THE WHITEHEAD PROBLEM AND BEYOND
(LECTURE NOTES FOR NMAG565)

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Abstract. These notes present a significant milestone of modern algebra due to Saharon Shelah: the independence (of ZFC + GCH) of the existence of non-free Whitehead groups, i.e., the undecidability of the Whitehead problem. The independence is proved by employing combinatorial properties of infinite cardinals, notably Shelah's Uniformization Principle (SUP) and the diamond prediction principles.

First, we prove in ZFC that all countable Whitehead groups are free. SUP is then employed to construct arbitrarily large non-free Whitehead groups. Finally, we show that it is consistent with ZFC + GCH that all Whitehead groups W are free: the proof is by induction on the cardinality, κ, of W, using the Weak Diamond Principle Φ when κ is a regular uncountable cardinal, and Shelah's Singular Compactness in the case when κ is singular.

Though undecidability of the Whitehead problem for groups is the main topic here, some of the results are proved in more general settings, providing thus tools for further applications of set-theoretic methods in homological algebra and representation theory.

Contents

1. The Whitead problem

In the late 1940s, Whitehead asked whether each (abelian) group A such that $\operatorname{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) = 0$ is free. This question became known as the *Whitehead Problem.* Stein [25] provided a positive answer for A countable, but it was only in 1974 that Shelah [21] proved that the answer is independent of ZFC for groups of cardinality $\aleph_1$. Soon after, making use of his celebrated Singular Compactness Theorem [22], Shelah proved undecidability of the Whitehead problem for all groups A. This result, and the related more recent general works in the setting of modules over non-perfect rings, are the main topics of the present notes.

Shelah's solution to the Whitehead Problem was the starting point of a new branch of algebra dealing with applications of set-theoretic methods in representation and module theory. Let us stress that these applications are not restricted to independence results. They provide powerful techniques making it possible to work in ZFC with large representations/modules expressed as unions of chains, or direct limits of direct systems, of smaller modules. Besides almost free modules [6], more

Date: February 16, 2023.

recent applications include the structure theory of infinite dimensional tilting and Mittag-Leffler modules [11, Vol. 1], [2], [17], properties of indecomposable modules [11, Vol. 2], [16], approximations of modules [11, Vol. 1], [18], relative homological algebra [9], [19], [28], etc.

Definition 1.1. Let R be a ring. A module $M \in \text{Mod}\text{–}R$ is *Whitehead* provided that $\operatorname{Ext}^1_R(M,R) = 0$.

Remark 1.2. Clearly, any projective, and hence any free, module over any ring R is Whitehead. But the answer to the (generalized) Whitehead question of whether all Whitehead modules are projective depends on the ring R in case.

For example, if R is right self-injective then *all* modules are Whitehead, and if R is a cotorsion Dedekind domain, then all torsion-free modules are Whitehead. However, we will see below that the answer to the Whitehead question for $R = \mathbb{Z}$ (and more in general, for non-cotorsion PID's of cardinality $\leq \omega_1$) is independent of ZFC + GCH.

Lemma 1.3.
(1) *Let R be a right hereditary ring. Then all submodules of Whitehead modules are Whitehead.*
(2) *Let R be a Dedekind domain. Then all Whitehead modules are torsion-free.*

Proof. 1. If $N \subseteq M$ and M is Whitehead, then we have the exact sequence $0 = \operatorname{Ext}^1_R(M,R) \to \operatorname{Ext}^1_R(N,R) \to \operatorname{Ext}^2_R(M/N,R) = 0$ where the latter Ext-group is zero, because R is right hereditary.

2. Let I be any proper ideal of R. Since R is a Dedekind domain, I is projective and finitely generated, hence $I \oplus I' \cong R^n$ for some module I' and $0 < n < \omega$. The non-split short exact sequence $0 \to I \overset{\subseteq}{\to} R \to R/I \to 0$ shows that $\operatorname{Ext}^1_R(R/I, I) \neq 0$, whence also $\operatorname{Ext}^1_R(R/I,R) \neq 0$. So by part 1., if M is a Whitehead module, then R/I does not embed into M. In other words, M is torsion-free. □

For countable abelian groups, the following lemma (known as Pontryagins' Criterion) will be useful:

Lemma 1.4. *The following are equivalent for an abelian group A:*
(1) *A is ω_1-free (i.e., each countable subgroup of A is free),*
(2) *A is torsion-free, and each finite rank (pure) subgroup of A is free.*

Proof. Since finite rank torsion-free groups are countable, we are left to prove that 2. implies 1. Assume 2. and let B be a countable subgroup of A, generated by $\{b_n \mid n < \omega\}$. By induction on $n < \omega$, we can define a chain of pure subgroups $B_n \subseteq_* A$ such that $\sum_{m<n} b_m\mathbb{Z} \trianglelefteq B_n$ (see e.g. [6, IV.2.1]). Since each B_n is of finite rank (and pure in A), it is free by the assumption. As B_n is pure in B_{n+1}, the group B_{n+1}/B_n is finitely generated and torsion-free, and hence free. So B_n is a direct summand in B_{n+1}, and $\bigcup_{n<\omega} B_n$ is a free group containing B. Thus B is free, too. □

We now arrive at the classic result by Stein [25]. Its proof presented below follows [10, §99]. But first we recall the definition and basic properties of Prüfer groups:

Let p be a prime number. Let $\mathbb{Z}_{p^\infty}$ denote the *Prüfer p-group*, that is, $\mathbb{Z}_{p^\infty} = F/G$ where $F = \mathbb{Z}^{(\omega)}$ is the free group with the canonical basis $\{1_i \mid i < \omega\}$, and G is the subgroup of F generated by the elements $1_0.p$, and $1_i - 1_{i+1}.p$ for all $i < \omega$.

Lemma 1.5. *Let p be a prime number. Then $\mathbb{Z}_{p^\infty}$ is isomorphic to the p-torsion part of the torsion group $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$, where P is the set of all prime numbers.*

Moreover, $\mathbb{Z}_{p^\infty}$ is a uniserial group, its only proper subgroups being $(\mathbb{Z}^n + G)/G \cong \mathbb{Z}_{p^n}$ $(0 < n < \omega)$, and $\operatorname{End} \mathbb{Z}_{p^\infty} \cong \mathbb{J}_p$ is the ring of all p-adic integers.

Proof. By [10, §3 and §43, Ex. 3]. □

Theorem 1.6. *Let A be a Whitehead group. Then A is ω_1-free.*
In particular, each countable Whitehead group is free.

Proof. In view of Lemma 1.3, it suffices to prove the second claim. Since finitely generated torsion-free groups are free, Lemma 1.4 implies that we only have to prove that if W is a Whitehead group of finite rank, then W is finitely generated.

Assume this is not the case. Let n be the rank of W, so $\mathbb{Z}^n \trianglelefteq W \trianglelefteq \mathbb{Q}^n$. Then we have the exact sequence

$$\cdots \to H = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}) \to E = \mathrm{Ext}^1_{\mathbb{Z}}(T, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(W, \mathbb{Z}) = 0$$

where $T = W/\mathbb{Z}^n$ is a torsion group of cardinality ω. Clearly, $H \cong \mathbb{Z}^n$ is countable.

Let S be the socle of T. If S is not finitely generated, then S is an infinite direct sum of finite groups of prime order. Hence $F = \mathrm{Ext}^1_{\mathbb{Z}}(S, \mathbb{Z})$ is an infinite product of the non-zero groups $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z})$ for some primes p. In particular, F is uncountable. Since $S \subseteq T$ and $\mathbb{Z}$ is a hereditary ring, F is a homomorphic image of E. Hence E is uncountable, too, in contradiction with E being a homomorphic image of H.

If S is finitely generated, then $S \trianglelefteq T \trianglelefteq D$, where D is the injective (= divisible) hull of T. So D is a finite direct sum (say of $0 < m < \omega$ copies) of the Prüfer p-groups $\mathbb{Z}_{p^\infty}$ for some primes p. By induction on m we will show that T contains a copy of a Prüfer group. By Lemma 1.5, all proper subgroups of $\mathbb{Z}_{p^\infty}$ are finite, but T is infinite, so the assertion is clear for $m = 1$.

For the inductive step, we have $T \subseteq D = D' \oplus \mathbb{Z}_{p^\infty}$ and $T \nsubseteq D'$, for some prime p. If $T \cap D'$ is infinite, then it contains a copy of the Prüfer group by the inductive premise, and so does T. If $T \cap D'$ is finite, then $T/(T \cap D') \cong (T + D')/D'$ is an infinite subgroup of $D/D' \cong \mathbb{Z}_{p^\infty}$, whence $T/(T \cap D') \cong \mathbb{Z}_{p^\infty}$. Let $n < \omega$ be such that $p^n(T \cap D') = 0$. Then multiplication by p^n is an endomorphism of T whose kernel contains $T \cap D'$, so its non-zero image is a homomorphic image of, and hence isomorphic to, $\mathbb{Z}_{p^\infty}$.

Thus T contains a direct summand isomorphic to $\mathbb{Z}_{p^\infty}$. By Lemma 1.5, E has a direct summand isomorphic to

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) \cong \mathbb{J}_p.$$

However, $\mathbb{J}_p$ is uncountable, and so is E, a contradiction. □

Remark 1.7. The freeness of Whitehead groups can be proved in ZFC even within certain classes of groups much larger than the class of all countable groups: Let Γ_0 denote the class of all countable groups, and for each ordinal $\alpha > 0$, let Γ_α be the class of all torsion-free groups G containing a pure subgroup of finite rank, H, such that G/H is a direct sum of groups each of which belongs to a class Γ_β for some $\beta < \alpha$. In [14], Theorem 1.6 was extended as follows: if G is a Whitehead group such that $G \in \Gamma_\alpha$ for an ordinal α, then G is free.

2. Shelah's Uniformization Principle and the vanishing of Ext

Let R be a *non-right perfect ring*, that is, R is a ring containing a sequence of elements $(a_i \mid i < \omega)$ such that

$$(*) \quad Ra_0 \supsetneq Ra_1a_0 \supsetneq \dots Ra_n...a_0 \supsetneq Ra_{n+1}a_n...a_0 \supsetneq \dots$$

is a strictly decreasing chain of principal left ideals of R.

For example, $R = \mathbb{Z}$ is non-right perfect; in fact, so is any right noetherian ring which is not right artinian (by [1, 15.20, 15.22, and 28.4]). In particular, commutative noetherian rings are non-right perfect, iff their Krull dimension is at least 1 (by [9, 2.4.27]).

A distinctive feature of non-right perfect rings is the existence of *Bass modules*, i.e., countably presented flat modules of projective dimension 1:

Lemma 2.1. *Let R be non-right perfect and $(a_i \mid i < \omega)$ a sequence of elements of R such that the chain $(*)$ is strictly decreasing. Let $(1_i \mid i < \omega)$ be the canonical free basis of the free module $R^{(\omega)}$. Consider the short exact sequence*

$$0 \to R^{(\omega)} \overset{\nu}{\to} R^{(\omega)} \to B \to 0$$

where ν is defined by $\nu(1_i) = 1_i - 1_{i+1}.a_i$ for each $i < \omega$. Then B is a Bass module.

Proof. The short exact sequence does not split by [1, 28.2], whence proj.dim $B = 1$. However, it is easy to see that $\nu(R^{(n)})$ is a direct summand in $R^{(\omega)}$ for each $n < \omega$, so $B \cong R^{(\omega)}/\bigcup_{n<\omega} \nu(R^{(n)})$ is a direct limit of the countable direct system of projective modules $(R^{(\omega)}/\nu(R^{(n)}) \mid n < \omega)$, whence B is flat. $\square$

Throughout this section, we will assume that R is a non-right perfect ring. We will fix a sequence $(a_i \mid i < \omega)$ of elements of R such that the chain $(*)$ is strictly decreasing, as well as the corresponding Bass module B from Lemma 2.1. We will use this data to define particular large non-projective modules M such that the functor $\operatorname{Ext}^1_R(M, -)$ vanishes at all small modules.

