THE UNCOUNTABLE HADWIGER CONJECTURE AND CHARACTERIZATIONS OF TREES USING GRAPHS

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ABSTRACT. We prove that the existence of a non-special tree of size λ is equivalent to the existence of an uncountably chromatic graph with no K_{ω_1} minor of size λ, establishing a connection between the special tree number and the uncountable Hadwiger conjecture. Also characterizations of Aronszajn, Kurepa and Suslin trees using graphs are deduced. A new generalized notion of connectedness for graphs is introduced using which we are able to characterize weakly compact cardinals.

1. INTRODUCTION

The Hadwiger conjecture is a deep unsolved problem in finite graph theory with far-reaching consequences. It states that if G is a simple finite graph and the chromatic number of G is t, then the complete graph on t vertices is a minor of G. We are interested in generalizations of this conjecture to uncountable graphs.

Recently there has been some interest in the conjecture for infinite graphs. In [9] van der Zypen proved that there is a countable connected graph whose chromatic number is ω, but K_ω is not a minor of this graph, i.e. the straightforward generalization of the Hadwiger conjecture to infinite graphs fails. The proof can be generalized to limit cardinals. Later Komjáth [2] showed that the Hadwiger conjecture fails for every infinite cardinal κ, he proved that:

Theorem 1.1 (Komjáth [2]). *If κ is an infinite cardinal, then there is a graph of cardinality 2^κ, chromatic number κ^+, with no K_{κ^+} minor.*

Thus the Hadwiger conjecture fails also for uncountable graphs if there is no bound on the size of the witness. If we only consider graphs of size ω_1, the conjecture may hold.

Theorem 1.2 (Komjáth [2]). *If MA_{ω_1} holds, then every graph G with $|G| = \chi(G) = \omega_1$ contains a subdivision of K_{ω_1}.*

The proof of Theorem 1.1 uses a non-κ-special tree of size 2^κ (such a tree always exists) to construct a graph with the desired properties. On the other hand, starting with a graph a tree can be obtained, using a specific construction by Brochet and Diestel [1], which reflects many useful properties of the graph. Theorems 1.1 and 1.2 give rise to a cardinal invariant.

2020 *Mathematics Subject Classification.* 03E17, 03E35, 05C05, 05C63.

Key words and phrases. uncountable Hadwiger conjecture, special tree, Suslin tree, uncountable graphs.

This work has been supported by Charles University Research Center program No.UNCE/SCI/022.

Definition 1.3. The *uncountable Hadwiger conjecture number*, denoted $\mathfrak{hc}$, is the least size of an uncountably chromatic graph with no K_{ω_1} minor.

In what follows we will show that $\mathfrak{hc}$ is equal to the special tree number. For more information on the special tree number and its history see [8].

Definition 1.4. The *special tree number*, denoted $\mathfrak{st}$, is the least size of a non-special tree with no uncountable branch.

What Komjáth essentially proved in [2, Theorem 2] for the case of $\kappa = \omega$ can be succinctly written as:

Theorem 1.5 (Komjáth [2])**.** $\mathfrak{hc} \leq \mathfrak{st}$.

In this note we prove the other inequality, thus establishing that in fact $\mathfrak{hc} = \mathfrak{st}$, see Corollary 3.5.

Using the techniques in proving the previous theorem we also provide a characterization of Suslin trees in terms of graphs, proving that a Suslin tree exists if and only if there is a graph of size ω_1 with no uncountable independent set and no K_{ω_1} minor, see Theorem 4.2. This characterization has been independently discovered by P. Komjáth and S. Shelah in their recent paper [3].

In a similar fashion we define graph counterparts for Aronszajn and Kurepa trees using a generalized notion of connectedness for graphs. As a corollary we also deduce a new characterization of weakly compact cardinals using graphs.

Notation. We use standard set theoretic notation. A graph G is a pair (V, E), where V is an arbitrary set and $E \subseteq [V]^2$; the set V is called the vertex set and E the edge set of the graph. Unless otherwise specified G will always denote a graph whose vertex set is V and the edge set is E, in case of ambiguity we will use V_G and E_G instead. If κ is a cardinal, then K_κ denotes the complete graph on κ vertices, i.e. the graph $(\kappa, [\kappa]^2)$. Two subsets $X, Y \subseteq V$ are connected if there are vertices $x \in X$ and $y \in Y$ such that $\{x, y\} \in E$. A graph is connected if between any two vertices there is a path. A component of a graph is any maximal connected subgraph. A graph is κ-connected if it stays connected after removing $< \kappa$ many vertices. If X, Y, S are subsets of the vertex set we say that S separates X from Y in G if after removing S from the graph, $X \setminus S$ and $Y \setminus S$ lie in different components.

