Stabilita řešení

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Matematický ústav, Univerzita Karlova

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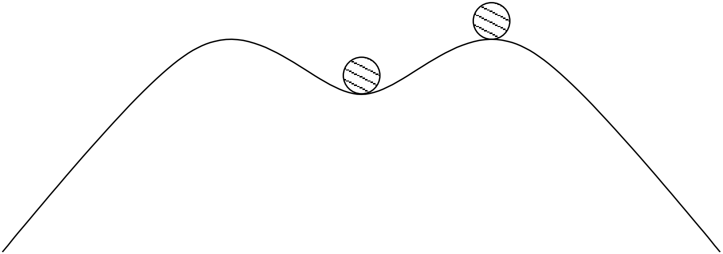
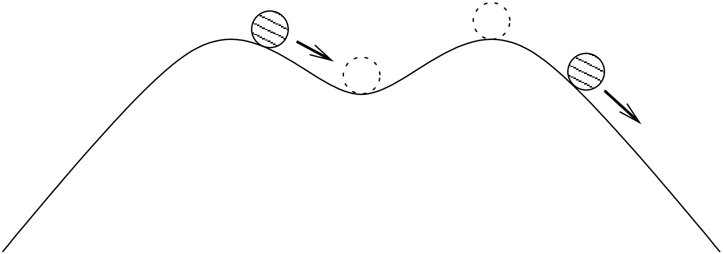
Obsah

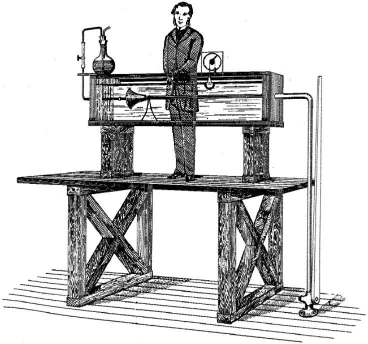
Stabilita řešení I



Stabilita řešení II



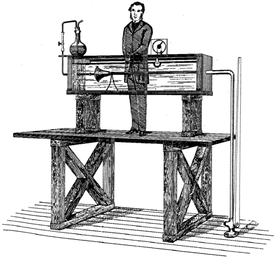
Reynolds experiment



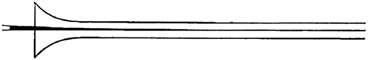
Obrázek: Reynolds experiment.

Osborne Reynolds. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. *Proc. R. Soc. Lond.*, 25:84–99, 1883

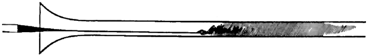
Reynolds experiment – základní pozorování



(a) Reynolds experiment.



(b) Laminární proudění.



(c) Turbulentní proudění.

[...] the internal motion of water assumes one or other of two broadly distinguishable forms—either the elements of the fluid follow one another along lines of motion which lead in the most direct manner to their destination, or the eddy about in sinous paths the most indirect possible.

Proč je to zajímavé?

Parabolický rychlostní profil

$$\boldsymbol{v} = \frac{\Delta_s R^2}{4\rho\nu}\left(1 - \left(\frac{r}{R}\right)^2\right)\boldsymbol{e}_{\hat{z}}$$

je řešením rovnic pro proudění. Do vzorce lze dosadit pro jakékoliv Δ_s. Jak je možné, že parabolický rychlostní profil v experimentu nevidíme pro jakékoliv Δ_s?

Gotthilf Heinrich Ludwig Hagen. Über die Bewegung des Wassers in Einen Cylindrischen Röhren. *Poggendorf's Annalen der Physik und Chemie*, pages 423–442, 1839
Jean Léonard Marie Poiseuille. Sur le mouvement des liquides de nature différente dans les tubes de très petits diamètres. *Annales de Chimie et Physique*, XXI:76–110, 1847