Our set-theoretic setting for this section will be as follows:

κ will denote a singular cardinal of cofinality ω such that $\kappa \geq \operatorname{card}(R)$, and E a subset of the (stationary) subset E_0 of κ^+, where $E_0 = \{\alpha < \kappa^+ \mid \operatorname{cf}(\alpha) = \omega\}$.

For each $\alpha \in E$, the term *ladder* (for α) will denote a strictly increasing chain of ordinals $\ell_\alpha = \{\ell_\alpha(i) \mid i < \omega\}$ such that $\alpha = \sup_{i<\omega} \ell_\alpha(i)$. So the ladder ℓ_α witnesses that α has cofinality ω; the ordinal $\ell_\alpha(i)$ will be called the ith *rung* of the ladder ℓ_α.

A set of ladders $\ell = \{\ell_\alpha \mid \alpha \in E\}$ will be called a *ladder system* for E. Notice that a particular ordinal can appear as a rung in many different ladders from ℓ, but any two distinct ladders in ℓ have only finitely many rungs in common.

Given a ladder system $\ell = \{\ell_\alpha \mid \alpha \in E\}$, we will define a module $M = F/G$ as follows.

F will denote the free module of rank κ^+ defined by $F = \bigoplus_{\alpha<\kappa^+} R_\alpha \oplus \bigoplus_{\alpha\in E} S_\alpha$, where $R_\alpha = R$ for each $\alpha < \kappa^+$ and $S_\alpha = R^{(\omega)}$ for each $\alpha \in E$. The canonical free generator of R_α will be denoted by 1_α, and the canonical free generators of S_α by $1_{\alpha,i}$ $(i < \omega)$.

G will be the submodule in F defined by $G = \sum_{\alpha\in E} G_\alpha$, where $G_\alpha = \sum_{i<\omega} g_{\alpha,i}R$ and $g_{\alpha,i} = 1_{\ell_\alpha(i)} - 1_{\alpha,i} + 1_{\alpha,i+1}a_i$. Then $\operatorname{Ann}(g_{\alpha,i}) = 0$, and since the rungs of the ladder ℓ_α are strictly increasing, $G_\alpha = \bigoplus_{i<\omega} g_{\alpha,i}R \cong R^{(\omega)}$. Since $\{\nu(1_i) \mid i < \omega\}$ is an R-independent set of elements of $R^{(\omega)}$, we infer that $G = \bigoplus_{\alpha\in E} G_\alpha$. It follows that $\mathcal{G} = \{g_{\alpha,i} \mid \alpha \in E, i < \omega\}$ is a free basis of the (free) module G.

Thus M has projective dimension ≤ 1.

A chain $\mathcal{M} = (M_\alpha \mid \alpha < \kappa^+)$ consisting of $\leq \kappa$-generated submodules of M is a called a κ^+-*filtration* of the module M provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for each $\alpha < \kappa^+$ (i.e., the chain is increasing), $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha < \kappa^+$ (i.e., the chain is continuous), and $M = \bigcup_{\alpha<\kappa^+} M_\alpha$.

Since κ^+ is a regular uncountable cardinal, it is easy to see that given any two κ^+-filtrations, $\mathcal{M} = (M_\alpha \mid \alpha < \kappa^+)$ and $\mathcal{M}' = (M'_\alpha \mid \alpha < \kappa^+)$ of M, the set $\{\alpha < \kappa^+ \mid M_\alpha = M'_\alpha\}$ is a club (= a closed and unbounded subset) in κ^+.

For example, $\mathcal{N} = (N_\alpha \mid \alpha < \kappa^+)$ where $N_\alpha = (\bigoplus_{\gamma<\alpha} R_\gamma \oplus \bigoplus_{\gamma\in E,\gamma<\alpha} S_\gamma + G)/G$ for each $\alpha < \kappa^+$, is a κ^+-filtration of M. We will call it the *canonical filtration* of M. Notice that the chain $\mathcal{N}$ is strictly increasing, so $\mathrm{card}(M) = \kappa^+$.

Lemma 2.2. *Assume E is a stationary subset of κ^+. Then M is not projective, so* $\mathrm{proj.dim}\, M = 1$.

Proof. Assume M is projective. By Kaplansky's Theorem [1, 26.2], M is a direct sum of countably generated projective modules, $M = \bigoplus_{\alpha<\kappa^+} Q_\alpha$. For each $\alpha < \kappa^+$, let $P_\alpha = \bigoplus_{\beta<\alpha} Q_\beta$. Then $\mathcal{P} = (P_\alpha \mid \alpha < \kappa^+)$ is a κ^+-filtration of M such that P_β/P_α is projective for all $\alpha < \beta < \kappa^+$. Let $\mathcal{N} = (N_\alpha \mid \alpha < \kappa^+)$ be the canonical filtration of M.

Then the set $C = \{\alpha < \kappa^+ \mid P_\alpha = N_\alpha\}$ is a club in κ^+. Since E is stationary in κ^+, there exist $\alpha \in C \cap E$ and $\beta \in C \cap E$ such that $\alpha < \beta$. In particular, $N_\beta/N_\alpha = P_\beta/P_\alpha$ is a projective module.

Consider the following submodules of the free module F:

$$X = \bigoplus_{\alpha\leq\gamma<\beta} R_\gamma \oplus \bigoplus_{\gamma\in E,\alpha<\gamma<\beta} S_\gamma, \quad Y = (\bigoplus_{\gamma<\alpha} R_\gamma \oplus \bigoplus_{\gamma\in E,\gamma<\alpha} S_\gamma) + G, \quad Z = Y + S_\alpha.$$

Notice that $N_\alpha = Y/G$ and $N_\beta = (X + Z)/G$.

We claim that $X \cap Z \subseteq Y$. Assume there exists $x = x_0 + x_1 \in (X \cap Z) \setminus Y$ with $x_0 \in \bigoplus_{\alpha\leq\gamma<\beta} R_\gamma$ and $x_1 \in X_1 = \bigoplus_{\gamma\in E,\alpha<\gamma<\beta} S_\gamma$. Then $x_1 \in \bigoplus_{\gamma\in E,\alpha<\gamma<\beta} \nu(S_\gamma)$. Let π denote the projection of F on to X_1. Then there are finitely many elements $g_1, \dots, g_m \in \mathcal{G}$ such that $x_1 = \pi(x)$ is generated by the $\pi(g_1), \dots, \pi(g_m)$, that is, $x_1 = \sum_{i\leq m} \pi(g_i).r_i$ for some $r_0, \dots, r_m \in R$. If $i \leq m$ is such that $g_i - \pi(g_i) \in \bigoplus_{\gamma<\alpha} R_\gamma$, then we let $g_i' = \pi(g_i)$, otherwise $g_i' = g_i$. Then $x' = x - \sum_{i\leq m} g_i'.r_i \in \bigoplus_{\alpha\leq\gamma<\beta} R_\gamma$. Since $G \subseteq Y$, also $g_i' \in Y$ for each $i \leq m$, whence $x' \in (X \cap Z) \setminus Y$. However, $(\bigoplus_{\alpha\leq\gamma<\beta} R_\gamma) \cap Z = 0$, so $x' = 0$, a contradiction. This proves our claim.

Since $X \cap Z \subseteq Y$, we have $(X + Y) \cap Z = Y$. Thus

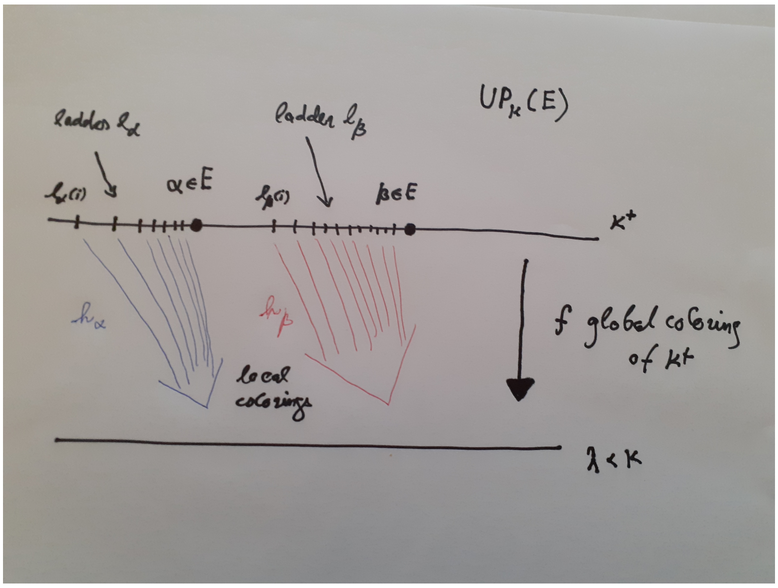
$$N_\beta/N_\alpha \cong (X + Z)/Y = (X + Y)/Y \oplus Z/Y.$$

Notice that $S_\alpha \cap Y = \nu(S_\alpha)$, whence $Z/Y \cong S_\alpha/(S_\alpha \cap Y) = S_\alpha/\nu(S_\alpha) \cong B$. Thus the non-projective Bass module B from Lemma 2.1 is isomorphic to a direct summand in the projective module N_β/N_α, a contradiction. $\square$

Next, we recall Shelah's Uniformization Principle (SUP), which is consistent with ZFC + GCH, see [7], and also [6, XIII.1.5]. (An illustrative picture for (SUP) appears at the next page.)

SUP For each singular cardinal κ of cofinality ω, the following holds:

SUP$_\kappa$ There exist a subset $E \subseteq E_0$ which is stationary in κ^+ and a ladder system ℓ, such that for each $\lambda < \kappa$ ('set of λ colors') and each set of functions $\{h_\alpha : \ell_\alpha \to \lambda \mid \alpha \in E\}$ ('local colorings' of the rungs of the ladders by λ colors) there exists $f : \kappa^+ \to \lambda$ ('global coloring' of all ordinals $< \kappa^+$ by λ colors) such that for each $\alpha \in E$, $f(\ell_\alpha(i)) = h_\alpha(\ell_\alpha(i))$ for almost all $i < \omega$. That is, for each $\alpha \in E$, the global (uniform) coloring f coincides with the local coloring h_α at all but finitely many rungs of the ladder ℓ_α.



Our main consistency result on vanishing of the Ext functor reads as follows:

Theorem 2.3. *Assume SUP. Let κ be a singular cardinal of cofinality ω such that $\kappa \geq \operatorname{card}(R)$, and $E \subseteq E_0$ be the stationary subset of κ^+, and ℓ the ladder system, provided by (SUP_κ). Let $M = F/G$ be the module constructed above for this setting. Let N be any module of cardinality $< \kappa$. Then $\operatorname{Ext}^1_R(M, N) = 0$.*

Proof. Since F is a free module, $\operatorname{Ext}^1_R(M,N) = 0$, iff each $x \in \operatorname{Hom}_R(G,N)$ extends to some $y \in \operatorname{Hom}_R(F,N)$.

Let $\lambda = \operatorname{card}(N)$. Then we can w.l.o.g assume that $\lambda = N$ and use x to define the local colorings h_α ($\alpha \in E$) as follows: for each $i < \omega$, $h_\alpha(\ell_\alpha(i)) = x(g_{\alpha,i})$.

Let $f : \kappa^+ \to \lambda$ be the global coloring provided by (SUP_κ). For each $\alpha \in E$, take $i_\alpha < \omega$ such that $f(\ell_\alpha(i)) = h_\alpha(\ell_\alpha(i))$ for all $i > i_\alpha$.

