) = 0$ for each module N of cardinality $< \kappa$.

Corollary

- **It is consistent with ZFC + GCH that there is no test set of epimorphisms for $\mathcal{P}_0$ over any non-right perfect ring R.**
- **In particular, it is consistent with ZFC + GCH that DBC fails for each non-right perfect ring R.**

Refinements of Faith's Problem

- Are the consistency results above actually provable in ZFC?
- If not, what is the border line between those non-right perfect rings, for which there is no test set of epimophisms for $\mathcal{P}_0$ in ZFC, and those, for which the existence of such set is independent of ZFC?
- What is the border line between those non-right perfect rings, for which DBC fails in ZFC, and those, for which it is independent of ZFC?

A positive consistency result

Assume the Axiom of Constructibility.

Let R be a non-right perfect ring, $\kappa = 2^{\text{card}(R)}$, F be the free module of rank κ, and M be a module of finite projective dimension.

Then M is projective, iff $\text{Ext}^i_R(M, F) = 0$ for all $i > 0$.

Corollary

- **It is consistent with ZFC + GCH that there is a test set of epimorphisms for $\mathcal{P}_0$ over any ring of finite global dimension.**
- **The assertion 'For each non-right perfect ring of finite global dimension, there exists a test set of epimorphisms for $\mathcal{P}_0$' is independent of ZFC + GCH.**

Further positive consistency results

Flatness can always be expressed by vanishing of Ext, in ZFC. So we have

Corollary

Let R be a ring such that each flat module has finite projective dimension. Then the existence of test set of epimorphisms for $\mathcal{P}_0$ is consistent with ZFC + GCH.

Corollary

The existence of test set of epimorphisms for $\mathcal{P}_0$ is independent of ZFC + GCH whenever R is a ring which is either

- n-Iwanaga-Gorenstein, for $n > 0$, or
- commutative noetherian with $0 < \mathrm{Kdim}(R) < \infty$, or
- almost perfect, but not perfect.

Back to Faith's Problem ...

Recall: By Hamsher'1967, DBC fails in ZFC already for all hereditary (= Dedekind) domains. Let's explore other hereditary rings ...

III. Transfinite extensions of simple artinian rings

Semiartinian rings

- A ring R is right semiartinian, if R is the last term of the right Loewy sequence of R, i.e., there are an ordinal σ and a strictly increasing sequence $(S_\alpha \mid \alpha \leq \sigma + 1)$, such that $S_0 = 0$, $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(R/S_\alpha)$ for all $\alpha \leq \sigma$, $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $S_{\sigma+1} = R$.
- R is von Neumann regular, if all (right R-) modules are flat.
- R has right primitive factors artinian (has right pfa for short) in case R/P is right artinian for each right primitive ideal P of R.

Let R be a regular ring.

- R is right semiartinian, iff it is left semiartinian, and the right and left Loewy sequences of R coincide.
- R has right pfa, iff it has left pfa, iff all homogenous completely reducible (left or right) modules are injective.

Structure of semiartinian regular rings with pfa

Let R be a right semiartinian ring and $(S_\alpha \mid \alpha \leq \sigma + 1)$ be the right Loewy sequence of R with $\sigma \geq 1$. The following conditions are equivalent:

- R is regular with pfa.
- for each $\alpha \leq \sigma$ there are a cardinal λ_α, positive integers $n_{\alpha\beta}$ $(\beta < \lambda_\alpha)$ and skew-fields $K_{\alpha\beta}$ $(\beta < \lambda_\alpha)$ such that $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$, as rings without unit. Moreover, λ_α is infinite iff $\alpha < \sigma$.
The pre-image of $M_{n_{\alpha\beta}}(K_{\alpha\beta})$ in this isomorphism coincides with the βth homogenous component of $\mathrm{Soc}(R/S_\alpha)$, and it is finitely generated as right R/S_α-module for all $\beta < \lambda_\alpha$.

$P_{\alpha\beta}$:= a representative of simple modules in the βth homogenous component of $S_{\alpha+1}/S_\alpha$. $Zg(R) := \{P_{\alpha\beta} \mid \alpha \leq \sigma, \beta < \lambda_\alpha\}$ is a set of representatives of all simple modules, and also the Ziegler spectrum of R. The Cantor-Bendixson rank of $Zg(R)$ is σ.

Transfinite extensions of simple artinian rings

$\mathbf{R/S_\sigma}$	$\mathbf{M_{n_{\sigma 0}}(K_{\sigma 0})} \oplus \quad \dots \quad \oplus \mathbf{M_{n_{\sigma,\lambda_\sigma - 1}}(K_{\sigma,\lambda_\sigma - 1})}$	
...	... ...	
$\mathbf{S_{\alpha+1}/S_\alpha}$	$\mathbf{M_{n_{\alpha 0}}(K_{\alpha 0})} \oplus \quad \dots \quad \oplus \mathbf{M_{n_{\alpha\beta}}(K_{\alpha\beta})} \oplus \dots$	$\beta < \lambda_\alpha$
...	... ...	
$\mathbf{S_2/S_1}$	$\mathbf{M_{n_{10}}(K_{10})} \oplus \quad \dots \quad \oplus \mathbf{M_{n_{1\beta}}(K_{1\beta})} \oplus \dots$	$\beta < \lambda_1$
$\mathbf{S_1 = Soc(R)}$	$\mathbf{M_{n_{00}}(K_{00})} \oplus \quad \dots \quad \oplus \mathbf{M_{n_{0\beta}}(K_{0\beta})} \oplus \dots$	$\beta < \lambda_0$

$\sigma + 1 =$ **Loewy length** of R (at least 2).

