

# Implicitly constituted boundary conditions, particularly threshold BC

## Boundary conditions

First let us define the tangential  $\mathbf{z}_\tau$  and normal component  $\mathbf{z}_n$  of the vector  $\mathbf{z}$  on the boundary with its unit outer normal  $\mathbf{n}$ :

$$\mathbf{z}_n = (\mathbf{z} \cdot \mathbf{n})\mathbf{n}, \quad \mathbf{z}_\tau = \mathbf{z} - \mathbf{z}_n, \quad \mathbf{z}_n \cdot \mathbf{z}_\tau = 0. \quad (1)$$

We present four different boundary conditions (BCs). In all cases we suppose that the fluid does not flow through the boundary, i.e.  $\mathbf{v}_n = \mathbf{0}$ .

**No slip BC** Fluid perfectly sticks on the boundary:  $\mathbf{v}_\tau = \mathbf{0}$ .

**Full slip BC** Fluid perfectly slips on the boundary:  $(\mathbf{T}\mathbf{n})_\tau = \mathbf{0}$ .

**Partial Navier slip BC** Combination of no slip and full slip BC:  $\gamma(\mathbf{T}\mathbf{n})_\tau + \mathbf{v}_\tau = \mathbf{0}$  for  $\gamma \geq 0$ . If  $\gamma = 0$  we obtain no slip BC,  $\gamma = \infty$  is equivalent to full slip BC.

**Threshold BC** Fluid perfectly sticks to the boundary. When the tangential stress exceeds a value  $\sigma$ , it behaves as a Navier slip.

$$\begin{aligned} \mathbf{v}_\tau = \mathbf{0} & \quad \Leftrightarrow \quad |(\mathbf{T}\mathbf{n})_\tau| \leq \sigma \\ \gamma(|(\mathbf{T}\mathbf{n})_\tau| - \sigma) \frac{(\mathbf{T}\mathbf{n})_\tau}{|(\mathbf{T}\mathbf{n})_\tau|} + \mathbf{v}_\tau = \mathbf{0} & \quad \Leftrightarrow \quad |(\mathbf{T}\mathbf{n})_\tau| > \sigma. \end{aligned} \quad (2)$$

This BC is a boundary equivalent of Bingham fluid and it can reduce to first three BCs. Particularly, if  $\sigma = \infty$  it reduces to no slip BC,  $\sigma = 0$  gives Navier slip and  $\sigma = 0 \& \gamma = \infty$  gives full slip BC.

## Analytical solution of Poiseuille Flow for Stokes with threshold BC

We find the analytical solution of Poiseuille Flow for Stokes problem with different threshold BCs at bottom and the top, see Figure 1. We suppose that fluid velocity is in the form  $\mathbf{v} = (u(y), 0)$ , then the balance of mass is automatically satisfied and the Cauchy stress tensor  $\mathbf{T}$  is in the form

$$\mathbf{T} = -p\mathbf{I} + 2\mu_s\mathbf{D} \Leftrightarrow \mathbf{T} = \begin{pmatrix} -p & \mu_s \frac{\partial u}{\partial y} \\ \mu_s \frac{\partial u}{\partial y} & -p \end{pmatrix}$$

and the balance of linear momentum  $\text{div } \mathbf{T} = 0$  gives

$$\frac{\partial p}{\partial x} = \mu_s \frac{\partial^2 u}{\partial y^2} \quad (3)$$

$$\frac{\partial p}{\partial y} = 0. \quad (4)$$

From (4) we know that  $p = p(x)$ , and so the left-hand-side of (3) is only a function of  $x$  and the right-hand-side is only a function of  $y$ . This implies that both sides of the equation have to be equal to a

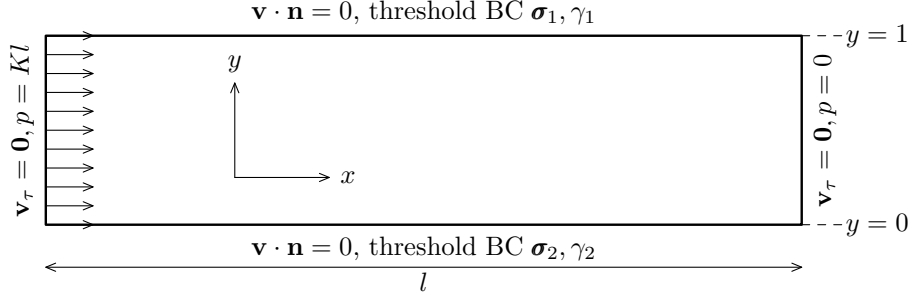


Figure 1: Poiseuille flow for Stokes problem with threshold BCs.