Turbulentní a laminární proudění – kvantitativní popis

Friction factor:

$$\lambda \approx \frac{\text{pressure drop}}{\text{volumetric flow rate}}$$

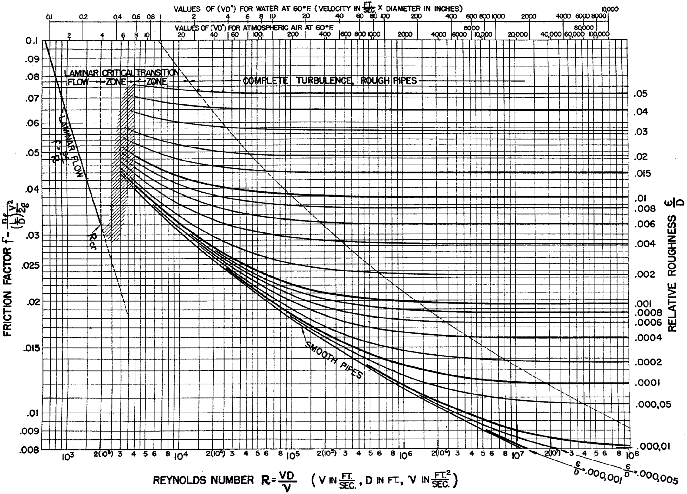
Reynolds číslo:

$$\mathrm{Re} = \frac{U_{\mathsf{max}} R}{\nu}$$

Pro laminární proudění (parabolický rychlostní profil):

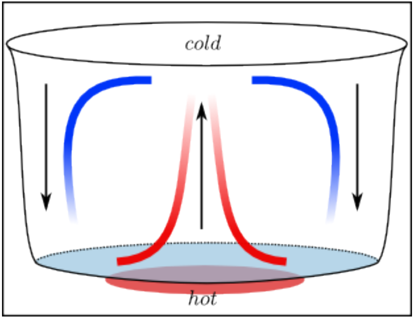
$$\lambda = \frac{64}{\mathrm{Re}}$$

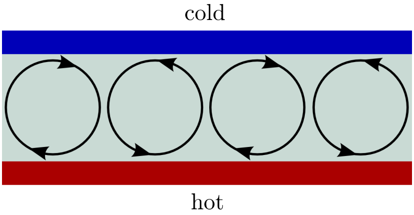
Moody diagram



Lewis F. Moody. Friction factors for pipe flow. *Transactions of ASME*, 66:671–684, November 1944

Rayleigh–Bénard I





http://www.mis.mpg.de/applan/research/rayleigh.html

Rayleigh–Bénard II

https://www.youtube.com/watch?v=5ApSJe4FaLI

Rayleigh–Bénard III

http://faraday.physics.uiowa.edu/astro/8B10.40.htm

Nanotrubky I

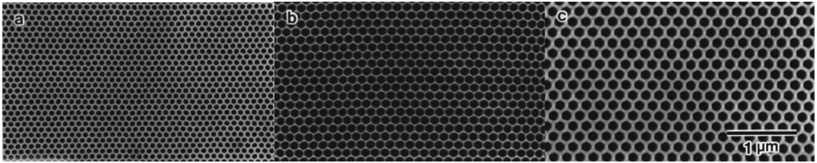


FIG. 5. SEM micrographs of cell configuration in barrier layer with different intervals of 100 nm (a), 150 nm (b), and 200 nm (c). Anodization voltage were 40 V (a), 60 V (b), and 80 V (c). Anodization was conducted in 0.3 M oxalic acid of 17 °C for (a) and (b), and 0.04 M oxalic acid of 3 °C for (c). Thickness of the oxide films was approximately 3 μm.

Hideki Masuda, Haruki Yamada, Masahiro Satoh, Hidetaka Asoh, Masashi Nakao, and Toshiaki Tamamura. Highly ordered nanochannel-array architecture in anodic alumina. *Appl. Phys. Lett.*, 71(19):2770–2772, 1997