The chromatic number of a graph G, denoted $\chi(G)$, is the least cardinal μ such that there is a proper coloring of G with μ colors. A proper coloring is a function $c : V \to \mu$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$.

If G and H are graphs, we say that G is a minor of H if there are pairwise disjoint non-empty subsets $(X_u)_{u \in V_G}$ of V_H such that each induces a connected subgraph in H and for every $\{u, v\} \in E_G$, X_u and X_v are connected in H. Note also that being a minor is a transitive property. The graph H is a subdivision of G if H is constructed from G by replacing edges with paths. Note that if H is a subdivision of G, then G is a minor of H.

A tree is a poset $(T, \leq)$ such that each set of the form $\{s \in T \mid s < t\}$ for some $t \in T$ is well-ordered, the order type of this set is called the height of the node t, denoted as $\mathrm{ht}(t)$. For a tree the set T_α is the set of nodes of height α, sometimes referred to as level α of T, for α limit it will be useful to denote $T_{<\alpha}$, the set of nodes of height less than α. By $\mathrm{pred}(t)$ we denote the set of nodes in the tree which are below t in the tree order. By a branch in a tree we mean a maximal chain.

A tree is called κ-special if there is a function $f : T \to \kappa$ which is injective on chains. An ω-special tree will be simply called a special tree.

A κ-Suslin tree is a tree of size κ with no κ-branches and no antichains of size κ. A κ-Aronszajn tree is a tree of height κ with no κ-branch and levels of size less than κ. A κ-Kurepa tree is a tree of height κ with at least κ^+ different branches but levels of size less than κ. When $\kappa = \omega_1$ we omit the cardinal specification.

The comparability graph of a tree T is a graph whose vertex set is the domain of the tree and the edge set is $\big\{\{s,t\} \in [T]^2 \mid s \leq t \vee t \leq s\big\}$.

If T is a tree, a graph is called a T-graph if it is isomorphic to a graph whose vertex set is T, is a subgraph of the comparability graph of T and for every $t \in T$ the vertices connected to t are cofinal in $\{s \in T \mid s < t\}$, if this set has a maximum we want t to be connected to it.

Acknowledgement. The author is grateful to Chris Lambie-Hanson for constructive discussions on the topic which greatly improved the exposition of this paper.

2. Limit cardinals

First we show that the following result by van der Zypen can be easily generalized to cover every limit cardinal.

Theorem 2.1 (van der Zypen [9])**.** *There is a countable connected graph whose chromatic number is ω but K_ω is not a minor of this graph.*

Proposition 2.2. *Suppose κ is a limit cardinal. There exists a connected graph whose size and chromatic number is κ, but K_κ is not a minor of this graph.*

Proof. Suppose $(\kappa_\alpha \mid \alpha < \mu)$ is an increasing cofinal sequence of cardinals in κ, where μ may be equal to κ. Let the vertex set of the graph be the union $\{0\} \cup \{(\alpha, \beta) \mid \alpha \in \mu \wedge \beta \in \kappa_\alpha\}$. The edge set is defined as follows: connect the vertex 0 to every other vertex and for $\alpha < \mu$ connect (α, β) to (α, β') for every $\beta < \beta' < \kappa_\alpha$. In simple terms the graph is the disjoint union of complete graphs of increasing size, cofinal in κ, plus a vertex connected to every other vertex.

The chromatic number of this graph is clearly bounded by κ but also larger than κ_α for every $\alpha < \mu$ as the complete graph K_{κ_α} embeds into the graph. Thus the chromatic number is exactly κ.

Now assume $\{C_\gamma \mid \gamma < \kappa\}$ is a collection of pairwise disjoint connected subgraphs forming a K_κ minor.

Suppose first that 0 does not belong to any of the sets C_γ, then necessarily $\bigcup_{\gamma<\kappa} C_\gamma$ is a subset of some complete graph K_{κ_α}, this is of course a contradiction as $\kappa_\alpha < \kappa$ and C_γ contains at least one element.