We will define $y \in \operatorname{Hom}_R(F,N)$ at the canonical free generators 1_α of the R_α ($\alpha < \kappa^+$) and the canonical free generators $1_{\alpha,i}$ of the S_α ($\alpha \in E$, $i < \omega$) as follows:

If $\alpha < \kappa^+$ and there exist $\beta \in E$ and $i > i_\beta$ such that $\alpha = \ell_\beta(i)$, then we put $y(1_\alpha) = f(\alpha)$. Otherwise, we let $y(1_\alpha) = 0$.

Assume $\alpha \in E$. If $i > i_\alpha$, then we put $y(1_{\alpha,i}) = 0$. For $0 \leq i \leq i_\alpha$, we define $y(1_{\alpha,i})$ by downward induction, distinguishing two cases, as follows:

Case I: there exist $\beta \in E$ and $j > i_\beta$ such that $\ell_\beta(j) = \ell_\alpha(i)$. Then we put $y(1_{\alpha,i}) = f(\ell_\alpha(i)) - x(g_{\alpha,i}) + y(1_{\alpha,i+1}).a_i$.

Case II: there are no such $\beta \in E$ and $j > i_\beta$. Then we let $y(1_{\alpha,i}) = -x(g_{\alpha,i}) + y(1_{\alpha,i+1}).a_i$.

It remains to verify that $x(g_{\alpha,i}) = y(1_{\ell_\alpha(i)}) - y(1_{\alpha,i}) + y(1_{\alpha,i+1}).a_i (= y(g_{\alpha,i}))$ for all $\alpha \in E$ and $i < \omega$.

First, if $i > i_\alpha$, then $y(1_{\ell_\alpha(i)}) = f(\ell_\alpha(i)) = h_\alpha(\ell_\alpha(i)) = x(g_{\alpha,i})$, while $y(1_{\alpha,i}) = y(1_{\alpha,i+1}) = 0$, whence $x(g_{\alpha,i}) = y(g_{\alpha,i})$.

If $0 \leq i \leq i_\alpha$, but there exist $\beta \in E$ and $j > i_\beta$ such that $\ell_\beta(j) = \ell_\alpha(i)$, then we are in Case I, whence $y(1_{\ell_\alpha(i)}) = f(\ell_\alpha(i))$, while $y(1_{\alpha,i}) = f(\ell_\alpha(i)) - x(g_{\alpha,i}) + y(1_{\alpha,i+1}).a_i$. So $x(g_{\alpha,i}) = y(g_{\alpha,i})$.

If $0 \leq i \leq i_\alpha$, but there are no such $\beta \in E$ and $j > i_\beta$, then we are in Case II, whence $y(1_{\ell_\alpha(i)}) = 0$, while $y(1_{\alpha,i}) = -x(g_{\alpha,i}) + y(1_{\alpha,i+1}).a_i$. So again, $x(g_{\alpha,i}) = y(g_{\alpha,i})$. □

Corollary 2.4. *Assume SUP. Then for each cardinal $\tau > 0$ there exists a non-free abelian group M_τ such that* $\mathrm{Ext}^1_{\mathbb{Z}}(M_\tau, \mathbb{Z}^{(\tau)}) = 0$. *In particular, M_τ is a Whitehead group.*

Remark 2.5. There is no analog of Theorem 2.3 for right perfect rings. For those rings, one can test for projectivity in ZFC: by [11, 8.8], if R is right perfect and $M \in \mathrm{Mod}\text{–}R$, then M is projective, iff $\mathrm{Ext}^1_R(M, N) = 0$ for each simple module N. So if simp–R denotes a representative set (up to isomorphism) of the class of all simple modules and $N' = \bigoplus_{N \in \mathrm{simp}\text{–}R} N$, then for each module M, M is projective, iff $\mathrm{Ext}^1_R(M, N') = 0$.

3. Diamond, weak diamond, and the non-vanishing of Ext

Now, we turn to a famous combinatorial principle discovered by Jensen in [13], the Diamond Principle $\diamondsuit$. We will formulate it in the following way which is more adapted to our applications, as a principle that predicts functions between κ-filtered sets:

$\diamondsuit$ For each regular uncountable cardinal κ and each stationary subset E of κ, the following holds:

$\diamondsuit_\kappa(\mathbf{E})$ Let A be a set of cardinality κ and B a set of cardinality $\leq \kappa$. Let $(A_\gamma \mid \gamma < \kappa)$ be a κ-filtration of the set A, and $(B_\gamma \mid \gamma < \kappa)$ a κ-filtration of the set B. Then there exists a sequence of functions $(f_\gamma \mid \gamma \in E)$ such that for each $\gamma \in E$, $f_\gamma \in {}^{A_\gamma}B_\gamma$, and for each function $f : A \to B$, the set $D(f) = \{\gamma \in E \mid f \restriction A_\gamma = f_\gamma\}$ is stationary in κ.

In fact, we will employ only a weaker version of $\diamondsuit$ called the Weak Diamond Principle Φ. That principle only predicts colors of functions between κ-filtered sets given by 2-colorings:

$\mathbf{\Phi}$ For each regular uncountable cardinal κ and each stationary subset E of κ, the following holds:

$\mathbf{\Phi}_\kappa(\mathbf{E})$ Let A be a set of cardinality κ and B a set of cardinality $\leq \kappa$. Let $(A_\gamma \mid \gamma < \kappa)$ be a κ-filtration of the set A, and $(B_\gamma \mid \gamma < \kappa)$ a κ-filtration of the set B. For each $\gamma \in E$, let $c_\gamma : {}^{A_\gamma}B_\gamma \to 2$. Then there exists a function $c : E \to 2$, such that for each $f \in {}^{A}B$, the set $C(f) = \{\gamma \in E \mid f \restriction A_\gamma \in {}^{A_\gamma}B_\gamma$ and $c(\gamma) = c_\gamma(f \restriction A_\gamma)\}$ is stationary in κ.

By a classic result of Gödel, the Axiom of Constructibility (V = L) is consistent with ZFC + GCH; Jensen [13] proved that $\diamondsuit$ is a consequences of V = L (see also [6, VII.§1 and §3]):

Theorem 3.1. *Assume $V = L$. Then $\diamondsuit$ holds.*

We also recall the following easy facts:

Lemma 3.2. (1) *Assume that $\diamondsuit_\kappa(\kappa)$ holds for $\kappa = \lambda^+$. Then $2^\lambda = \lambda^+$.*
(2) *$\diamondsuit$ implies the GCH.*

Proof. 1. Let $A_\gamma = \gamma$ and $B_\gamma = 2$ for all $\gamma < \kappa$, so $A = \kappa$ and $B = 2$. Let $(f_\gamma \mid \gamma < \kappa)$ be the sequence of functions provided by $\diamondsuit_\kappa(\kappa)$. Let X be a subset of λ and $f : A \to 2$ be the characteristic function of X, where X is viewed as a subset of κ, i.e., $f(\gamma) = 1$, iff $\gamma \in X$ for each $\gamma < \kappa$.

By $\diamondsuit_\kappa(\kappa)$, the set $\{\gamma < \kappa \mid f \restriction \gamma = f_\gamma\}$ is stationary in κ, so it contains some $\delta \geq \lambda$. Then $\{\gamma < \lambda \mid f_\delta(\gamma) = 1\} = X$. Thus for each $X \subseteq \lambda$ there exists $\delta < \kappa$ such that $f_\delta \restriction \lambda$ is the characteristic function of X. It follows that $2^\lambda \leq \kappa = \lambda^+$.

2. By part 1. $\square$

Lemma 3.3. *Let κ be regular uncountable cardinal and E be a stationary subset of κ. Assume $\diamondsuit_\kappa(E)$. Then $\Phi_\kappa(E)$ holds, too.*

Proof. Let A be a set of cardinality κ and B a set of cardinality $\leq \kappa$. Let $(A_\gamma \mid \gamma < \kappa)$ be a κ-filtration of the set A, and $(B_\gamma \mid \gamma < \kappa)$ a κ-filtration of the set B. For each $\gamma \in E$, let $c_\gamma : {}^{A_\gamma} B_\gamma \to 2$. Let $(f_\gamma \mid \gamma \in E)$ be the sequence of functions provided by $\diamondsuit_\kappa(E)$.

Define a function $c : E \to 2$ by $c(\gamma) = c_\gamma(f_\gamma)$ for each $\gamma \in E$. Let $f \in {}^A B$. By $\diamondsuit_\kappa(E)$, the set $D(f) = \{\gamma \in E \mid f \restriction A_\gamma = f_\gamma\}$ is stationary in κ. However, if $\gamma \in D(f)$, then $c(\gamma) = c_\gamma(f_\gamma) = c_\gamma(f \restriction A_\gamma)$, so $D(f) \subseteq C(f) = \{\gamma \in E \mid f \restriction A_\gamma \in {}^{A_\gamma} B_\gamma$ and $c(\gamma) = c_\gamma(f \restriction A_\gamma)\}$, and $C(f)$ is stationary in κ, too. □

By Lemma 3.2(1), $\diamondsuit_{\omega_1}(\omega_1)$ implies CH. However, this is not true of $\Phi_{\omega_1}(\omega_1)$: by a result of Devlin and Shelah [4], $\Phi_{\omega_1}(\omega_1)$ is equivalent to $2^\omega < 2^{\omega_1}$ (see also [6, VI.1.9]).

The consequences of Φ that we are going to prove contradict the consequences of SUP proved in Section 2. This is not surprising in view of the following lemma:

Lemma 3.4. *Let κ be a singular cardinal of cofinality ω. Assume $\Phi_{\kappa^+}(E)$ holds for each stationary subset E of κ^+ such that $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}$. Then SUP_κ fails.*

Proof. Let E be any stationary subset of κ^+ such that $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}$ and $\ell = \{\ell_\alpha \mid \alpha \in E\}$ be an arbitrary ladder system for E. Let $\lambda = 2$. Let $A_\gamma = \gamma$ and $B_\gamma = 2$ for all $\gamma < \kappa^+$, so $A = \kappa^+$ and $B = 2$.

For each $\alpha \in E$, we define $c_\alpha : {}^\alpha 2 \to 2$ by $c_\alpha(x) = 1$, if the set $S(x) = \{i < \omega \mid x(\ell_\alpha(i)) = 0\}$ is infinite, while $c_\alpha(x) = 0$ otherwise. By $\Phi_{\kappa^+}(E)$, there exists a function $c : E \to 2$, such that for each $f \in {}^{\kappa^+} 2$, the set $C(f) = \{\alpha \in E \mid c(\alpha) = c_\alpha(f \restriction \alpha)\}$ is stationary in κ^+.

We will define the local colorings, $\{h_\alpha : \ell_\alpha \to 2 \mid \alpha \in E\}$, of the rungs of the ladders in ℓ as the constant functions: $h_\alpha(\ell_\alpha(i)) = c(\alpha)$ for each $i < \omega$.

Assume there exists a global coloring $f : \kappa^+ \to 2$ such that for each $\alpha \in E$, $f(\ell_\alpha(i)) = h_\alpha(\ell_\alpha(i))$ for almost all $i < \omega$.

Take $\alpha \in C(f)$. Assume $f(\ell_\alpha(i)) = 0$ for infinitely many $i < \omega$. Then $c(\alpha) = 0$, whence $c_\alpha(f \restriction \alpha) = 0$, and $S(f \restriction \alpha)$ is finite, a contradiction. If $f(\ell_\alpha(i)) = 0$ only for finitely many $i < \omega$, then $c(\alpha) = 1 = c_\alpha(f \restriction \alpha)$, so $S(f \restriction \alpha)$ is infinite, which is also a contradiction.

