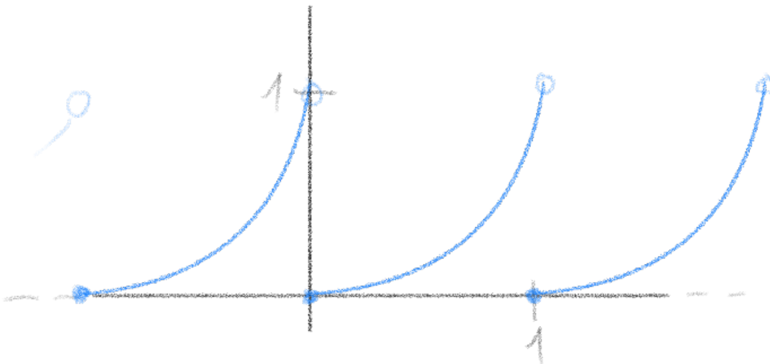
Bonusové cvičení — Téma: Fourierovy řady

1) Nalezněte Fourierovu řadu funkce $f(t)=t^2$ na intervalu $[0,1]$.

Řešení

$T=1$. Funkci f rozšíříme 1-periodicky na $\mathbb{R}$ takto:



Spočteme koeficienty z definice

$$a_k = \frac{2}{T}\int_a^{a+T} f(t)\cos\left(\frac{2\pi k t}{T}\right)dt =$$

$$= 2\int_0^1 t^2 \cos(2\pi k t)\,dt \overset{P.P.}{=} 2\left[t^2\,\frac{\sin(2\pi k t)}{2\pi k}\right]_0^1 - \frac{2}{\pi k}\int_0^1 t\,\sin(2\pi k t)\,dt$$

$$t^2 \downarrow 2t, \quad \cos(2\pi k t) \uparrow \frac{\sin(2\pi k t)}{2\pi k},\ k \geq 1; \qquad t \downarrow 1, \quad \sin(2\pi k t) \uparrow -\frac{\cos(2\pi k t)}{2\pi k}$$

$$\overset{P.P.}{=} \underbrace{\frac{\sin(2\pi k)}{\pi k}}_{=0} - \frac{2}{\pi k}\left(\left[-t\,\frac{\cos(2\pi k t)}{2\pi k}\right]_0^1 + \int_0^1 \frac{\cos(2\pi k t)}{2\pi k}\,dt\right) =$$

$$= -\frac{2}{\pi k}\left(-\frac{\overbrace{\cos(2\pi k)}^{=1}}{2\pi k} + \frac{1}{4\pi^2 k^2}\Big[\sin(2\pi k t)\Big]_0^1\right) =$$

$$= \frac{1}{\pi^2 k^2} - \frac{\overbrace{\sin(2\pi k)}^{=0}}{2\pi^3 k^3} = \boxed{\frac{1}{\pi^2 k^2}}, \quad k \in \mathbb{N}$$

$$a_0 = 2\int_0^1 t^2\,dt = 2\left[\frac{t^3}{3}\right]_0^1 = \boxed{\frac{2}{3}}$$

$$b_k = \frac{2}{T}\int_a^{a+T} f(t)\sin\left(\frac{2\pi k t}{T}\right)dt =$$

$$= 2\int_0^1 t^2 \sin(2\pi k t)\,dt \overset{P.P.}{=} 2\left[-t^2\,\frac{\cos(2\pi k t)}{2\pi k}\right]_0^1 + \frac{2}{\pi k}\int_0^1 t\,\cos(2\pi k t)\,dt$$

$$t^2 \downarrow 2t, \quad \sin(2\pi k t) \uparrow -\frac{\cos(2\pi k t)}{2\pi k}; \qquad t \downarrow 1, \quad \cos(2\pi k t) \uparrow \frac{\sin(2\pi k t)}{2\pi k}$$

$$= -\frac{\overset{=1}{\cos(2\pi k)}}{\pi k} + \frac{2}{\pi k}\left(\left[t\,\frac{\sin(2\pi k t)}{2\pi k}\right]_0^1 - \int_0^1 \frac{\sin(2\pi k t)}{2\pi k}\,dt\right) =$$

$$= -\frac{1}{\pi k} + \frac{\overset{=0}{\sin(2\pi k)}}{\pi^2 k^2} - \frac{1}{\pi^2 k^2}\left[\frac{-\cos(2\pi k t)}{2\pi k}\right]_0^1 =$$

$$= -\frac{1}{\pi k} + \frac{1}{2\pi^3 k^3}\left(\underset{=1}{\cos(2\pi k)} - \underset{=1}{\cos(0)}\right) = \boxed{-\frac{1}{\pi k}}, \ k\in\mathbb{N}$$

Výsledek zapíšeme podle definice Fourierovy řady

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{2k\pi t}{T}\right) + b_k\sin\left(\frac{2k\pi t}{T}\right)\right) =$$

$$= \frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2}\cos(2k\pi t) - \frac{1}{\pi k}\sin(2k\pi t)$$