$\lambda_\alpha =$ **number of homogenous components** of the αth layer of R $(\alpha \leq \sigma)$. *Infinite except for* $\alpha = \sigma$.

$n_{\alpha\beta} =$ (finite) **dimension** of βth homogenous component in αth layer.

$K_{\alpha\beta} =$ **endomorphism skew-field** of a simple module in βth homogenous component of the αth layer $(\alpha \leq \sigma,\ \beta < \lambda_\alpha)$.

The hereditary setting

Lemma

If R has countable Loewy length and the αth layer of the socle sequence of R is countably generated for each $0 < \alpha < \sigma$, then R is hereditary. In particular, R is always hereditary in case it has Loewy length 2.

The simplest example

The K-algebra of all eventually constant sequences in K^ω over a field K.

$\mathbf{R/S_1 = K}$ $\qquad\qquad\qquad\qquad$ $\mathbf{K}$

$\mathbf{S_1 = K^{(\omega)}}$ $\qquad$ $\mathbf{K \oplus \quad ... \quad \oplus K \oplus \quad ... \quad ...}$

Further hereditary examples

$\mathbf{R/S_n = K}$			$\mathbf{K}$			
$\mathbf{S_n/S_{n-1} = K^{(\omega)}}$	$\mathbf{K} \oplus$	...	$\oplus \mathbf{K} \oplus$	...		
...	...	...	...	...		
$\mathbf{S_3/S_2 = K^{(\omega)}}$	$\mathbf{K} \oplus$	...	$\oplus \mathbf{K} \oplus$	...		
$\mathbf{S_2/S_1 = K^{(\omega)}}$	$\mathbf{K} \oplus$	...	$\oplus \mathbf{K} \oplus$	...		
$\mathbf{Soc(R) = K^{(\kappa)}}$	$\mathbf{K} \oplus$	...	$\oplus \mathbf{K} \oplus$	...	...	...

R is a hereditary K-subalgebra of K^κ of Loewy length $n+1$.

The simplest non-hereditary example

$\mathbf{R/S_2 = K}$
$\mathbf{S_2/S_1 = K^{(\omega_1)}}$
$\mathbf{Soc(R) = K^{(\omega)}}$

		$\mathbf{K}$			
$\mathbf{K} \oplus$	...	$\oplus \mathbf{K} \oplus$	...	...	...
$\mathbf{K} \oplus$	...	$\oplus \mathbf{K} \oplus$	...		

R is a K-subalgebra in K^ω of Loewy length 3.
(The second socle S_2 is not projective.)

IV. Dual Baer Criterion for small transfinite extensions

A weaker version of R-projectivity

R = semiartinian regular ring with pfa of Loewy length $\sigma + 1$, M a module.

Consider the following conditions:

1. M is R-projective.
2. For each $0 < \alpha \leq \sigma$, each homomorphism $f : M \to S_{\alpha+1}/S_\alpha$ factorizes through the projection $\pi_\alpha : S_{\alpha+1} \to S_{\alpha+1}/S_\alpha$.
3. M is weakly R-projective, i.e., each homomorphism $f : M \to S_{\alpha+1}/S_\alpha$ with a finitely generated image factorizes through the projection π_α.

Then $1 \Longrightarrow 2 \Longrightarrow 3$. If σ is finite, then $1 \iff 2$.

- If σ is finite, then all countably generated weakly R-projective modules are projective.
- Weakly R-projective modules are closed under submodules.
- S_2 from the simplest non-hereditary example above is weakly R-projective, but not R-projective.

The hereditary case revisited

Let R be a semiartinian regular ring with pfa. Then R is small, if R is of finite Loewy length, has countably generated consecutive Loewy factors, and $\mathrm{card(R)} \leq 2^\omega$. Note: A small ring is hereditary.

A consistency result

Assume the Axiom of Constructibility. Let R be small. Then the notions of a projective, R-projective, and weakly R-projective module coincide.
In particular, DBC holds for all modules.

An independence result

Let R be small. Then the validity of DBC is independent of ZFC + GCH.

References I

R.Baer, *Abelian groups that are direct summands of every containing abelian group*, Bull. AMS 46(1940), 800-806.

F.Sandomierski, *Relative Injectivity and Projectivity*, Penn State U. 1964.

R.M.Hamsher, *Commutative rings over which every module has a maximal submodule*, Proc. AMS **18**(1967), 1133-1137.

F.W.Anderson, K.R.Fuller, *Rings and Categories of Modules*, LNM 13, Springer-Verlag, N.Y. 1974.

C.Faith, *Algebra II. Ring Theory*, GMW 191, Springer-Verlag, Berlin 1976.

References II

P.C.Eklof, S.Shelah, *On Whitehead modules*, J. Algebra 142(1991), 492-510.

J.Trlifaj, *Whitehead test modules*, Trans. AMS 348(1996), 1521-1554.

P.C.Eklof, S.Shelah, J.Trlifaj, *On the cogeneration of cotorsion pairs*, J. Algebra 277(2004), 572-578.

J.Šťovíček, J.Trlifaj, *All tilting modules are of countable type*, Bull. LMS 39(2007), 121-132.

H.Alhilali, Y.Ibrahim, G.Puninski, M.Yousif, *When R is a testing module for projectivity?*, J. Algebra 484(2017), 198-206.

J.Trlifaj, *Faith's problem on R-projectivity is undecidable*, Proc. AMS 147(2019), 497-504.

J.Trlifaj, *The dual Baer criterion for non-perfect rings*, arXiv:1901.01442v2.

J.Šaroch, J.Trlifaj, *Test sets for factorization properties of modules*, manuscript.

Riccardo, thanks for all, and happy birthday to you!