Nanotrubky II

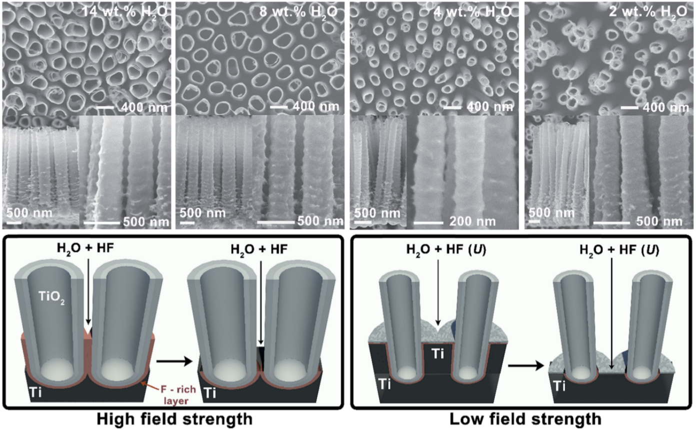


Fig. 2 SEM top and cross-section views for TiO_2 nanotubes formed at 60 V for 16 h in diethylene glycol electrolyte containing water content from 2–14 wt%. The high-electric-field model assumes the formation of space in-between the tubes by the dissolution of fluoride rich layers, *i.e.*, TiF_x, TiF_xO_y or similar (bottom left). The low electric-field-model assumes the generation of tubes of smaller diameter when the population of nucleation sites is low; the titanium is present in-between the bottom tube cells (bottom right).

Damian Kowalski, Jeremy Mallet, Jean Michel, and Michael Molinari. Low electric field strength self-organization of anodic tio2 nanotubes in diethylene glycol electrolyte. *J. Mater. Chem. A*, 3:6655–6661, 2015

Nanotrubky III

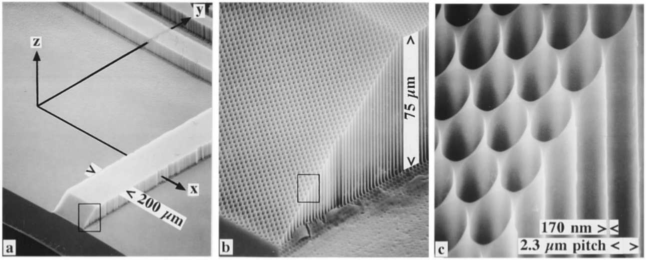
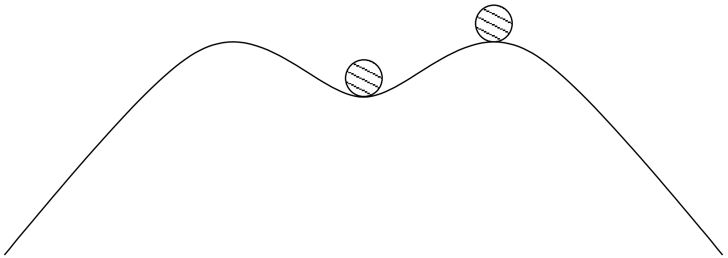
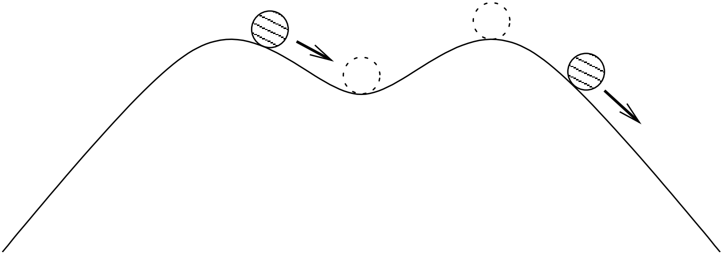


FIG. 1. Scanning electron micrographs of a patterned and micromachined layer of macroporous silicon forming a two-dimensional triangular lattice. (a) The 200 μm wide and 75 μm high bars of porous silicon are produced by micromechanically etching the porous layer. (b) A tenfold magnification of the inset in (a) is shown. The edges of the bars were formed by micromechanically etching the layer. (c) A tenfold magnification of the inset in (b) is shown. In order to reveal the microstructure, one side of the bar was polished to show a plane 45° inclined to the sample surface. The lattice constant of the macropore array is 2.3 μm, the pore diameter 2.13 μm, and the thinnest parts of the pore walls 170 nm.