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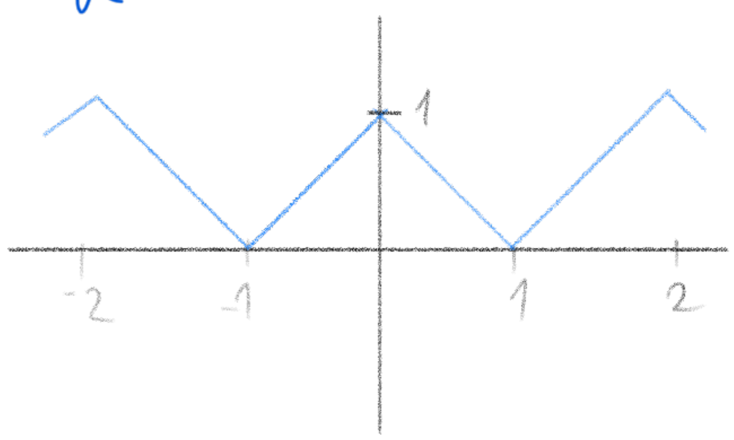
2) Nalezněte Fourierovu řadu funkce $f(t) = 1-|t|$ na intervalu $[-1,1)$.

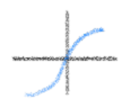
Řešení

$T=2$. Funkci f rozšíříme 2-periodicky na $\mathbb{R}$:



Funkce f je sudá.

Pro výpočet koeficientů můžeme využít symetrii, zde na b_k:



$$b_k = \frac{2}{T}\int_a^{a+T} f(t)\sin\left(\frac{2\pi k t}{T}\right)dt = \int_{-1}^{1} \underbrace{f(t)\cdot\underbrace{\sin(\pi k t)}_{\text{lichá}}}_{\text{lichá}}\,dt = 0$$

$$a_k = \frac{2}{T}\int_a^{a+T} f(t)\cos\left(\frac{2\pi k t}{T}\right)dt = \int_{-1}^{1} f(t)\cdot\cos(\pi k t)\,dt =$$

$$= \int_{-1}^{0}(1+t)\cos(\pi k t)\,dt + \int_0^1 (1-t)\cos(\pi k t)\,dt =$$

$$= \int_{-1}^{1} \cos(\pi k t)\,dt + \int_{-1}^{0} \underset{\substack{\downarrow \\ 1}}{t} \underset{\substack{\uparrow \\ \frac{\sin(\pi k t)}{\pi k}}}{\cos(\pi k t)}\,dt - \int_{0}^{1} \underset{\substack{\downarrow \\ 1}}{t} \underset{\substack{\uparrow \\ \frac{\sin(\pi k t)}{\pi k}}}{\cos(\pi k t)}\,dt = \qquad k \geq 1$$

$$= \left[\frac{\sin(\pi k t)}{\pi k}\right]_{-1}^{1} + \left[t\frac{\sin(\pi k t)}{\pi k}\right]_{-1}^{0} - \left[\frac{-\cos(\pi k t)}{\pi^2 k^2}\right]_{-1}^{0} - \left[\frac{t\sin(\pi k t)}{\pi k}\right]_{0}^{1} + \left[\frac{-\cos(\pi k t)}{\pi^2 k^2}\right]_{0}^{1}$$

$$= \underbrace{\frac{\sin(\pi k)}{\pi k}}_{=0} - \underbrace{\frac{\sin(-\pi k)}{\pi k}}_{=0} - \underbrace{\frac{\sin(-\pi k)}{\pi k}}_{=0} + \frac{1}{\pi^2 k^2} - \frac{\overbrace{\cos(-\pi k)}^{=(-1)^k}}{\pi^2 k^2} - \underbrace{\frac{\sin(\pi k)}{\pi k}}_{=0} - \frac{\overbrace{\cos(\pi k)}^{=(-1)^k}}{\pi^2 k^2} + \frac{1}{\pi^2 k^2}$$

$$= \frac{2}{\pi^2 k^2} - \frac{2(-1)^k}{\pi^2 k^2} = \frac{2}{\pi^2 k^2}\left(1-(-1)^k\right)$$

$$a_0 = \int_{-1}^{1} f(t)\,dt = \int_{-1}^{0}(1+t)\,dt + \int_{0}^{1}(1-t)\,dt = \left[t+\frac{t^2}{2}\right]_{-1}^{0} + \left[t-\frac{t^2}{2}\right]_{0}^{1} =$$

$$= 1-\frac{1}{2}+1-\frac{1}{2} = 1$$

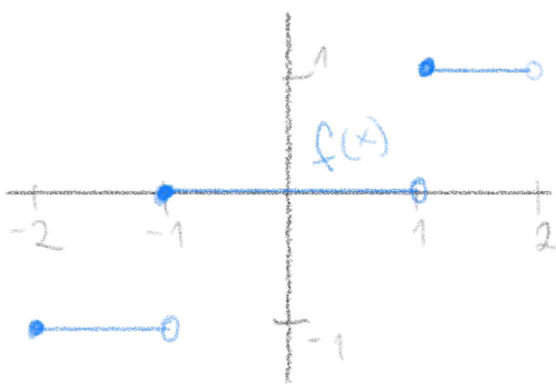
Výsledek zapíšeme podle definice Fourierovy řady

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{2k\pi t}{T}\right) + b_k \sin\left(\frac{2k\pi t}{T}\right)\right) =$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2}\left(1-(-1)^k\right)\cdot\cos(k\pi t) =$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2}\left(1+(-1)^{k+1}\right)\cdot\cos(k\pi t).$$

3) Nalezněte Fourierovu řadu funkce zadané grafem



Dále nalezněte součet Fourierovy řady na intervalu $[6,10)$.