So assume $0 \in C_{\gamma_0}$ for some $\gamma_0 < \kappa$. Now given any $\gamma \neq \gamma_0$ we must have that $C_\gamma \subseteq K_{\kappa_\alpha}$ for some κ_α otherwise they cannot be connected by an edge as 0 is in C_{γ_0} and it is the only vertex connecting the disjoint cliques. Hence we get a contradiction as in the previous case. □

Thus the Hadwiger conjecture fails unconditionally for ω and in general for every limit cardinal.

3. On the $\mathfrak{hc}$ number

We prove some basic results about $\mathfrak{hc}$ and show its connection to $\mathfrak{st}$.

Proposition 3.1. (1) $\omega_1 \leq \mathfrak{hc} \leq \mathfrak{st} \leq \mathfrak{c}$,
(2) MA *implies* $\mathfrak{hc} = \mathfrak{c}$,
(3) *for any* $\kappa > \omega_1$ *of uncountable cofinality it is consistent that* $\omega_1 = \mathfrak{hc} < \mathfrak{c} = \kappa$.

Proof. Clearly $\mathfrak{hc}$ cannot be countable as the graph must be uncountably chromatic. The upper bound comes from Theorem 1.5.

Assuming full MA in the proof of Theorem 1.2 we obtain that in any uncountably chromatic graph of size $< \mathfrak{c}$ a subdivision of K_{ω_1} can be found.

For (3) consider the comparability graph of a Suslin tree. This defines a graph of size ω_1 with no uncountable independent set as independent sets in comparability graphs clearly translate to antichains in the tree, hence the graph cannot be countably chromatic. The graph also has no K_{ω_1} minor as this would define an uncountable branch, see [2, Theorem 2] for a proof. As is well known [4, Corollary V.4.14] the continuum can be arbitrarily large while a Suslin tree exists. □

We can say something about the cofinality of $\mathfrak{hc}$ as well.

Proposition 3.2. *The cofinality of* $\mathfrak{hc}$ *is uncountable.*

Proof. Suppose κ is a cardinal and $(\kappa_n \mid n \in \omega)$ is an increasing sequence of cardinals converging to κ. Assume that $\kappa_n < \mathfrak{hc}$ for all $n \in \omega$. We will show that κ is also strictly less than $\mathfrak{hc}$.

Take a graph G of size κ whose chromatic number is ω_1 and which has no K_{ω_1} minor. Partition the vertex set into countably many parts of increasing size corresponding to the cofinal sequence $(\kappa_n \mid n \in \omega)$ and consider the induced subgraphs $\{G_n \mid n \in \omega\}$. Now each graph has size less than $\mathfrak{hc}$, thus it is either countably chromatic or contains K_{ω_1} as a minor. As the entire graph has no K_{ω_1} minor the latter is impossible, so it must be the case that G_n is countably chromatic for every $n \in \omega$. Fix the countable colorings $c_n : G_n \to \omega$, we will show there is a countable coloring of the entire G. Define $c : G \to \omega \times \omega$ as $c(v) = (n, k)$ if $v \in G_n$ and $c_n(v) = k$. Now if $\{u, v\}$ is an edge in G and $c(u) = c(v)$, then by the equality in the first coordinate both vertices are part of the same G_n and by the equality in the second coordinate $c_n(u) = c_n(v)$. However, since these were induced subgraphs the edge $\{u, v\}$ is present in G_n so c_n is not a good coloring, a contradiction. □

Before proving the main theorem we need a result about T-graphs. The proof can be found in [7, §2].

Proposition 3.3. *Suppose* κ *is an uncountable regular cardinal. If* G *is a* T*-graph and* T *has a* κ*-branch, then* K_κ *is a minor of* G.

Our main result about the uncountable Hadwiger cojecture number will follow from the following more general theorem.

Theorem 3.4. *Suppose* κ *and* λ *are infinite cardinals. The existence of a graph of size* λ *whose chromatic number is* κ^+ *and has no* K_{κ^+} *minor is equivalent to the existence of a non-*κ*-special tree of size* λ *with no branches of length* κ^+.

Proof. Given a non-κ-special tree of size λ with no branches of length κ^+ the proof of [2, Theorem 2] shows that the comparability graph of this tree has chromatic number κ^+ and has no K_{κ^+} minor, it is clear that the size of the comparability graph is the same as the size of the tree it is constructed from.