constant, let us denote it  $C$ . Then  $p = Cx + C_0$  and from the boundary conditions  $p(0) = Kl > 0, p(l) = 0$  we get that  $C = -K$  and find a general solution for  $p$  and  $u$

$$p = K(l - x) \text{ and } u = \frac{-K}{2\mu_s}y^2 + C_1y + C_2. \quad (5)$$

The fluid flows from left to the right and it can partially slip to the boundary, this means that  $u$  is a non-negative concave function of  $y$ , i.e.

$$u \geq 0, \quad \frac{\partial u}{\partial y} \Big|_{y=0} \geq 0, \quad \frac{\partial u}{\partial y} \Big|_{y=1} \leq 0,$$

which gives

$$C_2 \geq 0, \quad 0 \leq C_1 \leq \frac{K}{\mu_s}. \quad (6)$$

Now, we rewrite the threshold BC for our problem. First we compute the tangential part of the Cauchy stress tensor  $(\mathbf{Tn})_\tau$

$$\begin{aligned} y = 0 : (\mathbf{Tn})_\tau &= -T_{xy} = -\eta C_1 \leq 0 \Rightarrow |(\mathbf{Tn})_\tau| = \eta C_1 \\ y = 1 : (\mathbf{Tn})_\tau &= T_{xy} = -K + \eta C_1 \leq 0 \Rightarrow |(\mathbf{Tn})_\tau| = K - \eta C_1. \end{aligned}$$

Then the threshold BC is in the form

$$\begin{aligned} y = 0 : \quad & C_2 = u(0) = 0 \quad \Leftrightarrow \quad \mu_s C_1 \leq \sigma_1 \\ & C_2 = u(0) = \gamma_1(\mu_s C_1 - \sigma_1) \quad \Leftrightarrow \quad \mu_s C_1 > \sigma_1 \\ y = 1 : \quad & \frac{-K}{2\mu_s} + C_1 + C_2 = u(1) = 0 \quad \Leftrightarrow \quad (K - \mu_s C_1) \leq \sigma_2 \\ & \frac{-K}{2\mu_s} + C_1 + C_2 = u(1) = \gamma_2(K - \mu_s C_1 - \sigma_2) \quad \Leftrightarrow \quad (K - \mu_s C_1) > \sigma_2. \end{aligned}$$

The problem reduces into four variants:

**Variant 1**

$$\begin{aligned}\sigma_1 \geq \mu_s C_1 &\Leftrightarrow C_2 = 0 \\ \sigma_2 \geq (K - \mu_s C_1) &\Leftrightarrow \frac{-K}{2\mu_s} + C_1 + C_2 = 0.\end{aligned}$$

The solution is

$$\sigma_1 \geq \frac{K}{2} \quad \& \quad \sigma_2 \geq \frac{K}{2} \quad \Leftrightarrow \quad u = \frac{-K}{2\mu_s}(y^2 - y).$$

**Variant 2**

$$\begin{aligned}\sigma_1 \geq \mu_s C_1 &\Leftrightarrow C_2 = 0 \\ \sigma_2 < (K - \mu_s C_1) &\Leftrightarrow \frac{-K}{2\mu_s} + C_1 + C_2 = \gamma_2(K - \mu_s C_1 - \sigma_2).\end{aligned}$$

The solution is

$$\sigma_2 < \frac{K}{2} \quad \& \quad \sigma_1 \geq \frac{\frac{K}{2} + \mu_s \gamma_2 (K - \sigma_2)}{1 + \gamma_2 \mu_s} \quad \Leftrightarrow \quad u = \frac{-K}{2\mu_s} y^2 + \frac{\frac{K}{2\mu_s} + \gamma_2 (K - \sigma_2)}{1 + \gamma_2 \mu_s} y.$$

**Variant 3**

$$\begin{aligned}\sigma_1 < \mu_s C_1 &\Leftrightarrow C_2 = \gamma_1(\mu_s C_1 - \sigma_1) \\ \sigma_2 \geq (K - \mu_s C_1) &\Leftrightarrow \frac{-K}{2\mu_s} + C_1 + C_2 = 0.\end{aligned}$$