This proves that SUP_κ fails. □

In particular, if SUP_κ holds, then $\diamondsuit_{\kappa^+}(E)$ fails for some stationary subset E of κ^+ with $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}$. However, the validity of $\diamondsuit_{\kappa^+}(E)$ for other stationary subsets of κ^+ is a rather weak statement - it follows already from $2^\kappa = \kappa^+$. The general result is due to Shelah [23] (see also [15] and [11, 18.15]):

Lemma 3.5. *Let λ be a cardinal such that $2^\lambda = \lambda^+$. Then $\diamondsuit_{\lambda^+}(E)$ holds for each stationary subset E of λ^+ such that $E \subseteq \{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) \neq \mathrm{cf}(\lambda)\}$.*

Remark 3.6. The notion of a ladder ℓ_α can easily be extended to witness cofinality of ordinals α of cofinality $> \omega$. Also SUP can be extended accordingly: by [6, XIII.3.11], it is consistent with ZFC + GCH that for every successor cardinal $\kappa = \mu^+$ there is a stationary subset E of κ consisting of ordinals of cofinality $\mathrm{cf}(\mu)$ and a ladder system ℓ on E which has λ-uniformization for each $\lambda < \mu$. As in Lemma 3.4, one can prove that this extension of SUP is inconsistent with Φ. Thus, Lemma 3.5 gives a rather tight restriction on uniformization under GCH.

We will use Φ to prove the following recent result from [20], guaranteeing consistency of non-vanishing of Ext for arbitrary rings R:

Theorem 3.7. *Let R be a ring. Let κ be a regular uncountable cardinal. Let A and B be modules such that* $\operatorname{card}(B) \leq \kappa$, *and A has a κ-filtration $\mathcal{A} = (A_\alpha \mid \alpha < \kappa)$ such that $\operatorname{Ext}^1_R(A_\alpha, B) = 0$ for each $\alpha < \kappa$. Assume that the set $S = \{\alpha < \kappa \mid \operatorname{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) \neq 0\}$ is stationary in κ, and $\Phi_\kappa(S)$ holds. Then $\operatorname{Ext}^1_R(A, B) \neq 0$.*

Before proving Theorem 3.7, we note some of its immediate consequences:

Corollary 3.8.
(1) *Let R be a right hereditary ring. Let κ be a regular uncountable cardinal and assume that $\Phi_\kappa(E)$ holds for each stationary subset of κ. Let A and B be modules such that A is κ-generated,* $\operatorname{Ext}^1_R(A, B) = 0$, *and* $\operatorname{card}(B) \leq \kappa$.
Then A has a κ-filtration $(A_\alpha \mid \alpha < \kappa)$ such that $\operatorname{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) = 0$ *for all $\alpha < \kappa$.*
(2) *Let κ be a regular uncountable cardinal and assume that $\Phi_\kappa(E)$ holds for each stationary subset E of κ. Assume moreover that each Whitehead group of cardinality $< \kappa$ is free. Then each Whitehead group of cardinality κ is free, too.*

Proof. 1. Since the module A is κ-generated, it has a κ-filtration $(A'_\alpha \mid \alpha < \kappa)$ (we can simply take a minimal set of generators, $\{x_\alpha \mid \alpha < \kappa\}$, of A and let $A'_\alpha = \sum_{\beta<\alpha} x_\beta R$ for each $\alpha < \kappa$). Possibly skipping some of the terms of this filtration, we can w.l.o.g. assume that if $\alpha < \kappa$ is such that there exists $\alpha < \beta < \kappa$ with $\operatorname{Ext}^1_R(A'_\beta/A'_\alpha, B) \neq 0$, then already $\operatorname{Ext}^1_R(A'_{\alpha+1}/A'_\alpha, B) \neq 0$.

Since R is right hereditary, $\operatorname{Ext}^1_R(A, B) = 0$ implies that $\operatorname{Ext}^1_R(A'_\alpha, B) = 0$ for each $\alpha < \kappa$. By Theorem 3.7, the set $S = \{\alpha < \kappa \mid \operatorname{Ext}^1_R(A'_{\alpha+1}/A'_\alpha, B) \neq 0\}$ is not stationary in κ. So there is a club $C \subseteq \kappa$ such that $E \cap C = \emptyset$. Let $z : \kappa \to C$ be a strictly increasing continuous function whose image is C, and let $A_\alpha = A'_{z(\alpha)}$ for each $\alpha < \kappa$. Then $(A_\alpha \mid \alpha < \kappa)$ is a κ-filtration of A such that $\operatorname{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) = 0$ for all $\alpha < \kappa$.

2. This follows from part 1. by taking $R = \mathbb{Z}$, $B = \mathbb{Z}$, and $A =$ a Whitehead group of cardinality κ. The point is that the κ-filtration $(A_\alpha \mid \alpha < \kappa)$ of A constructed in 1. has the property that for each $\alpha < \kappa$, $A_{\alpha+1}/A_\alpha$ is a Whitehead group of cardinality $< \kappa$. Hence $A_{\alpha+1}/A_\alpha$ is free by the assumption, so $A_{\alpha+1} = A_\alpha \oplus C_\alpha$ for a free group C_α, whence $A \cong \bigoplus_{\alpha<\kappa} C_\alpha$ is free. $\square$

Proof. of Theorem 3.7:

First, using an analog of the Horseshoe Lemma (cf. [11, 7.1]), we can extend the κ-filtration $\mathcal{A}$ into a continuous well-ordered system of short exact sequences $\mathcal{E}_\alpha : 0 \to K_\alpha \overset{\subseteq}{\to} F_\alpha \overset{\pi_\alpha}{\to} A_\alpha \to 0$ where F_α is a free module of rank $< \kappa$ and the three components of connecting maps $\varepsilon_\alpha : \mathcal{E}_\alpha \to \mathcal{E}_{\alpha+1}$ are the inclusion of K_α into $K_{\alpha+1}$, the split inclusion $\nu_\alpha : F_\alpha \hookrightarrow F_{\alpha+1}$, and the inclusion $\mu_\alpha : A_\alpha \hookrightarrow A_{\alpha+1}$, respectively.

Then $\varinjlim_{\alpha<\kappa} \mathcal{E}_\alpha$ is the short exact sequence $0 \to K \overset{\subseteq}{\to} F \to A \to 0$ where F is free of rank κ. We can also choose a κ-filtration $(V_\alpha \mid \alpha < \kappa)$ of a set V of free generators of F, such that V_α is a set of free generators of F_α for each $\alpha < \kappa$.

Since $\operatorname{Ext}^1_R(A_\alpha, B) = 0$ for $\alpha < \kappa$, for each homomorphism $f : K_\alpha \to B$ we can fix an extension $f^e \in \operatorname{Hom}_R(F_\alpha, B)$ with $f^e \restriction K_\alpha = f$. Furthermore, for each $\alpha \in S$, $\operatorname{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) \neq 0$, so we can choose $k_\alpha \in \operatorname{Hom}_R(A_\alpha, B)$ that cannot be extended to $A_{\alpha+1}$.

Consider any κ-filtration $(B_\alpha \mid \alpha < \kappa)$ of the set B. For each $\alpha \in S$, we define a 2-coloring $c_\alpha : {}^{V_\alpha}B_\alpha \to 2$ as follows: For each $x \in {}^{V_\alpha}B_\alpha$, we let $x' \in \mathrm{Hom}_R(F_\alpha, B)$ be the (unique) extension of x to F_α, and put $y = (x' \restriction K_\alpha)^e$. Then $y - x'$ is zero on K_α, hence it defines a (unique) homomorphism from A_α to B. We let $c_\alpha(x) = 1$, iff this homomorphism can be extended to $A_{\alpha+1}$.

Now, $\Phi_\kappa(S)$ yields a $c : S \to 2$ for our choice of the 2-colorings c_α ($\alpha \in S$). In order to show that $\mathrm{Ext}^1_R(A, B) \neq 0$, we will recursively construct a homomorphism $f : K \to B$ which cannot be extended to an element of $\mathrm{Hom}_R(F, B)$.

First, $f_0 : K_0 \to B$ is the zero map. Assume $f_\alpha : K_\alpha \to B$ is already constructed for some $\alpha < \kappa$. We define $f_{\alpha+1} : K_{\alpha+1} \to B$ as follows:

We let $f'_\alpha = f^e_\alpha$ if $\alpha \notin S$ or $c(\alpha) = 0$; otherwise, we let $f'_\alpha = f^e_\alpha + k_\alpha\pi_\alpha$. In both cases, we extend f'_α arbitrarily to a homomorphism $f^+_\alpha : F_{\alpha+1} \to B$, and define $f_{\alpha+1}$ as $f^+_\alpha \restriction K_{\alpha+1}$. Since $\pi_\alpha \restriction K_\alpha = 0$, in both cases

$$f_{\alpha+1} \restriction K_\alpha = f^+_\alpha \restriction K_\alpha = f'_\alpha \restriction K_\alpha = f^e_\alpha \restriction K_\alpha = f_\alpha.$$

If $\alpha \leq \kappa$ is a limit ordinal, we put $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Finally, we let $f = f_\kappa : K \to B$.

Assume there exists $g \in \mathrm{Hom}_R(F, B)$ such that $g \restriction K = f$. By $\Phi_\kappa(S)$, there is a $\delta \in S$ such that $g \restriction V_\delta$ maps to B_δ and $c_\delta(g \restriction V_\delta) = c(\delta)$.

Notice that $f^+_\delta - g \restriction F_{\delta+1}$ is zero on $K_{\delta+1}$. Thus, there is a (unique) $h \in \mathrm{Hom}_R(A_{\delta+1}, B)$ such that $f^+_\delta - g \restriction F_{\delta+1} = h\pi_{\delta+1}$. Let $k = h\mu_\delta \in \mathrm{Hom}_R(A_\delta, B)$. Then

$$k\pi_\delta = h\mu_\delta\pi_\delta = h\pi_{\delta+1}\nu_\delta = (f^+_\delta - g \restriction F_{\delta+1}) \restriction F_\delta = f'_\delta - g \restriction F_\delta.$$

If $c(\delta) = 0$, then $(g \restriction K_\delta)^e - g \restriction F_\delta = f^e_\delta - g \restriction F_\delta = f'_\delta - g \restriction F_\delta = k\pi_\delta$. Thus k is the (unique) homomorphism from A_δ to B such that $(g \restriction K_\delta)^e - g \restriction F_\delta = k\pi_\delta$. As h is an extension of k to $A_{\delta+1}$, we infer that $c_\delta(g \restriction V_\delta) = 1$. However, $\delta \in S$, so $c_\delta(g \restriction V_\delta) = c(\delta) = 0$, a contradiction.

If $c(\delta) = 1$, then $k\pi_\delta = f'_\delta - g \restriction F_\delta = f^e_\delta + k_\delta\pi_\delta - g \restriction F_\delta$. So in this case $(g \restriction K_\delta)^e - g \restriction F_\delta = f^e_\delta - g \restriction F_\delta = (k - k_\delta)\pi_\delta$. As $c_\delta(g \restriction V_\delta) = c(\delta) = 1$, $k - k_\delta$ can be extended to $A_{\delta+1}$. Since $k = h\mu_\delta$, the same holds for k, and hence for k_δ. This contradicts our choice of k_δ. $\square$

We finish this section by showing that the converse of Corollary 3.8(1) holds in ZFC in the following strong form, called the Eklof Lemma [11, 6.2]:

Lemma 3.9. *Let R be any ring and $\mathcal{B}$ any class of modules. Let $A \in \mathrm{Mod}\text{–}R$ be the union of any increasing continuous chain $(A_\alpha \mid \alpha < \sigma)$ of its submodules (where σ is any ordinal), such that $\mathrm{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) = 0$ for each $\alpha < \sigma$ and each $B \in \mathcal{B}$. Then $\mathrm{Ext}^1_R(A, B) = 0$ for all $B \in \mathcal{B}$.*

Proof. Clearly, it suffices to prove the claim in the case when $\mathcal{B}$ is a singleton, that is, $\mathcal{B} = \{B\}$ for some $B \in \mathrm{Mod}\text{–}R$. Let $A_\sigma = A$. By induction on $\alpha \leq \sigma$, we will prove that $\mathrm{Ext}^1_R(A_\alpha, B) = 0$. The claim is then the case of $\alpha = \sigma$.

