Some representation theory arising from set-theoretic homological algebra

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I. Decomposition, deconstruction, and their limitations
A class of modules $\mathcal{C}$ is **decomposable**, provided that there is a cardinal $\kappa$ such that each module in $\mathcal{C}$ is a direct sum of strongly $< \kappa$-presented modules from $\mathcal{C}$.

[Kaplansky]
1. The class $\mathcal{P}_0$ of all projective modules is decomposable.

[Faith-Walker]
2. The class $\mathcal{I}_0$ of all injective modules is decomposable iff $R$ is a right noetherian ring.

[Huisgen-Zimmermann]
3. Mod-$R$ is decomposable iff $R$ is a right pure-semisimple ring. In fact, if $M$ is a module such that $\text{Prod}(M)$ is decomposable, then $M$ is $\Sigma$-pure-injective.
Such examples, however, are rare in general – most classes of modules are not decomposable.

**Example**

Assume that the ring $R$ is not right perfect, that is, there is a strictly decreasing chain of principal left ideals

$$Ra_0 \supsetneq \cdots \supsetneq Ra_n \cdots a_0 \supsetneq Ra_{n+1}a_n \cdots a_o \supsetneq \cdots$$

Then the class $\mathcal{F}_0$ of all flat modules is not decomposable.

**Example**

There exist arbitrarily large indecomposable flat abelian groups.
Transfinite extensions

Let \( \mathcal{A} \subseteq \text{Mod}^\to_R \). A module \( M \) is \( \mathcal{A} \)-filtered (or a transfinite extension of the modules in \( \mathcal{A} \)), provided that there exists an increasing sequence \( (M_\alpha \mid \alpha \leq \sigma) \) consisting of submodules of \( M \) such that \( M_0 = 0 \), \( M_\sigma = M \),

- \( M_\alpha = \bigcup_{\beta < \alpha} M_\beta \) for each limit ordinal \( \alpha \leq \sigma \), and
- for each \( \alpha < \sigma \), \( M_{\alpha+1}/M_\alpha \) is isomorphic to an element of \( \mathcal{A} \).

\textit{Notation:} \( M \in \text{Filt}(\mathcal{A}) \). A class \( \mathcal{A} \) is \textit{filtration closed} if \( \text{Filt}(\mathcal{A}) = \mathcal{A} \).

\textbf{Eklof Lemma}

\[ \perp \mathcal{C} = \text{KerExt}^1_R(\_\_ , \mathcal{C}) \] is filtration closed for each class of modules \( \mathcal{C} \).

In particular, so are the classes \( \mathcal{P}_n \) and \( \mathcal{F}_n \) of all modules of projective and flat dimension \( \leq n \), for each \( n < \omega \).
A class of modules $\mathcal{A}$ is **deconstructible**, provided there is a cardinal $\kappa$ such that $\mathcal{A} = \text{Filt}(\mathcal{A}^{<\kappa})$ where $\mathcal{A}^{<\kappa}$ denotes the class of all strongly $<\kappa$-presented modules from $\mathcal{A}$.

All decomposable classes are deconstructible.

For each $n < \omega$, the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ are deconstructible.

[Eklof-T.]

More in general, for each set of modules $S$, the class $\bot(S^{\bot})$ is deconstructible. Here, $S^{\bot} = \text{KerExt}_R^1(S, -)$. 

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A class of modules $\mathcal{A}$ is precovering if for each module $M$ there is $f \in \text{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \text{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ factorizes through $f$:

$$
\begin{array}{c}
A \\
\uparrow \\
\downarrow \\
A'
\end{array} \xrightarrow{f} 
\begin{array}{c}
M \\
\uparrow \\
\downarrow \\
f'
\end{array}
$$

The map $f$ is an $\mathcal{A}$–precover of $M$.

If $f$ is moreover right minimal (that is, $f$ factorizes through itself only by an automorphism of $A$), then $f$ is an $\mathcal{A}$–cover of $M$.