In the other direction assume G is a graph with the aforementioned properties. We can assume that G is connected as G must have a component with the required properties and also that the vertex set of G is λ. Note also that $\lambda \geq \kappa^+$. Using a construction by Brochet and Diestel, [1, Theorem 4.2.], we will build a tree T which will have the required properties. For completeness we reproduce the construction.

The construction is by induction. We will construct a tree level by level. To each node $t \in T$ we will associate two subsets $V_t \subseteq C_t$ of V both inducing connected subgraphs. In the end the graph obtained by contracting the sets V_t to a point will be a T-graph. For every α the following conditions will be satisfied:

(1) if $\alpha = \beta + 1$ and $t \in T_\alpha$ and s is the predecessor of t in T_β, then C_t is a component of $C_s \setminus V_s$ and $V_t = \{x_t\}$ such that $x_t \in C_t$ and x_t is connected to some element in V_s; also if C is a component of $C_s \setminus V_s$, then there is a unique node $r \in T_\alpha$ above s, such that $C = C_r$,

(2) if α is limit and $t \in T_\alpha$, then C_t is a component of $\bigcap_{s \in \mathrm{pred}(t)} C_s$ and V_t induces a non-empty connected subgraph of C_t such that the following set $\{s \in \mathrm{pred}(t) \mid V_s \text{ is connected to } V_t \text{ in } G\}$ is cofinal in $\mathrm{pred}(t)$; also if C is a component of $\bigcap_{s \in \mathrm{pred}(t)} C_s$, then there is a unique node $r \in T_\alpha$ above $\mathrm{pred}(t)$, such that $C = C_r$,

(3) if $t \in T_\alpha$, then $s \in \mathrm{pred}(t)$ if and only if $C_t \subsetneq C_s$.

The first step is to define the root $T_0 := \{t_0\}$ and $V_{t_0} := \{0\}$, where $0 \in \lambda = V$ and $C_{t_0} := V$. We now continue the construction by levels.

If we are at a successor stage, i.e. we have defined the tree up to level $\alpha + 1$ we do the following: for every node t in T_α consider the graph induced by $C_t \setminus V_t$ and let $\{C_i \mid i < \mu(t)\}$ be all of its components. In $T_{\alpha+1}$ the node t will have $\mu(t)$ many successors $\{t^i \mid i < \mu(t)\}$ and we put $C_{t^i} := C_i$. Let y be the least element of C_{t^i} (we assumed $V = \lambda$) and put $V_{t^i} := \{x_t^i\}$, where $x_t^i \in C_{t^i}$ is a vertex connected to V_t whose distance to y is minimal. Note that if $C_t \setminus V_t$ is empty, t will have no successors in T.

At limit stages we proceed similarly. Suppose that we have defined the tree up to level α and α is a limit ordinal. We consider every branch b of $T_{<\alpha}$ and determine the set $\bigcap_{s \in b} C_s$, if it is empty b will be a branch in T as well, otherwise let $\{C_i \mid i < \mu(b)\}$ be all of its components. We put nodes $(b^i)_{i<\mu(b)}$ into T_α, all of them extending b and pairwise incomparable. Fix a $j < \mu(b)$. The set C_{b^j} is simply C_j again. It is enough now to define the set V_{b^j}.

Claim. *There is a subset V' of C_j inducing a connected subgraph in G of size at most $\mathrm{cf}|\alpha|$ such that the set of nodes $\{t \in b | V_t \text{ is connected to } V' \text{ in } G\}$ is cofinal in b.*

Proof. First we show that $\{t \in b | V_t \text{ is connected to } C_j \text{ in } G\}$ is cofinal in b. Suppose s is a node in b. Note that $C_j \subsetneq C_s$ so there must be vertices $u \in C_j$ and $v \in C_s \setminus C_j$ which are connected. As $v \notin C_j$ let $s' \in b$ be the least node such that $v \notin C_{s'}$, note that we have $u \in C_{s'}$.

First suppose $\mathrm{ht}(s')$ is a successor ordinal, and the predecessor of s' is $\overline{s}$. We obtain that $u, v \in C_{\overline{s}}$ but $v \notin C_{s'}$ and since u is connected to v in G we must have that $v \in V_{\overline{s}}$. Hence $V_{\overline{s}}$ is connected to C_j.

