SUSLIN TREE AND $\omega_1 \to (\omega_1, \omega : 2)$

DÁVID UHRIK

Abstract. We give a self contained proof of the consistency of the existence of a Suslin tree and the partition relation $\omega_1 \to (\omega_1, \omega : 2)$, this result was proved by Todorcevic and Raghavan.

1. Introduction

It was Komjáth [1] who first proved the following.

Theorem 1.1 (Komjáth)**.** *It is consistent that* $\omega_1 \not\to (\omega_1, \omega+2)$ *while* $\omega_1 \to (\omega_1, \omega : 2)$.

Later Todorcevic and Raghavan [2] proved that the existence of a Suslin tree refutes the relation $\omega_1 \to (\omega_1, \omega + 2)$.

Theorem 1.2 (Todorcevic and Raghavan)**.** *If a Suslin tree exists, then* $\omega_1 \not\to (\omega_1, \omega + 2)$.

Here we provide a self contained proof that the relation $\omega_1 \to (\omega_1, \omega : 2)$ can be forced while also having a Suslin tree, this is also shown in [3]. This result also implies Komjáth's theorem.

SH denotes the Suslin hypothesis, i.e. $\neg\mathsf{SH}$ means that a Suslin tree exists.

2. The proof

First we need two lemmas.

Lemma 2.1. *Finite support iteration of productively ccc forcings is productively ccc*

Proof. We will proceed by induction. For $\alpha = 1$ there is nothing to prove.

Suppose we are at stage α and P_α is productively ccc and $\Vdash_{P_\alpha} \dot{Q}_\alpha$ is productively ccc. Put $P_{\alpha+1} := P_\alpha * \dot{Q}_\alpha$, we want to show that for any ccc poset R we have $P_{\alpha+1} \times R$ is ccc or equivalently that $P_{\alpha+1}$ forces $\check{R}$ to be ccc. The poset P_α is productively ccc so in V^{P_α} we have that R is ccc and Q_α is productively ccc hence in V^{P_α} it is true that $Q_\alpha \Vdash \check{R}$ is ccc, so $P_{\alpha+1}$ forces that $\check{R}$ is ccc.

Now let α be limit, suppose that for all $\beta < \alpha$ P_β is productively ccc, we will show that P_α is also productively ccc. Take R an arbitrary ccc poset, we want R to force that $\check{P}_\alpha$ is ccc. Now since we deal with finite support iteration it is true in V^R that P_α is the finite support iteration of the P_β's which stay ccc, hence P_α is ccc. □

Lemma 2.2. *Suppose* $V \models \neg\mathsf{SH}$ *and* P *is productively ccc. Then* $V^P \models \neg\mathsf{SH}$.

Date: September 5, 2019.

Proof. We will prove that if P and $P \times P$ are ccc then forcing with P does not add branches to ω_1-trees and since P is productively ccc we have that if S is a Suslin tree, then $P \Vdash \check{S}$ is ccc, i.e. S does not have an uncountable antichain after forcing with P.

Let T be an ω_1-tree in the ground model and suppose, for contradiction, that P forces a branch C through T. Inductively build an ω_1 sequence of pairs of conditions $((p_\alpha, q_\alpha) : \alpha < \omega_1)$ such that both p_α and q_α decide C up to level α in the same way but are incompatible otherwise. Now using the fact that $P \times P$ is ccc we find $\alpha < \beta < \omega_1$ such that (p_α, q_α) is compatible with (p_β, q_β), which is a contradiction because either p_β is compatible with p_α in which case q_β is incompatible with q_α since q_β decides C the same way as p_β up to β or vice versa bot not both at the same time. □

Proposition 2.3. $\omega_1 \to (\omega_1, \omega : 2) + \neg\mathsf{SH}$ *is consistent with* ZFC.

Proof. First, given a partition $[\omega_1]^2 = K_0 \cup K_1$ such that $(\omega : 2) \not\subseteq K_1$ we can consider the following poset adding an uncountable 0-homogeneous set.

$$P := \left\{p \in [\omega_1]^{<\omega} : [p]^2 \subseteq K_0\right\}$$

Claim. *P is productively ccc.*

Proof. Since the condition $(\omega : 2) \not\subseteq K_1$ is absolute, for any poset Q we have $V^Q \models (\omega : 2) \not\subseteq K_1$. Thus it is enough to show that $(\omega : 2) \not\subseteq K_1$ implies P is ccc.