If $\mathcal{A}$ provides for covers for all modules, then $\mathcal{A}$ is called a covering class.
The abundance of approximations

[Enochs], [Šťovíček]

- All deconstructible classes are precovering.
- All precovering classes closed under direct limits are covering.

In particular, the class $\perp (S\perp)$ is precovering for any set of modules $S$.

*Note*: If $R \in S$, then $\perp (S\perp)$ coincides with the class of all direct summands of $S$-filtered modules.

Flat cover conjecture

$F_0$ is covering for any ring $R$, and so are the classes $F_n$ for each $n > 0$.

The classes $P_n \ (n \geq 0)$ are precovering. . .
Let $R$ be a ring and $C$ be a class of countably presented modules. $\varinjlim_{\omega} C$ denotes the class of all **Bass modules** over $C$, that is, the modules $B$ that are countable direct limits of modules from $C$. W.l.o.g., such $B$ is the direct limit of a chain

$$F_0 \overset{f_0}{\rightarrow} F_1 \overset{f_1}{\rightarrow} \ldots \overset{f_{i-1}}{\rightarrow} F_i \overset{f_i}{\rightarrow} F_{i+1} \overset{f_{i+1}}{\rightarrow} \ldots$$

with $F_i \in C$ and $f_i \in \text{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

**The classic Bass module**

Let $C$ be the class of all countably generated projective modules. Then the Bass modules coincide with the countably presented flat modules. If $R$ is not right perfect, then a classic instance of such a Bass module $B$ arises when $F_i = R$ and $f_i$ is the left multiplication by $a_i$ ($i < \omega$) where $Ra_0 \supsetneq \ldots \supsetneq Ra_n \ldots a_0 \supsetneq Ra_{n+1} a_n \ldots a_o \supsetneq \ldots$ is strictly decreasing.
A module $M$ is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left $R$-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

The class of all flat Mittag-Leffler modules is denoted by $\mathcal{FM}$.

$\mathcal{P}_0 \subseteq \mathcal{FM} \subseteq \mathcal{F}_0$.
$\mathcal{FM}$ is filtration closed, and it is closed under pure submodules.

$M \in \mathcal{FM}$, iff each countable subset of $M$ is contained in a countably generated projective and pure submodule of $M$.
In particular, all countably generated modules in $\mathcal{FM}$ are projective.
Theorem (Angeleri-Šaroch-T.)

Assume that $R$ is not right perfect. Then the class $FM$ is not precovering, and hence not deconstructible.

Idea of proof: Choose a non-projective Bass module $B$ over $P_{0}^{<\omega}$, and prove that $B$ has no $FM$-precover.

The main tool: Tree modules.
II. Tree modules and their applications
Let $\kappa$ be an infinite cardinal, and $T_\kappa$ be the set of all finite sequences of ordinals $< \kappa$, so

$$T_\kappa = \{\tau : n \rightarrow \kappa \mid n < \omega\}.$$ 

Partially ordered by inclusion, $T_\kappa$ is a tree, called the tree on $\kappa$.

Let $\text{Br}(T_\kappa)$ denote the set of all branches of $T_\kappa$. Each $\nu \in \text{Br}(T_\kappa)$ can be identified with an $\omega$-sequence of ordinals $< \kappa$:

$$\text{Br}(T_\kappa) = \{\nu : \omega \rightarrow \kappa\}.$$ 

$|T_\kappa| = \kappa$ and $|\text{Br}(T_\kappa)| = \kappa^\omega$.

Notation: $\ell(\tau)$ denotes the length of $\tau$ for each $\tau \in T_\kappa$. 
Let $D := \bigoplus_{\tau \in T_{\kappa}} F_{\ell(\tau)}$, and $P := \prod_{\tau \in T_{\kappa}} F_{\ell(\tau)}$.