There is nothing to prove for $\alpha = 0$, as $A_0 = 0$. The induction step follows from the exactness of the sequence $0 = \mathrm{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) \to \mathrm{Ext}^1_R(A_{\alpha+1}, B) \to \mathrm{Ext}^1_R(A_\alpha, B) = 0$.

Let $\alpha \leq \sigma$ be a limit ordinal. Let $0 \to B \to I \xrightarrow{\pi} I/B \to 0$ be a short exact sequence in $\mathrm{Mod}\text{–}R$ such that I is an injective module. In order to prove that $\mathrm{Ext}^1_R(A_\alpha, B) = 0$, we have to show that for each $f \in \mathrm{Hom}_R(A_\alpha, I/B)$ there exists $g \in \mathrm{Hom}_R(A_\alpha, I)$ such that $f = \pi g$.

By induction on $\beta < \alpha$, we will construct a sequence of homomorphisms $g_\beta \in \mathrm{Hom}_R(A_\beta, I)$ such that $g_{\beta+1} \restriction A_\beta = g_\beta$ and $\pi g_\beta = f \restriction A_\beta$ for each $\beta < \alpha$. Then $\pi g = f$ will hold for $g = \bigcup_{\beta<\alpha} g_\beta$.

First, $g_0 = 0$. For the induction step, we first use the injectivity of I for extending g_β to some $\eta \in \mathrm{Hom}_R(A_{\beta+1}, I)$. Let $\delta = f \restriction A_{\beta+1} - \pi\eta$. By the induction premise,

$\delta \restriction A_\beta = \pi g_\beta - \pi(\eta \restriction A_\beta) = 0$. So there exists $\epsilon \in \mathrm{Hom}_R(A_{\beta+1}/A_\beta, I/B)$ such that $\delta = \epsilon\pi_\beta$ where $\pi_\beta : A_{\beta+1} \to A_{\beta+1}/A_\beta$ is the canonical projection.

Since $\mathrm{Ext}^1_R(A_{\beta+1}/A_\beta, B) = 0$, there also exists $\theta \in \mathrm{Hom}_R(A_{\beta+1}/A_\beta, I)$ such that $\epsilon = \pi\theta$. Let $g_{\beta+1} = \eta + \theta\pi_\beta$. Then $g_{\beta+1} \restriction A_\beta = \eta \restriction A_\beta = g_\beta$. Moreover,

$$\pi g_{\beta+1} = \pi\eta + \pi\theta\pi_\beta = \pi\eta + \epsilon\pi_\beta = \pi\eta + \delta = f \restriction A_{\beta+1}.$$

If $\beta < \alpha$ is a limit ordinal, we let $g_\beta = \bigcup_{\gamma<\beta} g_\gamma$. This completes our construction. □

4. Singular compactness

In this section, we will prove the following:

Theorem 4.1. *Let R be a right hereditary ring, λ be a singular cardinal, and M a λ-generated module such that each $< \lambda$-generated submodule of M is projective. Then M is projective.*

Before proving Theorem 4.1, we derive its corollary that proves consistency of a positive solution to the (generalized) Whitehead problem:

Corollary 4.2. *Assume Φ.*

(1) *Let R be a right hereditary ring of cardinality $\leq \omega_1$ such that each countably generated Whitehead module is projective. Then each Whitehead module is projective.*
(2) *Each Whitehead group is free.*

Proof. 1. Let M be a Whitehead module and κ be the minimal number of generators of M. By induction on κ, we will show that M is projective. This is true for $\kappa \leq \aleph_0$ by the assumption on R.

For κ regular uncountable, Corollary 3.8(1) yields a κ-filtration $(M_\alpha \mid \alpha < \kappa)$ of the module M such that $M_{\alpha+1}/M_\alpha$ is a Whitehead module for each $\alpha < \kappa$. By the inductive premise, $M_{\alpha+1}/M_\alpha$ is projective, so M_α is a direct summand in $M_{\alpha+1}$, and M is projective, too.

If κ is a singular cardinal, then the projectivity of M follows directly from the inductive premise by Lemma 1.3 and Theorem 4.1.

2. This follows by part 1 and Theorem 1.6. □

Remark 4.3. 1. Part 1. of Corollary 4.2 applies to other hereditary rings besides $\mathbb{Z}$, e.g., to all non-cotorsion PID's of cardinality $\leq \omega_1$ [6, XII.1.11], and to all simple countable von Neumann regular rings that are not completely reducible [27, 3.19].

However, in [8], a non-cotorsion PID of cardinality 2^{ω_1} was constructed (in ZFC) such that there exist non-free Whitehead modules – in fact, such that *each* ω_1-free module is Whitehead. So part 1. does not apply to all non-cotorsion PID's.

2. Recently, Clausen and Scholze have developed condensed mathematics in order to overcome the problem that categories of topological objects of various kinds are not abelian. In particular, for topological groups, this approach results in considering the abelian category, $\mathcal{CA}$, of condensed abelian groups which is an enrichment of the category Mod–$\mathbb{Z}$. Denote the enrichment of a group $A \in$ Mod–$\mathbb{Z}$ (equipped with discrete topology) by $\bar{A}$. If Whitehead groups are defined using the internal Ext functor on $\mathcal{CA}$ (that is, $A \in$ Mod–$\mathbb{Z}$ is Whitehead, if $\mathrm{Ext}^1_{\mathcal{CA}}(\bar{A}, \bar{\mathbb{Z}}) = 0$), then all Whitehead groups are free in ZFC, see [3, Session 8].

Theorem 4.1 is a consequence of a still more general result, the Singular Compactness Theorem by Shelah [22] in the setting of modules. The latter says that given a suitable notion of a "free" module, for each singular cardinal λ, a $\leq \lambda$-generated module M is "free", provided that M is κ-"free" for sufficiently many regular cardinals $\kappa < \lambda$.

The suitability of the notion of "free" is defined by a list of required properties, following the approach of [5] (see also [11, §7.4]):

First, a module M is "free" provided that there exists at least one "basis" $\mathcal{X}$ of M, which is a set of subsets of M. A non-empty set $B(M)$ of "bases" of M is attached to each "free" module M.

A submodule N of a "free" module M is called a *"free" factor* of M, provided that N is generated by some member of a "basis" of M; that is, $N = \langle X \rangle$ for some $\mathcal{X} \in B(M)$ and $X \in \mathcal{X}$.

Assume N is a "free" factor of a "free" module M. Then N is required to be "free", and a non-empty set $B(M,N)$ is given, such that $B(M,N)$ consists of pairs of "bases" of M and N respectively. We will write $\mathcal{Y} = \mathcal{X} \restriction N$ in case $(\mathcal{X}, \mathcal{Y}) \in B(M,N)$.

Let μ be an infinite cardinal. The list of the required properties reads as follows:

Properties 4.4. For each "free" module M, and each "basis" $\mathcal{X}$ of M, the following properties hold:

(P1) *(closedness)* $\emptyset \in \mathcal{X}$, and $\mathcal{X}$ is closed under arbitrary unions.

(P2) *(μ-Löwenheim-Skolem property)* If $X \in \mathcal{X}$ and $a \in M$, then there exists $Y \in \mathcal{X}$, such that $X \subseteq Y$, $a \in \langle Y \rangle$, and $\operatorname{card}(Y) \leq \operatorname{card}(X) + \mu$.

(P3) *(compatibility)* If $Y, X \in \mathcal{X}$ and $Y \subseteq X$, then there exists $\mathcal{Y} \in B(\langle X \rangle)$, such that $Y \in \mathcal{Y}$. In particular, $\langle Y \rangle$ is a "free" factor of $\langle X \rangle$.

(P4) *(basis extension)* If N is a "free" factor of M and $\mathcal{Y} \in B(N)$, then there exists $\mathcal{X} \in B(M)$, such that $\mathcal{Y} = \mathcal{X} \restriction N$.

(P5) *(free filtrations)* If $(D_\delta \mid \delta < \rho)$ is a continuous chain of "free" modules, such that for each $\delta < \rho$, D_δ is a "free" factor of $D_{\delta+1}$, then $\bigcup_{\delta<\rho} D_\delta$ is "free".

(P6) *(basis extension links)* If $(D_n \mid n < \omega)$ is a chain of "free" modules, such that for each $n < \omega$, D_n is a "free" factor of D_{n+1}, and $\mathcal{X}_n \in B(D_n)$ are such that $\mathcal{X}_n = \mathcal{X}_{n+1} \restriction D_n$ for each $n < \omega$, then $\bigcup_{n<\omega} \mathcal{X}_n$ is contained in some "basis" of $\bigcup_{n<\omega} D_n$.

In order to prove Theorem 4.1, we will make use of the following particular instance of the notions of "free", "basis", "free" factor, and $B(M,N)$:

Definition 4.5. Let R be a ring. A module M is "free", if it is projective. By Kaplansky's Theorem [1, 26.2], M is then a direct sum of countably generated projective submodules, that is, $M = \bigoplus_{i \in I} \langle G_i \rangle$ where G_i is a countable set of elements of M and $\langle G_i \rangle$ is a projective submodule of M for each $i \in I$. Let $\mathcal{X} = \{\bigcup_{j \in J} G_j \mid J \subseteq I\}$. Then $\mathcal{X}$ is "basis" of M, and each "basis" of M is obtained in this way from some direct sum decomposition of M into a direct sum of countably generated projective modules.

A submodule N of M is a "free" factor of M, provided that N is generated by some member of a "basis" of M, that is, provided that N is a direct summand in M. The set $B(M,N)$ is defined as the set of all pairs $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X}$ is a "basis" of M, $\mathcal{Y}$ is a "basis" of N, and $\mathcal{Y} = \{X \in \mathcal{X} \mid X \subseteq N\}$.

In other words, $\mathcal{Y} = \mathcal{X} \restriction N$, iff $\mathcal{X} = \{\bigcup_{j \in J} G_j \mid J \subseteq I\}$ where G_i is a countable set of elements of M such that $\langle G_i \rangle$ is a projective submodule of M for each $i \in I$, $M = \bigoplus_{i \in I} \langle G_i \rangle$, there is a subset $K \subseteq I$ such that $N = \bigoplus_{k \in K} \langle G_k \rangle$, and $\mathcal{Y} = \{\bigcup_{l \in L} G_l \mid L \subseteq K\}$. Notice that $(\mathcal{X}, \mathcal{Y}) \in B(M,N)$ implies $\mathcal{Y} \subseteq X$, but the converse need not hold in general.

Using elementary properties of direct sum decompositions, one easily verifies the following

Lemma 4.6. *For any ring R, the particular instances of the notions of "free", "basis", "free" factor, and $B(M,N)$ from Definition 4.5 satisfy Properties (P1)-(P6) from 4.4 for $\mu = \omega$.*

In order to state the general version of the Singular Compactness Theorem for modules, it remains to define the notion of a κ-"free" module:

Definition 4.7. Let κ be a regular uncountable cardinal and M be a module.

(1) M is *κ-"free"*, provided there exists a set $\mathcal{S}$ consisting of $< \kappa$-generated "free" submodules of M, such that $0 \in \mathcal{S}$, each subset of M of cardinality $< \kappa$ is contained in an element of $\mathcal{S}$, and $\mathcal{S}$ is closed under unions of well-ordered chains of length $< \kappa$.