Suppose $(p_\alpha : \alpha < \omega_1)$ is an uncountable antichain in P, note that two conditions p, q are incompatible if there exist $x \in p$ and $y \in q$ such that $\{x, y\} \in K_1$. Using the Δ-system lemma and standard counting arguments we may assume that:

(1) for any $\alpha, \beta < \omega_1$ we have $|p_\alpha| = |p_\beta|$,
(2) there exists an $r \in [\omega_1]^{<\omega}$ such that for any $\alpha < \beta < \omega_1$ we have $p_\alpha \cap p_\beta = r$ and $p_\alpha \setminus r < p_\beta \setminus r$.

Say that $p_\alpha \setminus r = \left\{p_\alpha^0, \dots, p_\alpha^{k-1}\right\}$. Consider the conditions $\{p_n : n < \omega\}$ and take a uniform ultrafilter $\mathcal{U}$ on ω. Now for any $n < \omega$ and $\omega \le \xi < \omega_1$ we can associate indices $i, j < k$ such that the following holds $\left\{p_n^i, p_\xi^j\right\} \in K_1$, put $U_{\xi,i,j} := \left\{n < \omega : \left\{p_n^i, p_\xi^j\right\} \in K_1\right\}$ and note that there are always indices $i_\xi, j_\xi < k$ such that $U_{\xi,i_\xi,j_\xi} \in \mathcal{U}$. So fix arbitrary $\xi_0, \xi_1 > \delta$ such that there are fixed $i, j < k$ with both $U_{\xi_0,i,j} \in \mathcal{U}$ and $U_{\xi_1,i,j} \in \mathcal{U}$ (there are uncountably many such ξ). Now we have that $U := U_{\xi_0,i,j} \cap U_{\xi_1,i,j}$ is infinite and the two sets $\left\{p_n^i : n \in U\right\}$ and $\left\{p_{\xi_0}^j, p_{\xi_1}^j\right\}$ witness that $(\omega : 2) \subseteq K_1$. □

To ensure that the generic filter is indeed uncountable we may need to get rid of certain "bad" conditions. To this end, call a condition p **bad** if the set $\{\alpha < \omega_1 : p \cup \{\alpha\} \in P\}$ is bounded in ω_1 and let b_p be its supremum, we want a $\delta < \omega_1$ such that no condition above δ is bad. Assume, for the sake of contradiction, that there is no such δ. We will construct an uncountable antichain $(p_\alpha : \alpha < \omega_1)$. Choose p_0 an arbitrary bad condition, suppose we have defined $(p_\alpha : \alpha < \beta)$ for some $\beta < \omega_1$, put $\gamma := \sup_{\alpha<\beta} b_{p_\alpha}$ and choose a bad condition p_β above γ, hence for each $\alpha < \beta$ there is an $x \in p_\alpha$ and $y \in p_\beta$ with $\{x, y\} \in K_1$, thus we get a contradiction.

Next, starting with a model of GCH and in particular of ¬SH we will iterate this construction in length ω_2 to consider every partition $[\omega_1]^2 = K_0 \cup K_1$. First we fix a bookkeeping function $g : \omega_2 \to \omega_2 \times \omega_2$ and then define a finite support iteration $P_{\omega_2} = (Q_\alpha : \alpha < \omega_2)$, in each step $\alpha < \omega_2$ we fix a bijection $A_\alpha : \omega_2 \to \wp([\omega_1]^2)$. Let $\alpha < \omega_2$ be arbitrary and suppose $g(\alpha) = (\xi, \eta)$, we consider the partition $A_\xi(\eta) \cup ([\omega_1]^2 \setminus A_\xi(\eta))$. Either $(\omega : 2) \subseteq [\omega_1]^2 \setminus A_\xi(\eta)$ in which case Q_α will be the trivial forcing or else we force with P with the appropriate partitions. Using Lemma 2.1 and 2.2 we are done. □

Remark. From the proof it is obvious that we can also have a Suslin tree and $\omega_1 \to (\omega_1, \omega : n)$ for any $n < \omega$.

References

[1] P. Komjáth. Distinguishing two partition properties of ω_1. *Fund. Math.*, 155(1):95–99, 1998.
[2] S. Todorcevic and D. Raghavan. Combinatorial dichotomies and cardinal invariants. *Math. Res. Lett.*, 21(2):379–401, 2014.
[3] S. Todorcevic and D. Raghavan. Suslin trees, the bounding number, and partition relations. *Israel J. Math.*, 225(2):771–796, 2018.

Email address: uhrik@karlin.mff.cuni.cz