The case when the height of s' is limit we obtain that $u, v \in \bigcap_{r \in \mathrm{pred}(s')} C_r$. However since $\{u, v\}$ forms an edge both vertices must be contained in the same component of $\bigcap_{r \in \mathrm{pred}(s')} C_r$, i.e. v would have to be an element of $C_{s'}$.

Let $b' \subseteq b$ be a cofinal subset of the branch of size $\mathrm{cf}|\alpha|$ such that each $s \in b'$ has the property that V_s is connected to C_j. For each $s \in b'$ choose a witness $x^i_s \in C_j$ connected to V_s. Let V' be any subset of C_i inducing a connected graph of size at most $\mathrm{cf}|\alpha|$ containing all the vertices x^i_s. □

The claim defines the sets V_{b^i} and the construction is finished.

It is clear that in each step of the induction we use up at least one vertex of G that we put in some V_t, so the length of the induction is some ordinal δ such that $|\delta| = \lambda$. However note that at successor steps of the induction we always chose a vertex with minimal distance from the least vertex which was not part of some V_t defined before. This implies that the vertex $\alpha < \lambda$ was the least vertex at the latest in step $\omega \cdot \alpha$ of the induction and was put into some V_t after finitely many steps, i.e. α belongs to some V_t for a t such that $\mathrm{ht}(t) < \omega \cdot \alpha + \omega$. From this we also obtain that the height of the constructed tree is at most λ.

We make a few observations. The sets $(V_t)_{t \in T}$ form a partition of V and each V_t induces a connected subgraph of G. The graph (T, F), where

$$\{s, t\} \in F \;\equiv\; V_s \text{ is connected to } V_t \text{ in } G$$

is a T-graph and also a minor of G.

To see that it is a T-graph, note that the way we defined the sets V_t it is clear that each t is F-related to its predecessor and if the height of t is limit the previous claim implies that it is cofinally often connected to its predecessors.

Thus it is enough to see that (T, F) is a subgraph of the comparability graph of T. Consider any $u, v \in G$ which are connected and the corresponding V_s and V_t so that $u \in V_s, v \in V_t$. Suppose s is incomparable with t.

In the first case assume that there is a limit α and a branch b in $T_{<\alpha}$ such that there are nodes $\bar{s}$ and $\bar{t}$ directly above b such that $\bar{s} \le s$ and $\bar{t} \le t$. We have that $u, v \in \bigcap_{r \in b} C_r$ and since they form an edge both u and v belong to the same component of $\bigcap_{r \in b} C_r$ and by the second induction hypothesis we obtain that $\bar{s} = \bar{t}$.

The case when the split happens on a successor level is proven analogously using the first induction hypothesis.

The following claim finishes the proof.

Claim. *T is a non-κ-special tree without κ^+-branches of size at most λ.*

Proof. The size of T is obviously at most λ. Suppose T has a κ^+-branch, then by Proposition 3.3 K_{κ^+} is a minor of (T, F) thus by transitivity it is also a minor of G which is impossible. Thus all branches have size $\le \kappa$ and so each V_t has size $\le \kappa$.

If T were κ-special there would be a specializing function $f : T \to \kappa$. Let $\{A_\alpha \mid \alpha \in \kappa\}$ be pairwise disjoint κ-sized subsets of κ. For every $t \in T$ let $g_t : V_t \to A_{f(t)}$ be any injection and define a coloring $c : G \to \kappa$ as follows: for every $u \in G$ there is a unique $t \in T$ such that $u \in V_t$ thus let $c(u)$ be $g_t(u)$. To see that c is proper, consider any two vertices $u, v \in G$ which are connected. If both are in the same V_t then as g_t is injective they clearly get different colors, else there are different $s, t \in T$ so that $u \in V_s$ and $v \in V_t$, since (T, F) is a T-graph we get that $s < t$ or vice versa, then however, $f(s) \neq f(t)$ and subsequently $c(u) \in A_{f(s)}$ and $c(v) \in A_{f(t)}$ and again they get different colors. Thus the chromatic number of G is at most κ, a contradiction.

□

Remark. From now on if a graph G is given we will freely denote by T_G the tree arising from the previous construction without explicitly mentioning so.

Using this theorem we can easily deduce the relationship between $\mathfrak{hc}$ and $\mathfrak{st}$, thus the inequality in Theorem 1.5 can be reversed.

Corollary 3.5. $\mathfrak{hc} = \mathfrak{st}$.