(2) M is *strongly κ-"free"*, provided there exists a set $\mathcal{T}$ consisting of $< \kappa$-generated "free" submodules of M, such that $0 \in \mathcal{T}$, and for each $N \in \mathcal{T}$ and each subset $X \subseteq M$ of cardinality $< \kappa$, there exists $N' \in \mathcal{T}$ such that $N \cup X \subseteq N'$ and N is a "free" factor of N'.

The sets $\mathcal{S}$ and $\mathcal{T}$ are said to *witness* the κ-"freeness" and strong κ-"freeness" of M, respectively.

Example 4.8. Let κ be a regular uncountable cardinal $> \mu$. Then each "free" module M is both κ-"free" and strongly κ-"free".

Indeed, if $\mathcal{X}$ is any "basis" of M, then the set of all submodules N of M of the form $N = \langle X \rangle$, where $X \in \mathcal{X}$ and $\mathrm{card}(X) < \kappa$, witnesses both the κ-"freeness" and the strong κ-"freeness" of M, by properties (P1), (P2), and (P3).

Let's have a closer look at these notions in the particular setting of Definition 4.5. Since in this setting, "free" means projective, the standard terminology for κ-"free" is *κ-projective*, and for strongly κ-"free", it is *strongly κ-projective.*

ω_1-projective modules over any ring can equivalently be characterized as the flat Mittag-Leffler modules by [11, 3.19]. In the hereditary setting, we have the following characterization:

Lemma 4.9. *Let κ be a regular uncountable cardinal and R be a right hereditary ring. Let $M \in$ Mod–R.*

(1) *M is κ-projective, if and only if all $< \kappa$-generated submodules of M are projective.*

(2) *If M is strongly κ-projective, then M is κ-projective.*

(3) *M is strongly κ-projective, iff M is κ-projective, and for each subset X of M of cardinality $< \kappa$ there exists a $< \kappa$-generated projective submodule P of M containing X, such that Q/P is projective for each $< \kappa$-generated submodule Q of M containing P.*

Proof. 1. The only-if claim is clear, since over a right hereditary ring, the class of all projective modules is closed under submodules. For the if-claim, it suffices to let $\mathcal{S}$ be the set of all $< \kappa$-generated submodules of M.

2. Since $\mathcal{T}$ consists of $< \kappa$-generated projective modules, and each subset $X \subseteq M$ of cardinality $< \kappa$ is contained in an element of $\mathcal{T}$, each $< \kappa$-generated submodule of M is projective, and part 1. applies.

3. Let $\mathcal{T}$ be a set witnessing the strong κ-projectivity of M. Let X be a subset of M of cardinality $< \kappa$. Since $0 \in \mathcal{T}$, there exists $P \in \mathcal{T}$ such that $X \subseteq P$. Let Q be any $< \kappa$-generated submodule M containing P and Y be a set of cardinality $< \kappa$ such that $Q = \langle Y \rangle$. Then there exists $N' \in \mathcal{T}$ such that $Y \subseteq N'$ and N'/P is

projective. Since R is right hereditary, also Q/P is projective. The κ-projectivity of M follows by part 2.

In order to prove the converse, let $\mathcal{T}$ be the set of all $< \kappa$-generated submodules P of M such that Q/P is projective for each $< \kappa$-generated submodule Q of M containing P. By part 1., $\mathcal{T}$ consists of projective modules and $0 \in \mathcal{T}$. Let $N \in \mathcal{T}$, let Y be a set of generators of N of cardinality $< \kappa$, and X be any subset of M of cardinality $< \kappa$. Then there exists a $< \kappa$-generated projective submodule P of M containing $X \cup Y$, such that Q/P is projective for each $< \kappa$-generated submodule Q of M containing P. Let $N' = P$. Then $N' \in \mathcal{T}$. Since $N \in \mathcal{T}$, the module N'/N is projective. Thus $\mathcal{T}$ witnesses the strong κ-projectivity of M. □

By part (1) of Lemma 4.9, in our original setting of groups, the notions of an ω_1-projective module and an ω_1-free group from Lemma 1.4(1) coincide. The best known example of an ω_1-free group which is not strongly ω_1-projective is the *Baer-Specker group* Z^ω, [24]:

Lemma 4.10. *Let $R = \mathbb{Z}$ and λ be any infinite cardinal. Then the group $\mathbb{Z}^\lambda$ is ω_1-free, but not strongly ω_1-projective (and hence not free).*

Proof. For a subset $J \subseteq \lambda$, we will denote by $\mathbb{Z}^J$ the direct summand in $\mathbb{Z}^\lambda$ consisting of all the $(x_\alpha \mid \alpha < \lambda) \in \mathbb{Z}^\lambda$ such that $x_\alpha = 0$ for all $\alpha \in \lambda \setminus J$.

By Lemma 1.4, in order to show that $\mathbb{Z}^\lambda$ is ω_1-free, it suffices to prove the (stronger) claim that each finite rank pure subgroup A of $\mathbb{Z}^\lambda$ is a free direct summand in $\mathbb{Z}^\lambda$.

First, we prove that each $x = (x_\alpha \mid \alpha < \lambda) \in \mathbb{Z}^\lambda$ is contained in a cyclic direct summand of $\mathbb{Z}^\lambda$. Let $1 \leq m < \omega$ be the greatest common divisor of all the x_α $(\alpha < \lambda)$. Let I be a finite subset of λ such that m is also the greatest common divisor of the x_i $(i \in I)$. Let $y = m^{-1}x \in \mathbb{Z}^\lambda$.

For each $z \in \mathbb{Z}^\lambda$, let z' be the restriction of z to I, that is, $z' \in \mathbb{Z}^I$ is such that $z'_i = z_i$ for each $i \in I$. Since the greatest common divisor of the $y'_i \in \mathbb{Z}$ $(i \in I)$ is 1, the group $\mathbb{Z}^I/y'\mathbb{Z}$ is torsion-free and finitely generated, hence it is free. So $y'\mathbb{Z}$ is a free direct summand in $\mathbb{Z}^I$. Let $x_1, \dots, x_k \in \mathbb{Z}^\lambda$ be such that $\{y', x'_1, \dots, x'_k\}$ is a free basis of $\mathbb{Z}^I$. Then $y\mathbb{Z} \oplus (\bigoplus_{i=1}^k x_i\mathbb{Z}) \oplus (\mathbb{Z}^{\lambda \setminus I}) = \mathbb{Z}^\lambda$, whence $y\mathbb{Z}$ is a direct summand in $\mathbb{Z}^\lambda$ containing x.

Now, let A be any pure subgroup of $\mathbb{Z}^\lambda$ of finite rank. By induction on its rank, n, we will prove that A is a free direct summand in $\mathbb{Z}^\lambda$. There is nothing to prove for $n = 0$. Let $0 \neq x \in A$ and let $y = m^{-1}x$ be as above. Since A is pure in $\mathbb{Z}^\lambda$ and $\mathbb{Z}^\lambda$ is torsion-free, necessarily $y \in A$. Also $\mathbb{Z}^\lambda = y\mathbb{Z} \oplus C$ for some $C \subseteq \mathbb{Z}^\lambda$ by the above. Then $A = y\mathbb{Z} \oplus (A \cap C)$, where $A \cap C$ has rank $n - 1$, and being a direct summand of the pure subgroup A, $A \cap C$ is also a pure subgroup in $\mathbb{Z}^\lambda$. So $A \cap C$ is a free direct summand in $\mathbb{Z}^\lambda$ by the inductive hypothesis. Hence A is free, and $C = (A \cap C) \oplus D$ for some $D \subseteq \mathbb{Z}^\lambda$, so $\mathbb{Z}^\lambda = A \oplus D$.

In order to prove that $\mathbb{Z}^\lambda$ is not strongly ω_1-projective, we will show that for each countable subgroup $\mathbb{Z}^{(\omega)} \subseteq H \subseteq \mathbb{Z}^\omega \subseteq \mathbb{Z}^\lambda$ there exists a countable group G such that $H \subseteq G \subseteq \mathbb{Z}^\omega$ and G/H is not free. This will suffice: since we have already proved that $\mathbb{Z}^\lambda$ is ω_1-free, by Lemma 4.9(1) and (3), we only have to show that there exists a countable subgroup X of $\mathbb{Z}^\lambda$ such that for each countable subgroup P of $\mathbb{Z}^\lambda$ containing X there exists a countable subgroup Q of $\mathbb{Z}^\lambda$ containing P such that Q/P is not free. However, we just let $X = \mathbb{Z}^{(\omega)}$, and for $H = P \cap \mathbb{Z}^\omega$ we find a corresponding $G \subseteq \mathbb{Z}^\omega$. Then putting $Q = P + G$, we see that $Q/P \cong G/(P \cap G) = G/H$ is not a free group.

Finally, let p be a prime integer. Since the group $\prod_{n<\omega} p^n\mathbb{Z} \subseteq \mathbb{Z}^\omega$ is uncountable, there exists $x = (x_n \mid n < \omega) \in Z^\omega \setminus H$ such that $x_n \in p^n\mathbb{Z}$ for each $n < \omega$. Let G be a countable pure subgroup of $\mathbb{Z}^\omega$ containing $H \cup \{x\}$. Since $\mathbb{Z}^{(\omega)} \subseteq H$, the element $0 \neq x + H \in G/H$ is divisible by p^n for each $n < \omega$. Thus, G/H is not a free group. □

Remark 4.11. 1. If R is not right hereditary, then our terminology may be misleading, as the implication from Lemma 4.9(2) need not hold in general: for each regular cardinal $\kappa > \omega_1$, there exists a ring R_κ and a module M_κ such that M_κ is strongly κ-projective, but not κ-projective, [26].

2. The question of the existence of a κ-projective, but not free, group of cardinality κ for a given regular uncountable cardinal can be translated (in ZFC) into a combinatorial statement, called NPT(κ), concerning existence of tranversals for families of size κ consisting of countable sets, cf. [6, VII.3.13]. NPT(κ) is known to fail for all (regular) weakly compact cardinals, [6, IV.3.2]. In view of Lemma 4.10, it may come as a surprise that NPT(κ) can be used to show in ZFC for each regular uncountable cardinal κ, that the existence of a κ-projective, but not free, group of cardinality κ implies also the existence of a strongly κ-projective, but not free, group of the same cardinality, [6, VII.3A].

3. The properties of the groups $\mathbb{Z}^\lambda$ proved in Lemma 4.10 are the same for each infinite cardinal λ. This contrasts with the properties of the groups $Z_\lambda = \mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}$, where $\mathbb{Z}^{<\lambda}$ denotes the subgroup of $\mathbb{Z}^\lambda$ consisting of all the sequences $x = (x_\alpha \mid \alpha < \lambda)$ whose support $\mathrm{supp}(x) = \{\alpha < \lambda \mid x_\alpha \neq 0\}$ has cardinality $< \lambda$.

By [6, IX.3.5], Z_λ is ω_1-free for each infinite cardinal λ of uncountable cofinality (in particular, for all $\lambda = \aleph_n$ where $1 \leq n < \omega$).

However, $Z_\omega = \mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}$ is not ω_1-free: Z_ω is a pure-injective torsion-free group by [6, V.1.16]. In fact, $Z_\omega \cong \mathbb{Q}^{2^\omega} \oplus \prod_{p\in\mathbb{P}} A_p$ where A_p denotes the p-adic completion of the group $\mathbb{J}_p^{(2^\omega)}$, $\mathbb{J}_p$ the group of all p-adic integers, and $\mathbb{P}$ the set of all prime integers (cf. [6, Ex. V.4]).