The solution is

$$\sigma_1 < \frac{K}{2} \quad \& \quad \sigma_2 \geq \frac{\frac{K}{2} + \mu_s \gamma_1 (K - \sigma_1)}{1 + \gamma_1 \mu_s} \quad \Leftrightarrow \quad u = \frac{-K}{2\mu_s} y^2 + \frac{\frac{K}{2\mu_s} + \gamma_1 \sigma_1}{1 + \gamma_1 \mu_s} y + \gamma_1 \frac{\frac{K}{2} - \sigma_1}{1 + \gamma_1 \mu_s}.$$

**Variant 4**

$$\begin{aligned}\sigma_1 < \mu_s C_1 &\Leftrightarrow C_2 = \gamma_1(\mu_s C_1 - \sigma_1) \\ \sigma_2 < (K - \mu_s C_1) &\Leftrightarrow \frac{-K}{2\mu_s} + C_1 + C_2 = \gamma_2(K - \mu_s C_1 - \sigma_2).\end{aligned}$$

The solution is

$$\begin{aligned}\sigma_1, \sigma_2 \text{ belong to } \textcircled{4}, \text{ see Figure 2} &\Leftrightarrow \\ u = \frac{-K}{2\mu_s} y^2 + \frac{\frac{K}{2\mu_s} + \gamma_1 \sigma_1 + \gamma_2 (K - \sigma_2)}{1 + \mu_s (\gamma_1 + \gamma_2)} y + \gamma_1 \frac{\frac{K}{2} + \gamma_2 \mu_s (K - \sigma_1 - \sigma_2) - \sigma_1}{1 + \mu_s (\gamma_1 + \gamma_2)}.\end{aligned}$$

The form of the solution  $u$  depends on the choice of  $\sigma_1$  and  $\sigma_2$ . Four variants are depicted in Figure 2. The solution for Variant 1 is no slip on both boundaries, Variant 2 corresponds to noslip at the bottom and Navier slip at the top, Variant 3 is no slip at the top, and Navier slip at the bottom and Variant 4 are two Navier slips.

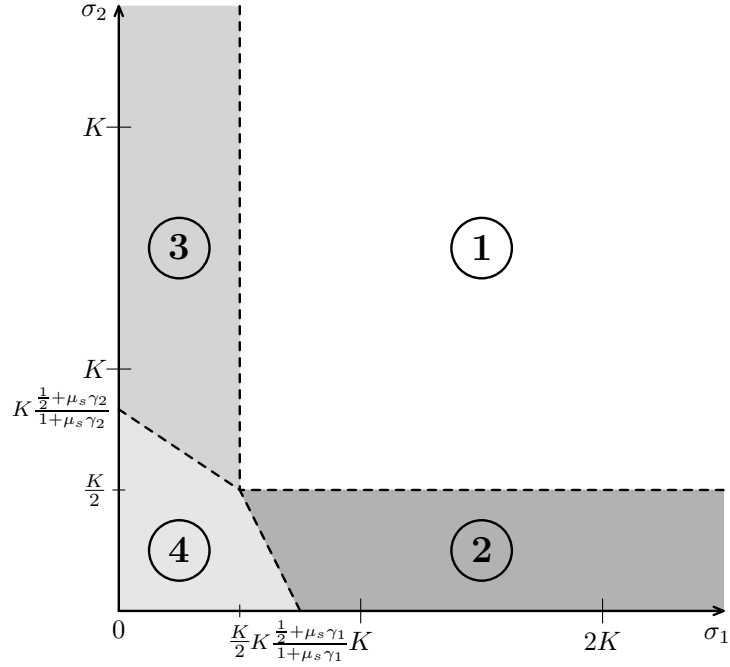


Figure 2: Four variants for  $\sigma_1, \sigma_2$ .

## Weakly-imposed Dirichlet boundary conditions – Nitsche’s method

We introduce a method that enables us to impose Dirichlet boundary conditions in a weak sense for arbitrary boundaries. It is called Nitsche’s method [3] and originally it was developed for diffusion-type problems. We use it for solving steady incompressible Stokes and  $p$ -Stokes flow with threshold BCs in the domain  $\Omega$ . Its boundary  $\partial\Omega$  consists of Neumann boundary  $\Gamma_N$  and Dirichlet boundary  $\Gamma_D$ , i.e.  $\partial\Omega = \Gamma_D \cup \Gamma_N$ ,  $\Gamma_N \cap \Gamma_D = \emptyset$  and we solve the following problem:

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \quad (7a)$$

$$\operatorname{div} (-p\mathbf{I} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)) = 0 \text{ in } \Omega \quad (7b)$$

$$\mathbf{v} = \mathbf{v}_D \text{ on } \Gamma_D \quad (7c)$$

$$-p\mathbf{n} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \mathbf{n} = \mathbf{t} \text{ on } \Gamma_N. \quad (7d)$$