Proof. We show that $\mathfrak{hc} \geq \mathfrak{st}$. Given a graph of size $\mathfrak{hc}$ with uncountable chromatic number and no K_{ω_1} minor the previous theorem implies that there is a non-special tree with no uncountable branch of size at most $\mathfrak{hc}$. □

The previous construction has many useful properties. Suppose κ is a regular cardinal. We have the following:

(1) if T_G has a κ-branch, then K_κ is a minor of G,
(2) if T_G is κ-special, then the chromatic number of G is at most κ,
(3) if T_G has a κ-sized antichain, then G has an independent set of size κ.

The first two items follow from Proposition 3.3 and the last claim in the main theorem. The last item will follow from the characterization in the next section.

We can further show that the implication in item (1) can be reversed but not in (2) nor (3).

Proposition 3.6. *Suppose κ is a regular cardinal. Let G be a graph, Then the following holds:*

(1) *if K_κ is a minor of G, then T_G has a branch of size at least κ,*
(2) *there is a countably chromatic graph G such that T_G has a κ-branch,*
(3) *there is a graph with an independent set of size κ such that T_G is isomorphic to $(\kappa, \in)$.*

Proof. For (1) suppose $\{U_\alpha \mid \alpha < \kappa\}$ are the disjoint sets of vertices forming a K_κ minor and assume also that there is no node in T_G at height κ, otherwise there clearly is a branch of length at least κ. For each α we define a node in the tree T_G, put $t_\alpha := \min\{t \in T_G \mid U_\alpha \cap V_t \neq \emptyset\}$ (the sets V_t are defined as in Theorem 3.4).

Claim. *For each α the node t_α is well-defined.*

Proof. Suppose s, t are incomparable such that $U_\alpha \cap V_s \neq \emptyset$ and also $U_\alpha \cap V_t \neq \emptyset$. Let x_s be an element of $U_\alpha \cap V_s$ and x_t an element of $U_\alpha \cap V_t$. As U_α is connected, there is a finite path $(x_i)_{i<n}$ in U_α such that $x_0 = x_s$ and $x_{n-1} = x_t$. To each x_i we van associate a node r_i such that $x_i \in V_{r_i}$ (this mapping need not be injective). Since x_i is connected to x_{i+1} we obtain that r_i is comparable with r_{i+1} for each $i < n-1$. Now there must exist some r_j such that $r_j \leq r_i$ for each $i \neq j$. We can prove this by induction. If $n \leq 3$ this is clear. Suppose $k < n$ and $(r_i)_{i<k}$ has a least element, say r_l. Now $r_l \leq r_{k-1}$ and r_k is comparable with r_{k-1}, hence r_k is comparable with r_l and the least element of $(r_i)_{i\leq k}$ is $\min\{r_k, r_l\}$.

Let r_j be the least element of $(r_i)_{i<n}$, then $r_j \leq s, t$ and $x_j \in V_{r_j} \cap U_\alpha$. □

We claim that for every $\alpha, \beta < \kappa$ the nodes t_α and t_β are comparable. If α and β are given consider the nodes $s_\alpha, s_\beta \in T_G$ such that $x \in U_\alpha$ and $y \in U_\beta$ are connected in G and $x \in V_{s_\alpha}$ and $y \in V_{s_\beta}$, so s_α is comparable with s_β. We then have that $t_\alpha, t_\beta \leq \max\{s_\alpha, s_\beta\}$, hence t_α and t_β are comparable. We are almost done but notice that the mapping $\alpha \mapsto t_\alpha$ need not be injective, however, as the sets $\{U_\alpha \mid \alpha < \kappa\}$ are pairwise disjoint and the sets V_t have size less than κ, this mapping has the property that the preimage of each node has size $< \kappa$ and so there must be κ many unique nodes t_α all comparable to each other forming a branch in T_G.

Now (2) is easy, just consider the complete graph on κ vertices and subdivide each edge once, this is clearly a bipartite graph, hence 2-colorable but from the previous part T_G will have a κ-branch.

The witness for (3) is simply the complete bipartite graph with both partitions of size κ, clearly, either partition forms an independent set of size κ. Note that this graph is κ-connected. Following the construction in Theorem 3.4 it is not hard to see that κ-connectivity implies that at no step of the construction does the graph split, i.e. the tree T_G is simply a single κ-branch. □

Let us conclude this section with an observation about the connectedness of the graphs which are counterexamples to the uncountable Hadwiger conjecture. The next proposition claims that a graph with no K_κ minor cannot be κ-connected.