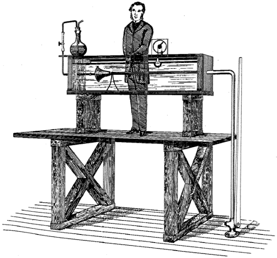
Stabilita řešení I

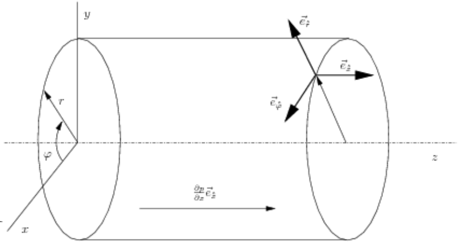


Stabilita řešení II



Reynolds experiment – matematický popis





Navier–Stokes rovnice, okrajové podmínky $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + [\nabla \boldsymbol{u}]\, \boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u}$$

$$\operatorname{div} \boldsymbol{u} = 0$$

Tlakový gradient ve směru $\boldsymbol{e}_{\hat{z}}$: $\frac{\partial p}{\partial z} = -\Delta_s$

Reynolds experiment – evoluční rovnice pro poruchu

Rychlostní pole rozložíme na základní rychlostní pole $\boldsymbol{V}$ a poruchu $\boldsymbol{v}$:

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{v}$$

Evoluční rovnice pro poruchu $\boldsymbol{v}$:

$$\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} + [\nabla \boldsymbol{V}]\boldsymbol{v} + [\nabla \boldsymbol{v}]\boldsymbol{V} + [\nabla \boldsymbol{v}]\boldsymbol{v} &= -\nabla p + \frac{1}{\mathrm{Re}}\Delta \boldsymbol{v} \\
\operatorname{div} \boldsymbol{v} &= 0 \\
\boldsymbol{v} \bullet \boldsymbol{n}\big|_{\partial\Omega} &= 0 \\
\boldsymbol{v} \bullet \boldsymbol{t}\big|_{\partial\Omega} &= 0 \\
\boldsymbol{v}\big|_{t=0} &= \boldsymbol{v}_0
\end{aligned}$$

Rayleigh–Bénard – matematický popis

Oberbeck–Boussinesq aproximace:

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + [\nabla \boldsymbol{u}]\, \boldsymbol{u} \right) = -\nabla p + \mu \Delta \boldsymbol{u} - \rho g \left(1 - \alpha (T - T_{\mathrm{ref}})\right) \boldsymbol{e}_{\hat{z}}$$

$$\operatorname{div} \boldsymbol{u} = 0$$

$$\rho c_v \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \bullet (\nabla)\, T \right) = \kappa \Delta T$$

Okrajové podmínky:

$$\boldsymbol{u}\big|_{\partial\Omega} = 0$$

$$T\big|_{\mathrm{hot\ wall}} = T_2$$

$$T\big|_{\mathrm{cold\ wall}} = T_1$$

Rayleigh–Bénard – evoluční rovnice pro poruchu

Rychlostní a teplotní pole rozložíme na základní rychlostní pole (bez proudění) a základní teplotní pole (lineární teplotní profil):

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{v}$$

$$\vartheta = T + \theta$$

Evoluční rovnice pro poruchu:

$$\frac{\partial \boldsymbol{v}}{\partial t} + [\nabla \boldsymbol{v}]\, \boldsymbol{v} = -\nabla p + \Delta \boldsymbol{v} + \mathrm{R}\theta \boldsymbol{e}_{\hat{z}}$$

$$\operatorname{div} \boldsymbol{v} = 0$$

$$\mathrm{Pr}\left(\frac{\partial \theta}{\partial t} + \boldsymbol{v} \bullet \nabla \theta\right) = \mathrm{R} v^{\hat{z}} + \Delta \theta$$

$$\boldsymbol{v}\big|_{\partial\Omega} = 0$$

$$\theta\big|_{\partial\Omega} = 0$$

$$\boldsymbol{v}\big|_{t=0} = \boldsymbol{v}_0$$

$$\theta\big|_{t=0} = \theta_0$$

Linearizace I

Úplný systém:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathbb{A}\boldsymbol{v} + \mathbb{N}(\boldsymbol{v}),$$
$$\boldsymbol{v}\big|_{t=t_0} = \boldsymbol{v}_0.$$