The standard weak formulation of this problem is: Find  $(p, \mathbf{v} - \mathbf{v}_D) \in L^2(\Omega) \times V(\Omega)$  such that

$$\int_{\Omega} \operatorname{div} \mathbf{v} q \, dx = 0 \quad \forall q \in L^2(\Omega) \quad (8)$$

$$\int_{\Omega} (-p\mathbf{I} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)) \cdot \nabla \boldsymbol{\varphi} \, dx - \int_{\Gamma_N} \mathbf{t} \cdot \boldsymbol{\varphi} = 0 \quad \forall \boldsymbol{\varphi} \in V(\Omega)^2,$$

where  $V(\Omega) = \overline{\{\mathbf{v} \in C^\infty(\Omega)^2, \mathbf{v}|_{\Gamma_D} = 0\}}^{W^{1,2}(\Omega)}$ , so the Dirichlet condition makes a restriction on the space where the solution lives. A numerical solution of this problem is done by Finite element method that is based on the weak formulation (8), and the Dirichlet boundary condition is again imposed on the Finite element subspace.

Such a strong imposing of the boundary condition can cause convergence problems !!! Further, it can be difficult to implement the problem with Dirichlet boundary condition for the normal component of velocity  $\mathbf{v}_n$  and Neumann condition for the tangential component  $\mathbf{v}_\tau$  for arbitrary curved boundary. Further, it is much simpler to implement the threshold boundary in a weak sense using Nitsche's method.

Nitsche's method for Stokes problem (7) was written for example in [2]. One way how to derive this method was done in [1], we show it here.

First, as in standard weak formulation, we multiply (7a) by a test function  $q \in L^2(\Omega)$  and integrate it over  $\Omega$ . Also, we multiply (7b) by  $\varphi \in W^{1,2}(\Omega)^2$  and integrate the result over  $\Omega$ , use per partes and split  $\partial\Omega$  into  $\Gamma_N$  and  $\Gamma_D$ , we obtain

$$\underbrace{- \int_{\Omega} \operatorname{div} \mathbf{v} q \, dx}_{I_1} + \int_{\Omega} -p \mathbf{I} \cdot \nabla \varphi \, dx + \underbrace{\mu_s \int_{\Omega} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \cdot \nabla \varphi \, dx}_{I_2} - \int_{\Gamma_D} [-p \mathbf{n} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \mathbf{n}] \cdot \varphi \, dS - \int_{\Gamma_N} \mathbf{t} \cdot \varphi \, dS = 0. \quad (9)$$

Now, we take care of  $I_2$ . Using

$$(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \cdot \nabla \varphi = \nabla \mathbf{v} \cdot (\nabla \varphi + (\nabla \varphi)^T) \quad (10)$$

we integrate  $I_2$  per partes

$$I_2 = -\mu_s \int_{\Omega} \operatorname{div} (\nabla \varphi + (\nabla \varphi)^T) \cdot \mathbf{v} \, dx + \mu_s \int_{\Gamma_D} (\nabla \varphi + (\nabla \varphi)^T) \mathbf{n} \cdot \mathbf{v} \, dS + \mu_s \int_{\Gamma_N} (\nabla \varphi + (\nabla \varphi)^T) \mathbf{n} \cdot \mathbf{v} \, dS,$$

on the boundary  $\Gamma_D$  we set the Dirichlet boundary condition  $\mathbf{v} = \mathbf{v}_D$  and once again integrate the first term per partes. We obtain

$$I_2 = \mu_s \int_{\Omega} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \cdot \nabla \varphi \, dx - \mu_s \int_{\Gamma_D} (\nabla \varphi + (\nabla \varphi)^T) \mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_D) \, dS. \quad (11)$$

If we do the same procedure with  $I_1$  we obtain

$$I_1 = - \int_{\Omega} \operatorname{div} \mathbf{v} q \, dx + \int_{\Gamma_D} q \mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_D) \, dS. \quad (12)$$

By inserting (11) and (12) into (9) gives

$$\begin{aligned} & - \int_{\Omega} \operatorname{div} \mathbf{v} q \, dx + \int_{\Omega} [-p \mathbf{I} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] \cdot \nabla \varphi \, dx - \int_{\Gamma_D} [-p \mathbf{I} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] \mathbf{n} \cdot \varphi \, dS \\ & \quad - \int_{\Gamma_D} [-q \mathbf{I} + \mu_s (\nabla \varphi + (\nabla \varphi)^T)] \mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_D) \, dS = \int_{\Gamma_N} \mathbf{t} \cdot \varphi \, dS. \end{aligned} \quad (13)$$