Proposition 3.7. *Suppose κ is an infinite cardinal. If G is κ-connected, then G contains a subdivision of K_κ.*

Proof. We will inductively choose elements $\{v_\alpha \mid \alpha < \kappa\} \subseteq V$ and finite sequences of vertices $\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \kappa\}$ so that for each pair of vertices v_α, v_β the sequence $v_\alpha{}^\frown\mathbf{p}_{\alpha\beta}{}^\frown v_\beta$ is a path and the collection of all these $\mathbf{p}_{\alpha\beta}$'s is pairwise disjoint. Clearly this forms a subdivision of K_κ.

Suppose we have constructed $\{v_\alpha \mid \alpha < \gamma\}$ for some $\gamma < \kappa$ and we also have the collection of pairwise disjoint paths $\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \gamma\}$. Consider the subgraph of G induced by the vertices $V_\gamma := V \setminus \bigcup\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \gamma\}$. By our assumption this still induces a connected graph, choose any vertex v from this set different from any vertex included in $\{v_\alpha \mid \alpha < \gamma\}$, this will be the vertex v_γ.

By induction again we choose the finite sequences $\mathbf{p}_{\alpha\gamma}$. Suppose we have constructed $\{\mathbf{p}_{\beta\gamma} \mid \beta < \alpha\}$ for some $\alpha < \gamma$. As paths are finite we have that the set $V_\gamma \setminus \bigcup\{\mathbf{p}_{\beta\gamma} \mid \beta < \alpha\}$ still induces a connected graph as we assume G is κ-connected, so choose any path between v_α and v_γ using only these vertices excluding the so far chosen $\{v_\alpha \mid \alpha < \gamma\}$ and this will be our $\mathbf{p}_{\alpha\gamma}$. □

4. Suslin trees

Before we state the characterization of Suslin trees we mention a related result by Wagon. He introduced a graph counterpart to a Suslin tree whose existence is actually equivalent to the existence of a Suslin tree, in [10, Theorem 2.2.] he proved the following:

Theorem 4.1 (Wagon [10])**.** *For an infinite cardinal κ, the following are equivalent:*

1. *a κ^+-Suslin tree exists,*
2. *there exists a triangulated graph G with $\alpha(G) = \kappa < \theta(G)$.*

The number $\alpha(G)$ denotes the supremum of the sizes of independent sets of G and $\theta(G)$ the least size of a family of cliques which cover the graph. A graph is triangulated if every induced cycle is a triangle.

In our characterization we drop the triangularity requirement and instead of considering the invariant $\theta(G)$ we forbid a specific minor. This result has been independently proven by P. Komjáth and S. Shelah [3].

Theorem 4.2. *Suppose κ is a regular cardinal. The existence of a κ-Suslin tree is equivalent to the existence of a graph of size κ, which has no independent set of size κ and has no K_κ minor.*

Proof. Given a κ-Suslin tree the comparability graph has the desired properties. It clearly has no independent set of size κ as this would translate to a κ sized antichain in the tree and a K_κ minor translates to a κ-branch, for details see [2, Theorem 2].

In the other direction we proceed as before. Given a graph G of size κ with no independent set of size κ and no K_κ minor we construct the tree T_G, the size of the tree is κ. Clearly the fact that the graph has no K_κ minor again translates to the fact that the tree has no κ-branch.

To see that T_G has no antichain of size κ we proceed by contradiction. Suppose $\{t_i \mid i \in \kappa\}$ is an antichain in T_G and consider the sets $\{V_{t_i} \mid i \in \kappa\}$. We showed that if s is incomparable with t then there is no edge connecting the sets V_s and V_t, thus choose any $x_i \in V_{t_i}$ as all of these are non-empty, now it is clear that $\{x_i \mid i \in \kappa\}$ forms an independent set in G, a contradiction. □

5. Narrow and Kurepa trees

In this section we will try to characterize Aronszajn and Kurepa trees similarly to what has been done in the previous section. First we introduce a generalized notion of connectedness for graphs.

Definition 5.1. Suppose κ, λ are infinite cardinals. A graph is (κ, λ)-connected if after the removal of less than κ vertices the number of components is non-zero and less than λ.