Linearizace:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathbb{A}\boldsymbol{v},$$
$$\boldsymbol{v}\big|_{t=t_0} = \boldsymbol{v}_0.$$

Linearizace II

Struktura rovnic po linearizaci:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathbb{A}(\mathrm{Re}, \boldsymbol{V})\boldsymbol{v}$$

Hledáme řešení ve tvaru:

$$\boldsymbol{v}(x, y, z, t) = \tilde{\boldsymbol{v}}(x, y, z)\mathrm{e}^{-\mathrm{i}\omega t} = \tilde{\boldsymbol{v}}(x, y, z)\mathrm{e}^{-\mathrm{i}\Re(\omega)t}\mathrm{e}^{\Im(\omega)t}$$

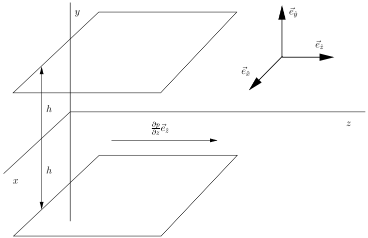
Problém pro vlastní čísla:

$$\mathrm{i}\omega\tilde{\boldsymbol{v}} = \mathbb{A}(\mathrm{Re}, \boldsymbol{V})\tilde{\boldsymbol{v}}$$

Řekneme, že základní rychlostní pole $\boldsymbol{V}$ je pro dané Reynolds číslo Re stabilní vůči infinitesimálním poruchám, právě když všechna vlastní čísla ω operátoru $\mathbb{A}$ platí

$$\Im(\omega) < 0.$$

Orr–Sommerfeld rovnice (proudění v rovinném kanálu) I



Hledáme řešení ve tvaru:

$$\boldsymbol{v}(y, z, t) = \tilde{\boldsymbol{v}}(y) e^{i(\alpha z - \omega t)}$$

Orr–Sommerfeld rovnice (proudění v rovinném kanálu) II

Rovnice pro $\tilde{v}^{\hat{y}}$, okrajové podmínky $\left.\tilde{v}^{\hat{y}}\right|_{y=\pm 1} = 0$, $\left.\frac{\mathrm{d}\tilde{v}^{\hat{y}}}{\mathrm{d}y}\right|_{y=\pm 1} = 0$:

$$\left(-\mathrm{i}\omega + i\alpha V^{\hat{z}}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}^{\hat{y}} - \mathrm{i}\alpha\frac{\mathrm{d}^2 V^{\hat{z}}}{\mathrm{d}y^2}\tilde{v}^{\hat{y}} = \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)^2\tilde{v}^{\hat{y}}$$

Struktura:

$$-\mathrm{i}\omega\mathbb{B}\tilde{v}^{\hat{y}} = \mathbb{C}\tilde{v}^{\hat{y}}$$

Lagrange interpolace

Lagrange interpolace:

$$p(x) = \sum_{j=0}^{n} f_j l_j(x) \qquad l_j(x) =_{\mathrm{def}} \frac{\prod_{k=0,k\neq j}^{n}(x - x_k)}{\prod_{k=0,k\neq j}^{n}(x_j - x_k)}$$

Zřejmě:

$$p(x_j) = f_j \qquad l_j(x_k) = \begin{cases} 1, & j = k, \\ 0, & \text{v ostatních případech} \end{cases}$$

Lloyd N. Trefethen. *Spectral methods in MATLAB*, volume 10 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000

Lloyd N. Trefethen. *Approximation theory and approximation practice*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013

Barycentrická interpolace I

Barycentric weight:

$$w_j =_{\mathrm{def}} \frac{1}{\prod_{k=0,k\neq j}^{n}(x_j - x_k)}$$

Polynom:

$$l(x) =_{\mathrm{def}} \prod_{k=0}^{n}(x - x_k) \qquad l_j(x) = l(x)\frac{w_j}{x - x_j}$$

Lagrange interpolace:

$$p(x) = \left(\sum_{j=0}^{n} f_j \frac{w_j}{x - x_j}\right) l(x)$$

Barycentrická interpolace II

Zřejmě:

$$1 = \sum_{j=0}^{n} l_j(x) = \left(\sum_{j=0}^{n} \frac{w_j}{x - x_j} \right) l(x)$$