This weak formulation is consistent with (7), but it is unstable. Nitsche [3] showed that the system stabilizes if a consistent penalizing term is added

$$\frac{\beta}{h} \int_{\Gamma_D} (\mathbf{v} - \mathbf{v}_D) \cdot \boldsymbol{\varphi} \, dS.$$

Parameter  $\beta > 0$  has to be large enough (depends on the problem that is solved, [2] used  $\beta = 10$ ),  $h$  is a characteristic size of the elements. All together we obtained a formulation of incompressible steady Stokes problem with weakly imposed Dirichlet condition:

$$\forall (q, \boldsymbol{\varphi}) \in L^2(\Omega) \times W^{1,2}(\Omega)^2 \quad \int_{\Omega} \operatorname{div} \mathbf{v} \, q \, dx = 0, \quad (14a)$$

$$\int_{\Omega} \mathbf{T} \cdot \nabla \boldsymbol{\varphi} \, dx - \int_{\Gamma_D} \mathbf{T} \mathbf{n} \cdot \boldsymbol{\varphi} \, dS - \int_{\Gamma_D} \mathbf{T}^{q, \boldsymbol{\varphi}} \mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_D) \, dS + \frac{\beta}{h} \int_{\Gamma_D} (\mathbf{v} - \mathbf{v}_D) \cdot \boldsymbol{\varphi} \, dS = \int_{\Gamma_N} \mathbf{t} \cdot \boldsymbol{\varphi} \, dS, \quad (14b)$$

where  $\mathbf{T} = -p\mathbf{I} + \mu_s(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$  is the Cauchy stress tensor and  $\mathbf{T}^{q, \boldsymbol{\varphi}} = -q\mathbf{I} + \mu_s(\nabla \boldsymbol{\varphi} + (\nabla \boldsymbol{\varphi})^T)$  is a stress tensor composed of the corresponding test functions.

In the derivation of this formulation we used the fact that the Cauchy stress tensor  $\mathbf{T}$  is a linear function of  $\nabla \mathbf{v}$  and so it is simply invertible. In case of non-invertible relation, or even fully implicit relation between  $\mathbf{T}$  and  $\nabla \mathbf{v}$ , the derivation can not be done in this way.

Ler us suppose that the Cauchy stress tensor  $\mathbf{T}$  is also unknown and we solve the Stokes problem in the form:

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \quad (15a)$$

$$\operatorname{div} \mathbf{T} = 0 \text{ in } \Omega \quad (15b)$$

$$\mathbf{T} = -p\mathbf{I} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \text{ in } \Omega \quad (15c)$$

$$\mathbf{v} = \mathbf{v}_D \text{ on } \Gamma_D \quad (15d)$$

$$-p\mathbf{n} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \mathbf{n} = \mathbf{t} \text{ on } \Gamma_N. \quad (15e)$$

In the standard weak formulation of this problem (15c) is multiplied by matrix test function  $\mathbf{S}$  and integrated over  $\Omega$ . In the Nitsche's method instead of using the stress tensor composed of the corresponding test functions  $\mathbf{T}^{q, \boldsymbol{\varphi}}$  we can directly use the test function  $\mathbf{S}$ . Then the generalized Nitsche's method is in the form

$$\forall (q, \boldsymbol{\varphi}, \mathbf{S}) \in L^2(\Omega) \times W^{1,2}(\Omega)^2 \times L^2(\Omega)^{2 \times 2} \quad \int_{\Omega} \operatorname{div} \mathbf{v} \, q \, dx = 0, \quad (16a)$$

$$\int_{\Omega} \mathbf{T} \cdot \nabla \boldsymbol{\varphi} \, dx - \int_{\Gamma_D} \mathbf{T} \mathbf{n} \cdot \boldsymbol{\varphi} \, dS + \frac{\beta}{h} \int_{\Gamma_D} (\mathbf{v} - \mathbf{v}_D) \cdot \boldsymbol{\varphi} \, dS = \int_{\Gamma_N} \mathbf{t} \cdot \boldsymbol{\varphi} \, dS, \quad (16b)$$

$$\int_{\Omega} [\mathbf{T} - (-p\mathbf{I} + \mu_s (\nabla \mathbf{v} + (\nabla \mathbf{v})^T))] \cdot \mathbf{S} \, dx - \int_{\Gamma_D} \mathbf{S} \mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_D) \, dS = 0. \quad (16c)$$

The advantage of the formulation (16) is that it can be used even for fully implicit relations for the stress tensor  $\mathbf{T}$ , particularly for example for explicit non-linear  $p$ -Stokes model or viscoelastic models we are interested in.