The version of Shelah's Singular Compactness Theorem that we are going to prove here is

Theorem 4.12. *Let R be a ring, μ be an infinite cardinal, λ a singular cardinal $> \mu$, and M be a $\leq \lambda$-generated module. Assume that M is κ-"free" for each regular cardinal $\mu < \kappa < \lambda$, and the notion of "free", "basis", "free" factor, and $B(M,N)$ satisfy Properties (P1)-(P6). Then M is "free".*

Notice that in view of Lemma 4.6, Theorem 4.12 implies Theorem 4.1.

The proof of Theorem 4.12 will proceed in two steps, following [5] and [6, §IV.3] (which in turn was inspired by [12]). For the first step, we need a set-theoretic fact:

Lemma 4.13. *Let κ be an infinite cardinal. Then there is a bijection $\psi : \kappa \to \kappa \times \kappa$ such that for all $\nu < \kappa$, if $\psi(\nu) = (\alpha, \tau)$ then $\alpha \leq \nu$.*

Proof. Since $\mathrm{card}(\kappa) = \mathrm{card}(\kappa \times \kappa)$, it suffices to prove that an arbitrary bijection $\phi : \kappa \to \kappa \times \kappa$ can be modified to a bijection ψ as in 4.13.

By induction on $\beta \leq \kappa$, we define a sequence of bijections $\psi_\beta : \kappa \to \kappa \times \kappa$ $(\beta < \kappa)$, and a continuous chain $(S_\beta \mid \beta \leq \kappa)$ of subsets $S_\beta \subseteq \kappa$ such that $\beta \subseteq S_\beta$ for each $\beta \leq \kappa$, $\mathrm{card}(S_\beta) < \kappa$ for each $\beta < \kappa$, and the following four conditions are satisfied for each $\beta < \kappa$:

(1_β) If $\psi_\beta(\nu) = (\alpha, \tau)$ and $\nu \in S_\beta$, then $\alpha \leq \nu$ (4.13 restricted to S_β),
(2_β) $\psi_\beta(\mu) = \psi_\nu(\mu)$ for each $\nu < \beta$ and $\mu \in S_\nu$ (compatibility of the sequence),
(3_β) $\psi_\beta(\nu) = \phi(\nu)$ for all $\nu \notin S_\beta$ (local relation to ϕ outside S_β), and
(4_β) $\phi(S_\beta) = \psi_\beta(S_\beta)$ (global relation to ϕ on S_β).

First, $\psi_0 = \phi$ and $S_0 = \emptyset$. In the inductive step for $\beta < \kappa$, we distinguish two cases:

Case 1: $\psi_\beta(\beta) = (\gamma, \rho)$ for some $\gamma \le \beta$ and $\rho < \kappa$. Then we define $\psi_{\beta+1} = \psi_\beta$ and $S_{\beta+1} = S_\beta \cup \{\beta\}$.

Case 2: $\psi_\beta(\beta) = (\gamma, \rho)$ for some $\gamma > \beta$ and $\rho < \kappa$. Then $\beta \notin S_\beta$. Moreover, ψ_β is a bijection, whence the set T_β consisting of all $\delta < \kappa$ such that $\psi_\beta(\delta) = (0, \tau)$ for some $\tau < \kappa$ has cardinality κ. Since $\mathrm{card}(S_\beta) < \kappa$, there exists $\delta_\beta \in T_\beta$ such that $\delta_\beta \ge \gamma$ and $\delta_\beta \notin S_\beta$. Let $S_{\beta+1} = S_\beta \cup \{\beta, \delta_\beta\}$, and define $\psi_{\beta+1}$ as ψ_β, but with swapped values at β and δ_β. That is, $\psi_{\beta+1}(\mu) = \psi_\beta(\mu)$ for all $\mu < \kappa^+$ different from β and δ_β, $\psi_{\beta+1}(\beta) = \psi_\beta(\delta_\beta)$, and $\psi_{\beta+1}(\delta_\beta) = \psi_\beta(\beta)$.

In either case, $\psi_{\beta+1}$ is clearly a bijection, and conditions $(1_{\beta+1})$-$(4_{\beta+1})$ hold by the inductive premise and by our construction of the $\psi_{\beta+1}$.

If $\beta \le \kappa$ is a limit ordinal, we define $S_\beta = \bigcup_{\gamma<\beta} S_\gamma (\supseteq \beta)$. For $\delta \notin S_\beta$, we let $\psi_\beta(\delta) = \phi(\delta)$, so (3_β) holds. For $\delta \in S_\beta$, let $\gamma < \beta$ be the least (non-limit) ordinal such that $\delta \in S_\gamma$. We define $\psi_\beta(\delta) = \psi_\gamma(\delta)$. Since $\delta \notin S_{\gamma-1}$, either $\delta = \gamma - 1$ or $\delta = \delta_{\gamma-1}$. So (1_β) follows from (1_γ) for $\gamma < \beta$. Moreover, in the case when $\delta = \gamma - 1$, $\psi_\beta(\gamma - 1) = \psi_\gamma(\gamma - 1)$, whence (2_β) follows from (2_γ) for $\gamma < \beta$. Finally, $\phi(S_\beta) = \bigcup_{\gamma<\beta} \phi(S_\gamma) = \bigcup_{\gamma<\beta} \psi_\gamma(S_\gamma) = \bigcup_{\gamma<\beta} \psi_\beta(S_\gamma) = \psi_\beta(S_\beta)$ by (4_γ) for $\gamma < \beta$, and by (2_β). Thus (4_β) holds. By (2_β), ψ_β is monic at S_β, and (3_β) implies that ψ_β is monic at $\kappa \setminus S_\beta$. By (3_β) and (4_β), ψ_β is surjective.

Finally, let $\psi = \psi_\kappa$. Then ψ is a bijection, and since $S_\kappa = \kappa$, condition (1_κ) is just the claim of 4.13. $\square$

Now, we can make the first step:

Lemma 4.14. *Let R be a ring, μ be an infinite cardinal, κ be a regular cardinal $> \mu$, and M be a κ^+-"free" module. Then M is strongly κ-"free".*

Proof. For any $\le \kappa$-generated "free" submodule N of M, we define the *N-Shelah game* for two players, I and II, with moves indexed by natural numbers, as follows: In the nth move, player I plays a subset X_n of M of cardinality $< \kappa$, and player II replies with a $< \kappa$-generated submodule N_n of M containing N. Player II wins, in case for each $n < \omega$, N_n is a "free" module containing $N_{n-1} \cup X_n$ such that N_{n-1} is a "free" factor of N_n (where $N_{-1} = N$); otherwise, player I wins.

A *winning strategy* for player I the N-Shelah game is a function s_N that gives the 0th move $X_0 = s_N(N)$ of player I, and then his nth move $X_n = s_N(N_0, \dots, N_{n-1})$ for each $0 < n < \omega$, so that player I wins, that is, after some move X_n of player I, there exists no "free" submodule N_n of M containing $N_{n-1} \cup X_n$ such that N_{n-1} is a "free" factor of N_n.

We claim that player I does not have a winning strategy in the 0-Shelah game. If so, then we can define $\mathcal{T}$ as the set of all $< \kappa$-generated "free" submodules N of M such that player I does not have a winning strategy in the N-Shelah game.

By our claim, $0 \in \mathcal{T}$. Let $N \in \mathcal{T}$ and X be a subset of M of cardinality $< \kappa$. Consider the N-Shelah game where the 0th move of player I is $X_0 = X$. Let N_0 be the 0th move of player II; it is available because $N \in \mathcal{T}$. In particular, $N \cup X \subseteq N_0$, and N is a "free" factor of N_0. Moreover, player I cannot have a winning strategy in the N_0-Shelah game (otherwise, he would also have a winning strategy for the N-Shelah game). Thus $N_0 \in \mathcal{T}$. Hence $\mathcal{T}$ witnesses that M is a strongly κ-"free" module.

It remains to prove our claim. We will do it by contradiction. Assume $s = s_0$ is a winning strategy for player I in the 0-Shelah game. Let $\mathcal{S}$ be the set witnessing that M is a κ^+-"free" module. By Lemma 4.13, there is a bijection $\psi : \kappa \to \kappa \times \kappa$ such that for all $\nu < \kappa$, if $\psi(\nu) = (\alpha, \tau)$ then $\alpha \le \nu$.

By induction on $\nu < \kappa$ we will define a continuous chain $(N_\nu \mid \nu < \kappa)$ consisting of $< \kappa$-generated submodules of M, and select from $\mathcal{S}$ a continuous chain of modules

$(F_\nu \mid \nu < \kappa)$ together with sets of generators, $\{g_\nu^\tau \mid \tau < \kappa\}$ of F_ν so that $N_\nu \subseteq F_\nu$ for each $\nu < \kappa$ as follows: First, let $N_0 = 0$, and let $F_0 \in \mathcal{S}$ be arbitrary.

If ν is a non-limit ordinal, we take N_ν so that $g_\alpha^\tau \in N_\nu$, where $\psi(\nu - 1) = (\alpha, \tau)$. This is possible since $\alpha < \nu$, so g_α^τ is already defined. Moreover, we can also assume that N_ν contains $s(0)$ and $s(N_{\alpha_1}, \dots, N_{\alpha_k})$ whenever $k \geq 1$, $\alpha_1 < \dots < \alpha_k < \nu$ and $s(N_{\alpha_1}, \dots, N_{\alpha_k})$ is defined. This is possible since there are $< \kappa$ such sequences of ordinals $< \nu$. Since $\mathcal{S}$ witnesses the κ^+-"freeness" of M, we can select $F_\nu \in \mathcal{S}$ such that $N_\nu \cup F_{\nu-1} \subseteq F_\nu$ and take a generating set $\{g_\nu^\tau \mid \tau < \kappa^+\}$ for F_ν.

If ν is a limit ordinal, we let $N_\nu = \bigcup_{\sigma<\nu} N_\sigma$, $F_\nu = \bigcup_{\sigma<\nu} F_\sigma \in \mathcal{S}$, and $\{g_\nu^\tau \mid \tau < \kappa\} = \bigcup_{\sigma<\nu}\{g_\sigma^\tau \mid \tau < \kappa\}$.

Let $F = \bigcup_{\nu<\kappa} N_\nu$. Clearly, $F \subseteq \bigcup_{\nu<\kappa} F_\nu$, and the opposite inclusion holds because ψ is a bijection: by construction, if $\mu < \kappa$ is such that $\psi(\mu) = (\alpha, \tau)$, then $g_\alpha^\tau \in N_{\mu+1}$. Thus $F = \bigcup_{\nu<\kappa} F_\nu \in \mathcal{S}$. Let $\mathcal{X}$ be a "basis" of F.

Let $C = \{\alpha < \kappa \mid N_\alpha = \langle X_\alpha \rangle$ for some $X_\alpha \in \mathcal{X}$ such that $\operatorname{card}(X_\alpha) < \kappa\}$. We claim that C is unbounded in κ: Indeed, if $\nu < \kappa$, then by induction on $n < \omega$, we can define a strictly increasing chain of ordinals $< \kappa$, $\nu = \nu_0 < \nu_1 < \dots$, and a chain of elements of $\mathcal{X}$, $X_0 \subseteq X_1 \subseteq \dots$, so that X_n has cardinality $< \kappa$, and $N_{\nu_n} \subseteq \langle X_n \rangle \subseteq N_{\nu_{n+1}}$ for each $n < \omega$. This is possible since $\mu < \kappa$, by the properties (P1) and (P2) of $\mathcal{X}$ from 4.4. Let $\alpha = \sup_{n<\omega} \nu_n$. Then $N_\alpha = \langle X \rangle$ where $X = \bigcup_{n<\omega} X_n \in \mathcal{X}$ by property (P1), so $\alpha \in C$, and $\nu < \alpha$.