Lagrange interpolace (barycentric formula):

$$p(x) = \frac{\sum_{j=0}^{n} f_j \frac{w_j}{x-x_j}}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}}$$

Derivování I

Máme:

$$\left\{ f(x)|_{x=x_j} \right\}_{i=0}^{n}$$

Chceme:

$$\left\{ \left. \frac{\mathrm{d}f}{\mathrm{d}x} \right|_{x=x_j} \right\}_{i=0}^{n}$$

Umíme:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\sum_{j=0}^{n} f_j \frac{w_j}{x-x_j}}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}} \right)$$

Aproximace derivace:

$$\left. \frac{\mathrm{d}f}{\mathrm{d}x} \right|_{x=x_j} \approx \left. \frac{\mathrm{d}p}{\mathrm{d}x} \right|_{x=x_j}$$

Derivování II

Derivace interpolačního polynomu:

$$p(x) = \sum_{j=0}^{n} f_j l_j(x) \implies \frac{\mathrm{d}p}{\mathrm{d}x}(x) = \sum_{j=0}^{n} f_j \frac{\mathrm{d}l_j}{\mathrm{d}x}(x)$$

Derivace „bázové“ funkce v interpolačních bodech:

$$\left.\frac{\mathrm{d}l_j}{\mathrm{d}x}\right|_{x=x_i} = \frac{w_j}{w_i}\frac{1}{x_i - x_j} \qquad \left.\frac{\mathrm{d}l_j}{\mathrm{d}x}\right|_{x=x_j} = -\sum_{i=0, i\neq j}^{n} \left.\frac{\mathrm{d}l_j}{\mathrm{d}x}\right|_{x=x_i}$$

Celkem:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_i} \approx \left.\frac{\mathrm{d}p}{\mathrm{d}x}\right|_{x=x_i} = \sum_{j=0}^{n} f_j \frac{w_j}{w_i}\frac{1}{x_i - x_j} = \sum_{j=0}^{n} \mathrm{D}^{(1)}{}_{ij} f_j$$

Derivování III

Chebyshev body, interval $[-1, 1]$, $x_i = \cos\left(\frac{(k-1)\pi}{N-1}\right)$, $i = 1, \ldots, N$:

$$\begin{bmatrix} \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_1} \\ \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_2} \\ \vdots \\ \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_N} \end{bmatrix} = \mathbb{D}^{(1)} \begin{bmatrix} f\big|_{x=x_1} \\ f\big|_{x=x_2} \\ \vdots \\ f\big|_{x=x_N} \end{bmatrix}$$

Polož $c_1 = c_N = 2$, $c_2 = \cdots = c_{N-1} = 1$:

$$\mathrm{D}^{(1)}{}_{11} = \frac{2(N-1)^2+1}{6} \qquad \mathrm{D}^{(1)}{}_{NN} = -\frac{2(N-1)^2+1}{6}$$

$$\mathrm{D}^{(1)}{}_{kj} = \frac{c_k}{c_j}\frac{(-1)^{j+k}}{(x_k - x_j)} \qquad \mathrm{D}^{(1)}{}_{jj} = -\frac{1}{2}\frac{x_j}{(1-x_j^2)}$$

Derivování IV

Pozor na konvenci číslování uzlovýchh bodů.

$$
D_N \; = \;
\left|
\begin{array}{c|c|c}
\hline
\dfrac{2N^2+1}{6} & 2\dfrac{(-1)^j}{1-x_j} & \dfrac{1}{2}(-1)^N \\
\hline
 & \dfrac{(-1)^{i+j}}{x_i - x_j} & \\
-\dfrac{1}{2}\dfrac{(-1)^i}{1-x_i} & \dfrac{-x_j}{2(1-x_j^2)} & \dfrac{1}{2}\dfrac{(-1)^{N+i}}{1+x_i} \\
 & \dfrac{(-1)^{i+j}}{x_i - x_j} & \\
\hline
-\dfrac{1}{2}(-1)^N & -2\dfrac{(-1)^{N+j}}{1+x_j} & -\dfrac{2N^2+1}{6} \\
\hline
\end{array}
\right|
$$