## Application of Nitsche's method to threshold BC

Let  $\partial\Omega = \Gamma_N \cup \Gamma_D$ , on  $\Gamma_N$  we have Neumann boundary condition (for example on the inlet). On  $\Gamma_D$  we have implicit boundary condition (weak Dirichlet), specifically threshold boundary condition

$$\begin{aligned} \mathbf{v} = \mathbf{0} &\Leftrightarrow |(\mathbf{Tn})_\tau| < \sigma, \\ \underbrace{|(\mathbf{Tn})_\tau| \mathbf{v} + \gamma(|(\mathbf{Tn})_\tau| - \sigma)(\mathbf{Tn})_\tau}_{\mathbf{b}_{eq}} = \mathbf{0} &\Leftrightarrow |(\mathbf{Tn})_\tau| \geq \sigma, \end{aligned}$$

where

$$\mathbf{z}_\tau = \mathbf{z} - (\mathbf{z} \cdot \mathbf{n})\mathbf{n}.$$

The weak formulation for Stokes is the following:

$$\begin{aligned} \forall (q, \boldsymbol{\varphi}, \mathbf{S}) \in L^2(\Omega) \times W^{1,2}(\Omega)^2 \times L^2(\Omega)^{2 \times 2} \quad & \int_{\Omega} \operatorname{div} \mathbf{v} q \, dx = 0, \\ \int_{\Omega} \mathbf{T} \cdot \nabla \boldsymbol{\varphi} \, dx - \int_{\Gamma_D} \mathbf{Tn} \cdot \boldsymbol{\varphi} \, dS + \frac{\beta}{h} \int_{\Gamma_D} \mathbf{b}_{eq} \cdot \boldsymbol{\varphi} \, dS &= \int_{\Gamma_N} \mathbf{t} \cdot \boldsymbol{\varphi} \, dS, \\ \int_{\Omega} [\mathbf{T} - (-p\mathbf{I} + \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T))] \cdot \mathbf{S} \, dx - \int_{\Gamma_D} \mathbf{Sn} \cdot \mathbf{b}_{eq} \, dS &= 0. \end{aligned}$$

Here  $\mathbf{b}_{eq}$  can stand for any implicit boundary condition, we treat it using Nitsche's method as an weak Dirichlet condition.

## Comparison of a numerical and analytical solution for Poiseuille flow

We use the Nitsche's method for a numerical simulation of Poiseuille flow with threshold boundary condition, see Figure 1. Parameters  $l = 4$  and  $K = 1$  are used in this computation.

We compare the numerical solution of the problem that was computed in Section with the analytical solution. In our problem we use  $\mu_s = 1\text{Pa s}$ ,  $\gamma_1 = \gamma_2 = 1$  and compute the problem for  $6 \times 6$  different  $(\sigma_1, \sigma_2) \in (0.2i, 0.2j)$ ,  $\forall i, j = 0, \dots, 5$ .

Figure 3 shows that the numerical solution corresponds to the analytical given by Figure 1.

In Figure 4 one can see the solution of velocity  $u$  and component of Cauchy stress  $T_{xy}$  for  $\sigma_1 = 0.6$  and  $\sigma_2 = 0.2$  (Variant 2).

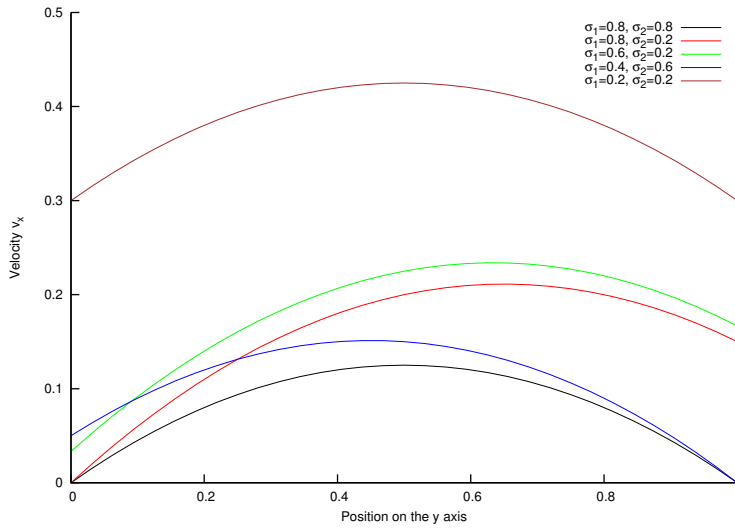


Figure 3: Graph of solution of velocity  $v_x = 0$  for  $x = 2$  for different  $\sigma_1, \sigma_2$ .