Finally, we show how player II can defeat the strategy s: for each $n < \omega$, he plays N_{α_n} for some $\alpha_n \in C$ so that $\alpha_0 < \alpha_1 < \dots$ as follows: first, since C is unbounded, there is $\alpha_0 \in C$ such that $s(0) \subseteq N_{\alpha_0} = \langle X_{\alpha_0} \rangle$. Similarly, in the $(n+1)$th move, player II takes $\alpha_{n+1} \in C$ such that $\alpha_n < \alpha_{n+1}$, $X_{\alpha_n} \subseteq X_{\alpha_{n+1}}$, and $s(N_{\alpha_0}, \dots, N_{\alpha_{n-1}}) \subseteq N_{\alpha_{n+1}} = \langle X_{\alpha_{n+1}} \rangle$. Since $X_{\alpha_n} \subseteq X_{\alpha_{n+1}}$, N_{α_n} is a a "free" factor of $N_{\alpha_{n+1}}$ by property (P3). □

Lemma 4.9(3) yields a much simpler proof of Lemma 4.14 in the particular setting of Definition 4.5 for right hereditary rings:

Lemma 4.15. *Let R be a right hereditary ring, κ a regular uncountable cardinal $> \mu$, and M a κ^+-projective module. Then M is strongly κ-projective.*

Proof. By Lemma 4.9(1), M is κ-projective. Assume M is not strongly κ-projective. Then by Lemma 4.9(3), there exists a subset X of M of cardinality $< \kappa$ such that for each $< \kappa$-generated projective submodule P of M containing X, there exists a $< \kappa$-generated submodule Q of M containing P such that Q/P is not projective. This makes it possible to construct, by induction on $\alpha < \kappa$, a κ-filtration $\mathcal{N} = (N_\alpha \mid \alpha < \kappa)$ such that $N_0 = 0$, $N_1 = \langle X \rangle$, and $N_{\alpha+1}/N_\alpha$ is not projective for each $0 < \alpha < \kappa$. Let $N = \bigcup_{\alpha<\kappa} N_\alpha$. Then N is $\leq \kappa$-generated, but not projective, in contradiction with the assumption that M is κ^+-projective. Indeed, if N were projective, then the κ-filtration $\mathcal{N}$ of N would have to coincide on a club C in κ with the κ-filtration induced by the direct sum decomposition of N into a direct sum of countably generated projective modules. Since R is right hereditary, this would contradict the fact that consecutive factors of $\mathcal{N}$ are not projective. □

In view of Lemma 4.14, the proof of Theorem 4.12 will be complete once we prove

Lemma 4.16. *Let R be a ring, μ an infinite cardinal, $\lambda > \mu$ a singular cardinal, and M a λ-generated module, such that M is strongly κ^+-"free" for all cardinals $\mu < \kappa < \lambda$. Then M is "free".*

Proof. Let $\tau = \operatorname{cf}(\lambda)$. Then $\tau < \lambda$ by assumption, and there exists an increasing continuous sequence of cardinals, $(\kappa_\nu \mid \nu < \tau)$, whose supremum is λ, and such that $\kappa_0 > \mu$ and $\kappa_0 > \tau$. Since M λ-generated, we can choose a generating subset G of

M of cardinality λ, and an increasing continuous chain of subsets of G, $(G_\nu \mid \nu < \tau)$, such that $\mathrm{card}(G_\nu) = \kappa_\nu$ for each $\nu < \tau$ and $G = \bigcup_{\nu<\tau} G_\nu$.

By induction on $n < \omega$, we will construct, for all $\nu < \tau$, the following objects: a subset C_ν^n of M of cardinality $\leq \kappa_\nu$, a $\leq \kappa_\nu$-generated "free" submodule F_ν^n of M, a "basis" $\mathcal{X}_\nu^n$ of F_ν^n, and an element $X_\nu^n \in \mathcal{X}_{\nu+1}^n$ of cardinality $\leq \kappa_\nu$.

We will require that these objects satisfy, for all $n < \omega$ and $\nu < \tau$, the following conditions:

(C1) $G_\nu \subseteq F_\nu^n \subseteq \langle C_\nu^n \rangle \subseteq F_\nu^{n+1}$;
(C2) F_ν^n is a "free" factor of F_ν^{n+1}, and $\mathcal{X}_\nu^n = \mathcal{X}_\nu^{n+1} \restriction F_\nu^n$;
(C3) $C_\rho^{n-1} \subseteq C_\nu^n$ for each $\rho \leq \nu$;
(C4) $\langle X_\nu^n \rangle \subseteq \langle X_\nu^{n+1} \rangle$ and $X_\nu^n \subseteq C_\nu^n$;
(C5) $C_\nu^{n-1} \subseteq \langle X_\nu^{n+1} \rangle$.

Moreover, we will require the following condition:

(C6) $(C_\nu \mid \nu < \tau)$ is a continuous chain of submodules of M, where $C_\nu = \bigcup_{n<\omega} \langle C_\nu^n \rangle$ for each $\nu < \tau$.

Assume that the construction above is possible. Then, by (C1), $C_\nu = \bigcup_{n<\omega} F_\nu^n$ and $\bigcup_{\nu<\tau} C_\nu = M$. By (C2), and the properties (P5) and (P6), C_ν is "free", and $\bigcup_{n<\omega} \mathcal{X}_\nu^n$ is contained in a "basis" of C_ν, say $\mathcal{X}_\nu$. Moreover, by (C4) and (C5), C_ν is generated by $X_\nu = \bigcup_{n<\omega} X_\nu^n$, and $X_\nu \in \mathcal{X}_{\nu+1}$ by property (P1). So C_ν is a "free" factor of $C_{\nu+1}$. Finally, (C5) and property (P5) yield that M is "free".

For the construction, we first fix, for each $\nu < \tau$ a set $\mathcal{T}_\nu$ witnessing the strong κ_ν^+-"freeness" of M. At the nth stage of the construction, we will define for all $\nu < \tau$ the modules $F_\nu^n \in \mathcal{T}_\nu$, the "bases" $\mathcal{X}_\nu^n$ of F_ν^n, subsets C_ν^{n-1} of M of cardinality $\leq \kappa_\nu$, $X_\nu^n \in \mathcal{X}_{\nu+1}^n$ of cardinality $\leq \kappa_\nu$, and sets $\{u_{\nu,\alpha}^n \mid \alpha < \kappa_\nu\}$ of generators of F_ν^n as follows:

For $n = 0$, we choose $F_\nu^0 \in \mathcal{T}_\nu$ so that $G_\nu \subseteq F_\nu^0$, $\mathcal{X}_\nu^0 \in B(F_\nu^0)$, and let $C_\nu^{-1} = X_\nu^0 = \emptyset$.

In the inductive step, we first define $C_\nu^n = X_\mu^n \cup \bigcup_{\rho\leq\nu} C_\rho^{n-1} \cup \{u_{\rho,\alpha}^n \mid \rho < \tau, \alpha < \kappa_\nu\}$. Since C_ν^n contains $\{u_{\nu,\alpha}^n \mid \alpha < \kappa_\nu\}$, by the inductive premise $F_\nu^n \subseteq \langle C_\nu^n \rangle$. So we can take $F_\nu^{n+1} \in \mathcal{T}_\nu$ so that $C_\nu^n \subseteq F_\nu^{n+1}$ and F_ν^n is a "free" factor of F_ν^{n+1}. By property (P4), we can choose $\mathcal{X}_\nu^{n+1} \in B(F_\nu^{n+1})$ so that $\mathcal{X}_\nu^n = \mathcal{X}_\nu^{n+1} \restriction F_\nu^n$. Then clearly conditions (C1)-(C3) hold true for n.

Next, we take $X_\nu^{n+1} \in \mathcal{X}_{\nu+1}^{n+1}$ of cardinality $\leq \kappa_\nu$ so that $\langle X_\nu^n \rangle \subseteq \langle X_\nu^{n+1} \rangle$ and $C_\nu^n \cap F_{\nu+1}^{n+1} \subseteq \langle X_\nu^{n+1} \rangle$. This is possible by properties (P1) and (P2). Thus (C4) holds for n.

Since $C_\nu^{n-1} \subseteq C_\nu^n$, and $C_\nu^{n-1} \subseteq C_{\nu+1}^n \subseteq F_{\nu+1}^{n+1}$ by (C1), we have $C_\nu^{n-1} \subseteq C_\nu^n \cap F_{\nu+1}^{n+1} \subseteq \langle X_\nu^{n+1} \rangle$, and (C5) holds for n.

It remains to prove condition (C6). First, $(C_\nu \mid \nu < \tau)$ is a chain of submodules of M by (C3), so we only have to verify its continuity: Let $\gamma < \tau$ be a limit ordinal. By (C1), $C_\gamma = \bigcup_{n<\omega} F_\gamma^n = \bigcup_{n<\omega} \langle \{u_{\gamma,\alpha}^n \mid \alpha < \kappa_\gamma\} \rangle = \bigcup_{n<\omega} \bigcup_{\nu<\gamma} \langle \{u_{\gamma,\alpha}^n \mid \alpha < \kappa_\nu\} \rangle$, where the latter equality holds because $\kappa_\gamma = \sup_{\nu<\gamma} \kappa_\nu$. Note that C_ν^n contains $u_{\rho,\alpha}^n$ for *all* $\rho < \tau$ and $\alpha < \kappa_\nu$, so the latter union is contained in $\bigcup_{n<\omega} \bigcup_{\nu<\gamma} \langle C_\nu^n \rangle$. As $C_\nu = \bigcup_{n<\omega} \langle C_\nu^n \rangle$, we infer that $C_\gamma \subseteq \bigcup_{\nu<\gamma} C_\nu$. The opposite inclusion is obvious, so we conclude that $C_\gamma = \bigcup_{\nu<\gamma} C_\nu$, q.e.d. □

Remark 4.17. 1. The particular setting of Definition 4.5 can easily be generalized as follows: we take any set $\mathcal{C}$ consisting of $\leq \mu$ generated modules and call a module M "free", if it is isomorphic to a direct sum of modules from $\mathcal{C}$. Then the notions of a "basis", "free" factor, and $B(M, N)$ can easily be adapted so that properties (P1)-(P6) from 4.4 hold for μ, and Theorem 4.12 extends to this generalized setting (see [5, §2.II] for more details). The particular setting of projective modules from

Definition 4.5 is just the case when $\mu = \omega$ and $\mathcal{C}$ = a representative set of all countably generated projective modules.

The general form of 4.4 even makes it possible to consider settings far beyond projectivity or decomposition into direct sums of modules from a given set $\mathcal{C}$. Instead of direct sums (= possibly infinitely iterated, but split, extensions), one considers infinitely iterated, but not necessarily split, extensions of modules from $\mathcal{C}$. Then a module M is "free", if M is a transfinite extension of modules from $\mathcal{C}$. For more details on this setting, we refer to [5, 2.III] and [11, 7.4]; its applications are far reaching: one can prove structure results for Baer modules [11, §14.3], a finite type theorem for infinitely generated tilting modules [11, 13.46], etc.

2. In all the settings mentioned in 1., the relation $B(M, N)$ satisfies $(\mathcal{X}, \mathcal{Y}) \in B(M, N)$, iff $\mathcal{Y} = \{X \in \mathcal{X} \mid X \subseteq N\}$. Hence $\mathcal{Y} = \mathcal{X} \restriction N$ implies $\mathcal{Y} \subseteq \mathcal{X}$. In particular, the sets X^n_ν $(n < \omega)$ constructed in the proof of Lemma 4.16 can be chosen to form a chain. Thus, in order to prove Theorem 4.12 in these settings, it suffices to verify property (P1) of the "bases" $\mathcal{X}$ in the weaker form of continuity:

(P1′) *(continuity)* $\emptyset \in \mathcal{X}$, and $\mathcal{X}$ is closed under unions of arbitrary chains.

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