For $\nu \in \text{Br}(T_{\kappa})$, $i < \omega$, and $x \in F_i$, we define $x_{\nu i} \in P$ by

$$\pi_{\nu | i}(x_{\nu i}) = x,$$

$$\pi_{\nu | j}(x_{\nu i}) = g_{j-1} \ldots g_{i}(x) \text{ for each } i < j < \omega,$$

$$\pi_{\tau}(x_{\nu i}) = 0 \text{ otherwise},$$

where $\pi_{\tau} \in \text{Hom}_R(P, F_{\ell(\tau)})$ denotes the $\tau$th projection for each $\tau \in T_{\kappa}$.

Let $X_{\nu i} := \{x_{\nu i} \mid x \in F_i\}$. Then $X_{\nu i}$ is a submodule of $P$ isomorphic to $F_i$. 
Let $X_\nu := \sum_{i < \omega} X_{\nu i}$, and $G := \sum_{\nu} \text{Br}(T_\kappa) X_\nu$.

**Basic properties**

- $D \subseteq G \subseteq P$.
- There is a ‘tree module’ exact sequence
  \[
  0 \to D \to G \to B^{\text{Br}(T_\kappa)} \to 0.
  \]
- $G$ is a flat Mittag-Leffler module.
Proof of the Theorem

Assume there exists a $\mathcal{FM}$-precover $f : F \rightarrow B$ of the classic Bass module $B$. Let $K = \text{Ker}(f)$, so we have an exact sequence

$$0 \rightarrow K \hookrightarrow F \xrightarrow{f} B \rightarrow 0.$$

Let $\kappa$ be an infinite cardinal such that $|R| \leq \kappa$ and $|K| \leq 2^\kappa = \kappa^\omega$.

Consider the ‘tree module’ exact sequence

$$0 \rightarrow D \hookrightarrow G \rightarrow B^{(2^\kappa)} \rightarrow 0,$$

so $G \in \mathcal{FM}$ and $D$ is a free module of rank $\kappa$. Clearly, $G \in \mathcal{P}_1$.

Let $\eta : K \rightarrow E$ be a $\{G\}^\perp$-preenvelope of $K$ with a $\{G\}$-filtered cokernel.
Consider the pushout

\[
\begin{array}{ccccccccc}
0 & \to & K & \to & F & \to & B & \to & 0 \\
\downarrow & & \downarrow \subseteq & \downarrow f & & \downarrow & & \downarrow & \\
0 & \to & E & \to & P & \to & B & \to & 0 \\
\downarrow & & \downarrow \subseteq & \downarrow g & & \downarrow & & \downarrow & \\
Coker(\eta) & \to & Coker(\varepsilon) & \to & \to & & \to & & \to & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

Then \( P \in \mathcal{FM} \). Since \( f \) is an \( \mathcal{FM} \)-precover, there exists \( h : P \to F \) such that \( fh = g \). Then \( f = g\varepsilon = fh\varepsilon \), whence \( K + \text{Im}(h) = F \). Let \( h' = h \upharpoonright E \). Then \( h' : E \to K \) and \( \text{Im}(h') = K \cap \text{Im}(h) \).
Consider the restricted exact sequence

\[ 0 \longrightarrow \text{Im}(h') \longrightarrow \subseteq \longrightarrow \text{Im}(h) \xrightarrow{f \upharpoonright \text{Im}(h)} B \longrightarrow 0. \]

As \( E \in G^\perp \) and \( G \in \mathcal{P}_1 \), also \( \text{Im}(h') \in G^\perp \).

Applying \( \text{Hom}_R(\cdot, \text{Im}(h')) \) to the ‘tree-module’ exact sequence above, we obtain the exact sequence

\[ \text{Hom}_R(D, \text{Im}(h')) \longrightarrow \text{Ext}_R^1(B, \text{Im}(h'))^{2^\kappa} \longrightarrow 0 \]

where the first term has cardinality only \( \leq |K|^\kappa \leq 2^\kappa \), so the second term must be zero.