Lloyd N. Trefethen. *Spectral methods in MATLAB*, volume 10 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000
J. A. Weideman and S. C. Reddy. A MATLAB differentiation matrix suite. *ACM Trans. Math. Softw.*, 26(4):465–519, 2000

Orr–Sommerfeld rovnice – diskretizace

Rovnice pro $\tilde{v}^{\hat{y}}$, okrajové podmínky $\left.\tilde{v}^{\hat{y}}\right|_{y=\pm 1} = 0$, $\left.\frac{\mathrm{d}\tilde{v}^{\hat{y}}}{\mathrm{d}y}\right|_{y=\pm 1} = 0$:

$$\left(-\mathrm{i}\omega + i\alpha V^{\hat{z}}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}^{\hat{y}} - \mathrm{i}\alpha\frac{\mathrm{d}^2 V^{\hat{z}}}{\mathrm{d}y^2}\tilde{v}^{\hat{y}} = \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)^2\tilde{v}^{\hat{y}}$$

Plán:

- Rovnici vynutíme ve vnitřních interpolačních bodech $i = 2, \dots, N-1$.

  (V krajních bodech intervalu máme okrajovou podmínku $\left.\tilde{v}^{\hat{y}}\right|_{y=\pm 1} = 0$. Podmínka $\left.\frac{\mathrm{d}\tilde{v}^{\hat{y}}}{\mathrm{d}y}\right|_{y=\pm 1} = 0$ se vynucuje lehce komplikovanějším způsobem.)

- Derivace nahradíme maticovým násobením $\frac{\mathrm{d}}{\mathrm{d}x} \mapsto \mathbb{D}^{(1)}$.

Zkuste si sami

J. A. Weideman and S. C. Reddy. A MATLAB differentiation matrix suite. *ACM Trans. Math. Softw.*, 26(4):465–519, 2000
Lloyd N. Trefethen. *Spectral methods in MATLAB*, volume 10 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000

Přechodný růst I

Rovnice:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix} \begin{bmatrix} v \\ \eta \end{bmatrix}$$

Operátor:

$$\mathbb{A} =_{\mathrm{def}} \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}$$

Přechodný růst I

Rovnice:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

Operátor:

$$\mathbb{A} =_{\mathrm{def}} \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}$$

Operátor $\mathbb{A}$ je nenormální.

$$\mathbb{A}^\top \mathbb{A} \neq \mathbb{A}\mathbb{A}^\top$$

Vlastní čísla:

$$\lambda_{1,2} = -\frac{1}{\mathrm{Re}}$$

Báze v $\mathbb{R}^2$, $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$:

$$\left(\mathbb{A} - \lambda \mathbb{I}\right) \boldsymbol{v}_1 = \boldsymbol{v}_2$$

$$\left(\mathbb{A} - \lambda \mathbb{I}\right)^2 \boldsymbol{v}_1 = \left(\mathbb{A} - \lambda \mathbb{I}\right) \boldsymbol{v}_2 = 0$$

Přechodný růst II

Rovnice:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \mathbb{A}\boldsymbol{q}$$

Pozorování:

$$\boldsymbol{q}(t) = \mathrm{e}^{\mathbb{A}t}\boldsymbol{q}_0 = \mathrm{e}^{\mathbb{A}t}(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2) = \cdots = a_1\mathrm{e}^{\lambda t}\boldsymbol{v}_1 + (a_2 + a_1 t)\mathrm{e}^{\lambda t}\boldsymbol{v}_2$$

Nabízí se otázka jak odhadnout počáteční růst řešení. (Aneb pokusit se kvantifikovat nakolik je příslušný operátor nenormální.)