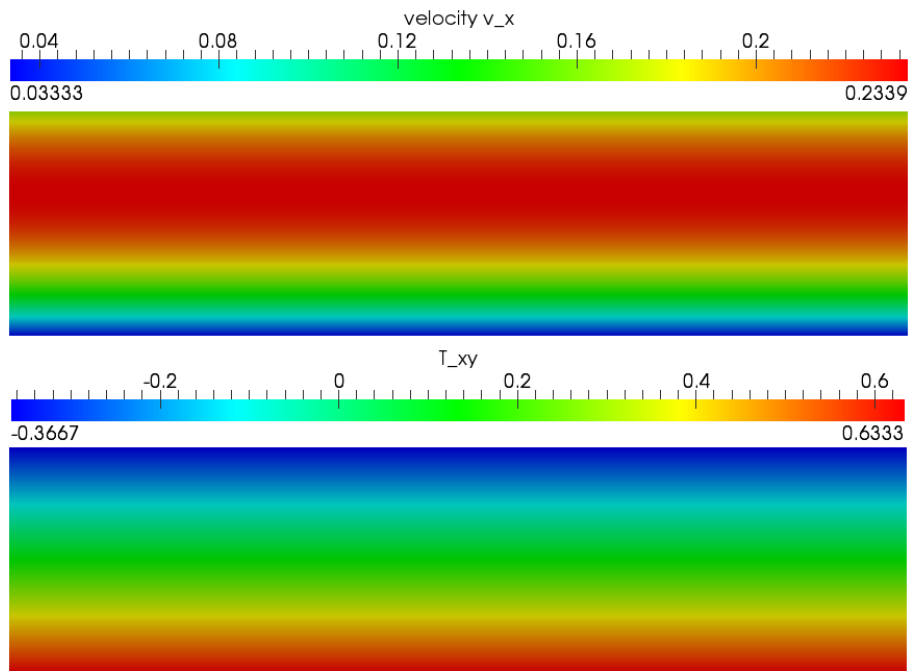


Figure 4: Solution of velocity  $v_x = u$  and  $T_{xy}$  for  $\sigma_1 = 0.6, \sigma_2 = 0.2$  (Variant 2).



## Computation in curved domains

We computed the problem also in a curved domain “knee-pipe” ( $R_1 = 1, R_2 = 1$ , straight 1 at inflow and straight 1 at outflow). We compute the problem both for Stokes and Navier-Stokes problem. The flow is driven by pressure  $p = 4$  at inflow on the top and free outflow on the right, there is threshold boundary condition at the walls. The parameters were:  $\mu = 1, \beta = 10, \sigma = 0.45$  and  $\rho = 200$  in case of Navier-Stokes.

We use the regularization for the norm in the form

$$|z|_\varepsilon = \sqrt{z^2 + \varepsilon}, \varepsilon > 0.$$

In the solution we start with  $\varepsilon = 10^{-3}$  and converge with it upto zero. In case of Navier-Stokes we converge also with  $\rho$  from zero to the value 200.

In Figure 5 there is a solution for Stokes problem in the left column and Navier-Stokes in the right column. It can be seen, that on the left the threshold is not exceeded and so there is a no-slip condition, on the right it is exceeded and it behaves as a Navier-slip. In case of Navier-Stokes the influence of the inertia can be observed.