This yields \( \text{Im}(h') \in B^\perp \). Then \( f \upharpoonright \text{Im}(h) \) splits, and so does the \( \mathcal{FM} \)-precover \( f \), a contradiction with \( B \notin \mathcal{FM} \). \( \square \)
Lemma (Šaroch)

Let $\mathcal{C}$ be a class of countably presented modules, and $\mathcal{L}$ the class of all ‘locally $\mathcal{C}$-free’ modules.

Let $B$ be a Bass module over $\mathcal{C}$ such that $B$ is not a direct summand in a module from $\mathcal{L}$.

Then $B$ has no $\mathcal{L}$-precovers.
III. A generalization via tilting theory
Large tilting modules

$T$ is a (large) tilting module provided that
1. $T$ has finite projective dimension,
2. $\Ext^i_R(T, T^{(\kappa)}) = 0$ for each cardinal $\kappa$, and
3. there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $r < \omega$, and for each $i < r$, $T_i \in \text{Add}(T)$, i.e., $T_i$ is a direct summand of a (possibly infinite) direct sum of copies of $T$.

$\mathcal{B} = \{T\}^\perp = \cap_{1<i} \ker \Ext^i_R(T, -)$ the right tilting class of $T$.

$\mathcal{A} = \perp \mathcal{B}$ the left tilting class of $T$.

$\mathcal{A} \cap \mathcal{B} = \text{Add}(T)$.

Right tilting classes coincide with the classes of finite type, that is, they have the form $S^\perp$ where $S$ is a set of strongly finitely presented modules of bounded projective dimension.

$\mathcal{A} = \text{Filt}(\mathcal{A}^{\leq \omega})$, hence $\mathcal{A}$ is precovering. Moreover, $\mathcal{A} \subseteq \varprojlim \mathcal{A}^{<\omega}$. 
**Σ-pure split tilting modules**

A module $M$ is **Σ-pure split** provided that each pure embedding $N' \hookrightarrow N$ with $N \in \text{Add}(M)$ splits.

**[Angeleri-T.]**

A tilting module $T$ is **Σ-pure split**, iff $\mathcal{A} = \lim_{\longrightarrow} \mathcal{A}^{<\omega}$, iff $\mathcal{A}$ closed under direct limits.

**Examples**

Let $T = R$. Then $T$ is a tilting module of projective dimension 0, and $T$ is **Σ-pure split** iff $R$ is a right perfect ring.

Each **Σ-pure injective** tilting module is **Σ-pure split**.

Each finitely generated tilting module over any artin algebra is **Σ-pure injective**.
Locally $T$-free modules

Let $R$ be a ring and $T$ a tilting module.

A module $M$ is **locally $T$-free** provided that $M$ possesses a set $\mathcal{H}$ of submodules such that

- $\mathcal{H} \subseteq \mathcal{A}^{\leq \omega}$,
- each countable subset of $M$ is contained in an element of $\mathcal{H}$,
- $\mathcal{H}$ is closed under unions of countable chains.

Let $\mathcal{L}$ denote the class of all locally $T$-free modules.

**Note:** If $M$ is countably generated, then $M$ is locally $T$-free, iff $M \in \mathcal{A}^{\leq \omega}$. 
For any ring $R$ and any tilting module $T$, we have

$$\mathcal{A} \subseteq \mathcal{L} \subseteq \lim_{\rightarrow} \mathcal{A}^{<\omega}.$$ 

**The 0-dimensional case**

Let $R$ be an arbitrary ring and $T = R$. Then

$$\mathcal{A} = \mathcal{P}_0 \subseteq \mathcal{L} = \mathcal{FM} \subseteq \lim_{\rightarrow} \mathcal{A}^{<\omega} = \mathcal{F}_0.$$
### Theorem

Let $R$ be a ring and $T$ be a tilting module. Then TFAE:

1. $\mathcal{L}$ is (pre)covering.
2. $\mathcal{L}$ is deconstructible.
3. $T$ is $\Sigma$-pure split.