Pseudospektrum

Pseudospektrum operátoru $\mathbb{A}$, $\Lambda_\varepsilon(\mathbb{A})$, $|\cdot|$ je 2-norma:

- $\Lambda_\varepsilon(\mathbb{A}) =_{\mathrm{def}} \left\{ z \in \mathbb{C} : \left| (z\mathbb{I} - \mathbb{A})^{-1} \right| \geq \frac{1}{\varepsilon} \right\}$
- $\Lambda_\varepsilon(\mathbb{A}) =_{\mathrm{def}} \{ z \in \mathbb{C} :$ existuje $\mathbb{E}, |\mathbb{E}| \leq \varepsilon,$ tak, že $z \in \Lambda(\mathbb{A} + \mathbb{E}) \}$
- $\Lambda_\varepsilon(\mathbb{A}) =_{\mathrm{def}} \{ z \in \mathbb{C} :$ existuje $\boldsymbol{v}, |\boldsymbol{v}| = 1,$ tak, že $|(\mathbb{A} - z\mathbb{I})\boldsymbol{v}| \leq \varepsilon \}$
- $\Lambda_\varepsilon(\mathbb{A}) =_{\mathrm{def}} \{ z \in \mathbb{C} : \sigma_{\min} [z\mathbb{I} - \mathbb{A}] \leq \varepsilon \}$

J. L. M. van Dorsselaer, J. F. B. M. Kraaijevanger, and M. N. Spijker. Linear stability analysis in the numerical solution of initial value problems. In *Acta numerica, 1993*, Acta Numer., pages 199–237. Cambridge Univ. Press, Cambridge, 1993

Spodní odhad na růst

Rovnice:

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = -\mathrm{i}\mathbb{A}\boldsymbol{q}$$

$$\boldsymbol{q}\big|_{t=0} = \boldsymbol{q}_0$$

Velikost:

$$\sup_{\boldsymbol{q}_0 \neq 0} \frac{|\boldsymbol{q}(t)|}{|\boldsymbol{q}_0|} = \left|\mathrm{e}^{-\mathrm{i}\mathbb{A}t}\right|$$

Odhad:

$$\sup_{t \geq 0} \left|\mathrm{e}^{-\mathrm{i}\mathbb{A}t}\right| \geq \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \sigma_{\varepsilon}(\mathbb{A})$$

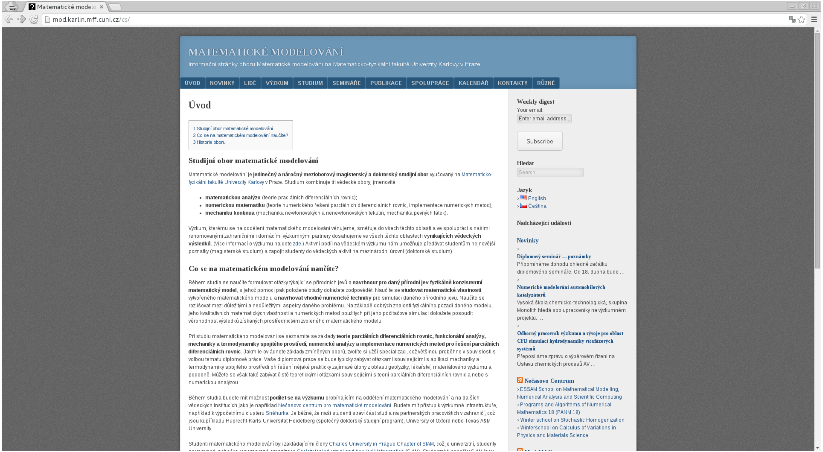
$$\sigma_{\varepsilon}(\mathbb{A}) =_{\mathrm{def}} \sup_{z \in \Lambda_{\varepsilon}(\mathbb{A})} \Im(z)$$

Lloyd N. Trefethen, Anne E. Trefethen, Satish C. Reddy, and Tobin A. Driscoll. Hydrodynamic stability without eigenvalues. *Science*, 261(5121):578–584, July 1993

Závěr

- Důležitou vlastností řešení je jeho stabilita.
- Stabilitu řešení lze zkoumat prostřednictvím spektra linearizovaného operátoru.
- Derivování je také „matice“. (S tímto tvrzením zacházejte opatrně.)
- Spektrální metoda je vhodný nástroj pro diskretizaci.

Matematické modelování



http://mod.karlin.mff.cuni.cz