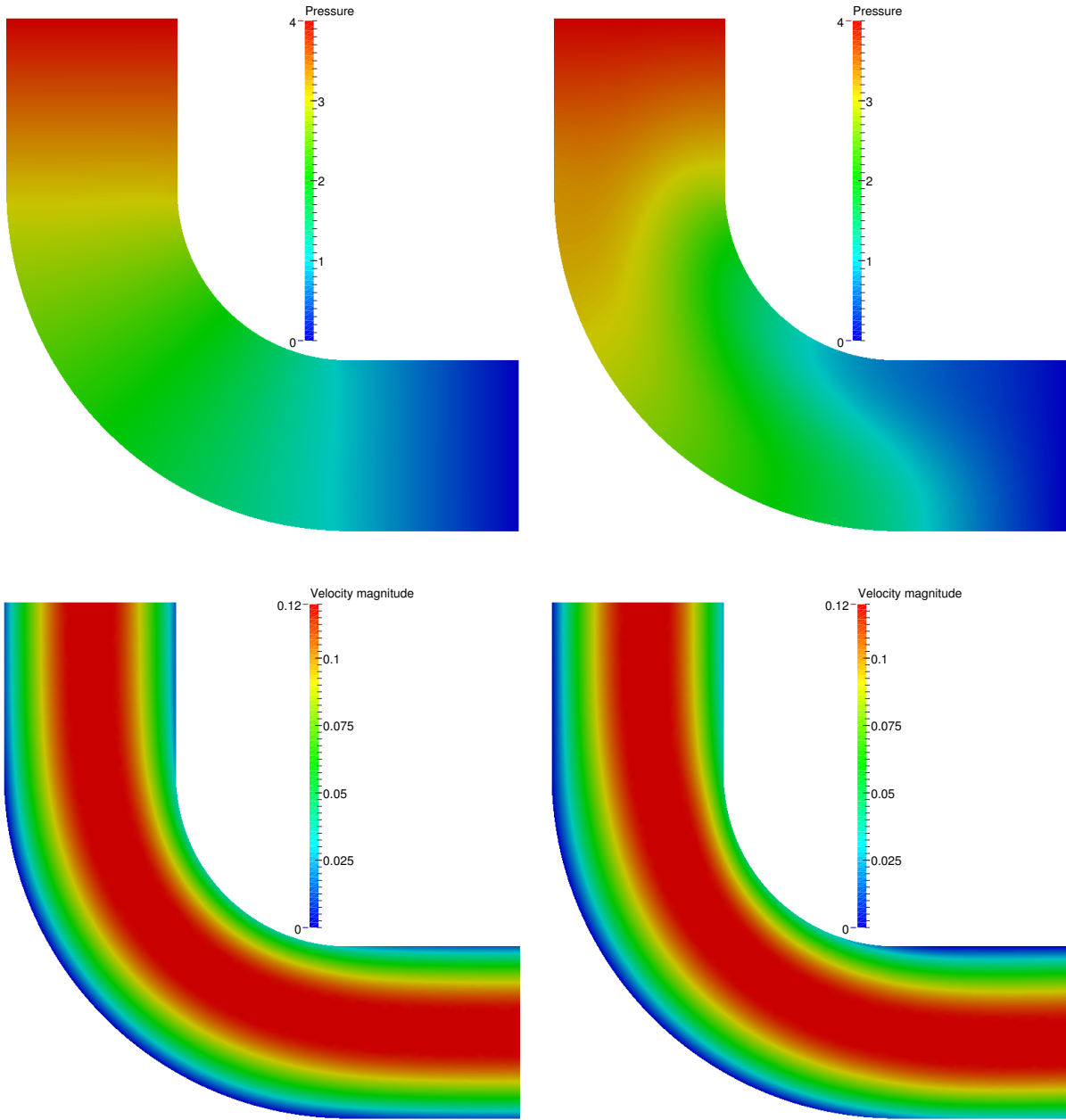


Figure 5: Solution of the curved flow for the velocity magnitude and pressure. In the left column is the solution for Stokes problem, in the right column for the Navier-Stokes.

## Other possibilities how to treat implicit boundary conditions

We solved the problem of implicit boundary condition by treating it as a weak Dirichlet BC using Nitsche's method. The other possibility can be to solve the implicit relation on the boundary and then this solution use as a standard Dirichlet condition.

However, there is some ambiguity: How one should treat the standard Navier-slip condition

$$\gamma(\mathbf{Tn})_\tau + \mathbf{v}_\tau = \mathbf{0}, \quad \gamma \geq 0.$$

In case of Navier-slip the solution for  $\mathbf{v}_\tau$  can be found very easily and treat this BC as a strong Dirichlet condition

$$\mathbf{v}_\tau = -\gamma(\mathbf{Tn})_\tau,$$

but we can found also a solution for  $(\mathbf{Tn})_\tau$  and use it as a weak Neumann condition by inserting it into the weak formulation

$$(\mathbf{Tn})_\tau = -\frac{1}{\gamma}\mathbf{v}_\tau.$$

**How to solve it?** Definition of new unknowns living on the boundary...

Solution? somehow have only one type of boundary condition different than using weakly imposed Dirichlet BC as done here.

Definition of solution in paper by Málek and Bulíček

## References

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