**Note:** The theorem on flat Mittag-Leffler modules stated earlier is just the particular case of $T = R$. 
The role of Bass modules, and Enochs’ Conjecture

**Theorem**

\( \mathcal{L} \) is (pre)covering, iff \( A \) is closed under direct limits, iff \( B \in A \) for each Bass module \( B \) over \( A^{<\omega} \) (i.e., \( \lim_{\rightarrow \omega} (A^{<\omega}) \subseteq A \)).

**Enochs’ Conjecture**

Let \( C \) be a class of modules. Then \( C \) is covering, iff \( C \) is precovering and closed under direct limits.

**Corollary**

*The Enochs’ Conjecture holds for all left tilting classes of modules.*
Let $R$ be an indecomposable hereditary finite dimensional algebra of infinite representation type. Then there is a partition of ind-$R$ into three sets:

$q$ ... the indecomposable preinjective modules

$p$ ... the indecomposable preprojective modules

$t$ ... the regular modules (the rest).

Then $p^\perp$ is a right tilting class (and $M \in p^\perp$, iff $M$ has no non-zero direct summands from $p$).

The tilting module $T$ inducing $p^\perp$ is called the **Lukas tilting module**. The left tilting class of $T$ is the class of all **Baer modules**. The locally $T$-free modules are called **locally Baer modules**.
Theorem

- The class of all Baer modules coincides with $\text{Filt}(p)$.
- The Lukas tilting module $T$ is countably generated, but has no finite dimensional direct summands, and it is not $\Sigma$-pure split.
  So the class $\mathcal{L}$ is not precovering (and hence not deconstructible).

The Bass modules behind the scene

The relevant Bass modules can be obtained as unions of the chains

$$
P_0 \xleftarrow{f_0} P_1 \xrightarrow{f_1} \ldots \xleftarrow{f_{i-1}} P_i \xrightarrow{f_i} P_{i+1} \xleftarrow{f_{i+1}} \ldots
$$

such that all the $P_i$ are preprojective (i.e., in $\text{add}(p)$), but the cokernels of all the $f_i$ are regular (i.e., in $\text{add}(t)$).
IV. Tree modules and the Auslander problem
**Almost split maps and sequences**

**Definition**

Let $R$ be a ring and $N$ be a module. A morphism of modules $f : M \to N$ is **right almost split**, provided that the following are equivalent for each morphism $g : P \to N$:

- $g$ factorizes through $f$,
- $g$ is not a split epimorphism.

Dually, **left almost split** morphisms $f' : N' \to M'$ are defined.

A short exact sequence of modules $0 \to N' \xrightarrow{f'} M \xrightarrow{f} N \to 0$ is **almost split**, if $f$ and $f'$ are right and left almost split morphisms, respectively.

**Theorem (Auslander)**

Let $N$ be an (indecomposable) finitely presented module with local endomorphism ring. Then there exists a right almost split morphism $f : M \to N$. If $N$ is not projective, then there even exists an almost split sequence as above.
Auslander’s problem and generalized tree modules

Auslander’1975, in Proc. 2nd Conf. Univ. Oklahoma
Are there further examples of right almost split morphisms in Mod-$R$?

A negative answer has recently been given using (generalized) tree modules:

Theorem (Šaroch’2015)

Let $R$ be a ring and $N$ be a module. TFAE:

1. There exists a right almost split morphism $f : M \to N$.
2. $N$ is finitely presented, and its endomorphism ring is local.

Corollary

Let $R$ be a ring and $0 \to P \to M \to N \to 0$ be an almost split sequence in Mod-$R$. Then $N$ is finitely presented with local endomorphism ring, and $P$ is pure-injective.
References


