A SHORT PROOF OF GALVIN'S THEOREM ABOUT GRAPHS ON ω_2

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ABSTRACT. We provide a short proof of Galvin's theorem: $\omega_2 \to (\omega_1, \omega + 2)$.

1. Introduction

We will prove a result by Galvin about graphs on ω_2 without uncountable independent sets, see [2, page 271]. We will use two well-known theorems, see [1] and [3].

Theorem 1.1 (Dushnik and Miller). *If κ is an uncountable cardinal, then $\kappa \to (\kappa, \omega + 1)$.*

Theorem 1.2 (Fodor's lemma). *Suppose κ is a regular uncountable cardinal and $S \subseteq \kappa$ is a stationary set. If $f : S \to \kappa$ is such that $f(\alpha) < \alpha$ for each $\alpha \in S$, then there is a stationary set $T \subseteq S$ and a $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for each $\alpha \in T$.*

2. The Proof

The result will follow from the following observation.

Lemma 2.1. *If κ is a regular cardinal, then $\kappa^+ \to (\kappa^+, \kappa : 1)$.*

Proof. Suppose a graph on κ^+ is given with no configuration $(\kappa : 1)$. Consider the set $S := \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$ and note that it is stationary. Now since there is no $(\kappa : 1)$ configuration we can define a function f so that for each $\alpha \in S$ we have $f(\alpha) < \alpha$ and for each $\gamma \in (f(\alpha), \alpha)$ there is no edge between γ and α. Hence by Fodor's lemma there is a stationary set $T \subseteq S$ and a fixed $\gamma < \kappa^+$ such that for all $\alpha \in T$ we have $f(\alpha) = \gamma$. Now the set $T \setminus \gamma$ is stationary and independent. Take $\alpha < \beta$ in $T \setminus \gamma$ now as $f(\beta) = \gamma < \alpha$ there can be no edge between α and β. □

Proposition 2.2. *Suppose κ is an uncountable regular cardinal and $\gamma \leq \kappa$. If $\kappa \to (\kappa, \gamma)$, then $\kappa^+ \to (\kappa, \gamma + 1)$.*

Proof. Given a graph on κ^+ with no independent set of size κ, from the previous lemma find a $(\kappa : 1)$ configuration and consider the subgraph on the first κ vertices, as this cannot be independent using the assumption that $\kappa \to (\kappa, \gamma)$ we obtain a complete subgraph of type γ and with the last vertex of the configuration this yields a $\gamma + 1$ subgraph. □

Corollary 2.3 (Galvin). $\omega_2 \to (\omega_1, \omega + 2)$.

Proof. Apply the previous proposition with $\kappa = \omega_1$ using the Dushnik-Miller theorem. □

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References

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[2] P. Erdős and A. Hajnal. Unsolved and solved problems in set theory. *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pp. 269–287. *Amer. Math. Soc., Providence, R.I.*, 1974.

[3] G. Fodor. Eine Bemerkung zur Theorie der regressiven Funktionen. *Acta Sci. Math. (Szeged)*, 17:139–142, 1956.

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