Řešení

$T = 4$, funkce je lichá. Pak

$$a_k = \frac{2}{T}\int_a^{a+T} f(t)\cos\left(\frac{2\pi k t}{T}\right)dt =$$

$$= \frac{1}{2}\int_{-2}^{2} \underbrace{\underbrace{f(t)}_{\text{lichá}}\,\underbrace{\cos\left(\frac{\pi k t}{2}\right)}_{\text{sudá}}}_{\text{lichá}} = 0\,,\quad k \in \mathbb{N}_0$$

$$b_k = \frac{2}{T}\int_a^{a+T} f(t)\sin\left(\frac{2\pi k t}{T}\right)dt = \frac{1}{2}\int_{-2}^{2} f(t)\sin\left(\frac{\pi k t}{2}\right)dt =$$

$$= \frac{1}{2}\left(\int_{-2}^{-1}(-1)\sin\left(\frac{\pi k t}{2}\right)dt + \int_{-1}^{1} 0\,dt + \int_{1}^{2} 1\cdot\sin\left(\frac{\pi k t}{2}\right)dt\right) =$$

$$= \frac{1}{2}\left(\left[\frac{\cos\left(\frac{\pi k t}{2}\right)}{\frac{\pi k}{2}}\right]_{-2}^{-1} + \left[-\frac{\cos\left(\frac{\pi k t}{2}\right)}{\frac{\pi k}{2}}\right]_{1}^{2}\right) =$$

$$= \frac{1}{\pi k}\left(+\underbrace{\cos\left(-\frac{\pi k}{2}\right)}_{=\cos\left(\frac{\pi k}{2}\right)} - \underbrace{\cos(-\pi k)}_{=\cos(\pi k) = -(-1)^k} - \cos(\pi k) + \cos\left(\frac{\pi k}{2}\right)\right) =$$

$$= \frac{1}{\pi k}\left(2\cos\left(\frac{\pi k}{2}\right) - 2\cos(\pi k)\right) = \frac{2\cos\left(\frac{\pi k}{2}\right) - 2(-1)^k}{\pi k}$$

Výsledek zapíšeme podle definice Fourierovy řady

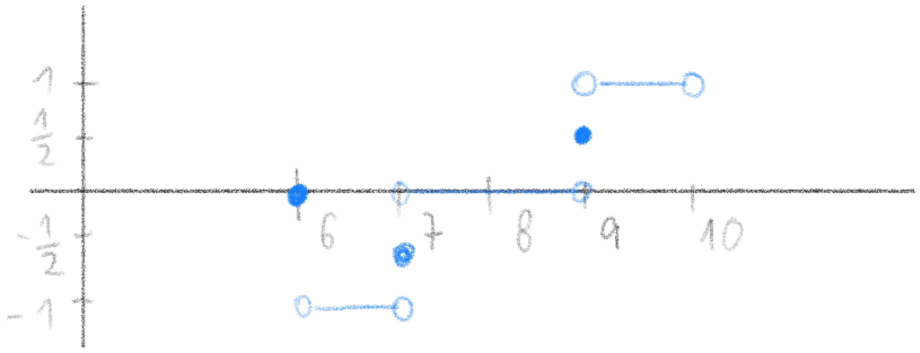
$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{2k\pi t}{T}\right) + b_k\sin\left(\frac{2k\pi t}{T}\right)\right) =$$

$$= \sum_{k=1}^{\infty}\frac{2\cos\left(\frac{\pi k}{2}\right) - 2\cdot(-1)^k}{\pi k}\cdot\sin\left(\frac{k\pi t}{2}\right)$$

Řada navíc konverguje na $\mathbb{R}$, neboť f i f' jsou po částech spojité na $[-2,2)$ a tedy

$$\mathcal{F}_f(t) = \sum_{k=1}^{\infty} \frac{2\cos\left(\frac{\pi k}{2}\right) - 2\cdot(-1)^k}{\pi k} \cdot \sin\left(\frac{k\pi t}{2}\right)$$

Na intervalu $[6,10)$ pak



$$\mathcal{F}_t(t) = \begin{cases} 0, & t = 6, \\ -1, & t \in (6,7), \\ -\frac{1}{2}, & t = 7, \\ 0, & t \in (7,9), \\ \frac{1}{2}, & t \in 9, \\ 1, & t \in (9,10). \end{cases}$$