Remark. Note that the proof of Proposition 2.2 shows that if κ is singular, then the constructed graphs are (κ, κ)-connected; this is not the case for regular κ.

Evidently, a graph is κ-connected when it is $(\kappa, 2)$-connected. This notion aims to stratify the property of connectedness for graphs. Typical examples of (κ, λ)-connected graphs come from trees and this notion is closely related to their width, e.g. the comparability graph of an Aronszajn tree is (ω_1, ω_1)-connected.

Proposition 5.2. *Suppose κ is an infinite cardinal. If G is a graph of size at least κ and G has no independent set of size κ, then G is (κ, κ)-connected.*

Proof. Suppose G is not (κ, κ)-connected, then there exists a set $X \subseteq V$ of size less than κ such that if X is removed from G, the graph splits into at least κ many components, $\{C_\gamma \mid \gamma < \kappa\}$. From each component choose a vertex, $x_\gamma \in C_\gamma$. Now $\{x_\gamma \mid \gamma < \kappa\}$ forms an independent set in G. □

Proposition 5.3. *Suppose $\kappa \geq \lambda$ are regular cardinals. If G is (κ, λ)-connected graph, then $(T_G)_{<\kappa}$ has levels of size $< \lambda$.*

Proof. We will proceed by induction, clearly by our assumption the set of roots of T_G has size $< \lambda$. Let $\alpha < \kappa$ and consider the level $(T_G)_\alpha$. Take the union $\bigcup_{t \in (T_G)_{<\alpha}} V_t$ (for definition of V_t see Theorem 3.4) and note that by construction of T_G and the induction hypothesis this set has size $< \kappa$ so removing these vertices from G leaves us with less than λ many components, so at stage α in the construction of T_G there are less than λ many components to consider and hence less than λ many nodes to extend $(T_G)_{<\alpha}$. □

Corollary 5.4. *Suppose κ is a regular cardinal. If there exists a cardinal $\lambda < \kappa$ such that G is (κ, λ)-connected, then K_κ is a minor of G.*

Proof. Given G with these properties consider the tree $(T_G)_{<\kappa}$. The size of the levels of this tree is less than λ but the size of the entire tree is κ. By a result of Kurepa [5] each tree of height κ whose levels have size less than λ has a cofinal branch, hence T_G has a branch of size at least κ so by Proposition 3.3 G has a K_κ minor. □

Considering the basic properties of T_G together with the last proposition we have the following.

Theorem 5.5. *Suppose κ is a regular cardinal. The existence of a κ-Aronszajn tree is equivalent to the existence of a (κ, κ)-connected graph of size κ, which has no K_κ minor.*

Proof. Let T be a κ-Aronszajn tree. We show that the comparability graph G_T of this tree has the desired properties. Note that removing less than κ many vertices from G_T we can assume all of them lie below some level α of T and hence if they are deleted we end up with less than κ many connected subgraphs as the levels of T have size less than κ. For the second property see the proof of [2, Theorem 2] using the fact that T has no κ-branch. We can also prove the converse. If G_T had the necessary properties then T would be κ-Aronszajn to begin with. Clearly, a κ-branch in T would imply the complete graph K_κ is a subgraph of G_T and if T had levels of size κ consider the first level $\alpha < \kappa$ which has κ many nodes, then removing those vertices from G_T which correspond to nodes in T of height less than α (there are less than κ many of those) we would end up with κ many disjoint connected components of G_T which is a contradiction as we assumed it is (κ, κ)-connected.

If G is (κ, κ)-connected of size κ with no K_κ minor, then by the previous proposition T_G has levels of size less than κ and by the properties of the construction of T_G, it can have no κ-branch as this would imply K_κ is a minor of G; from this we also have that T_G has height κ as G must have size κ as it is (κ, κ)-connected, hence T_G has size κ. We can prove the converse here as well, T_G being κ-Aronszajn implies that G is (κ, κ)-connected and has no K_κ minor. By Proposition 3.6 we get that since T_G has no cofinal branch, then G cannot have a K_κ minor, also if less than κ many vertices are removed from G, then there is a level $\alpha < \kappa$ such that all of these vertices are contained in the sets V_t for $t \in (T_G)_{<\alpha}$, however, as T_G has levels of size less than κ then removing all of these vertices leaves us with less than κ many cones in T_G all of which define connected subgraphs of G. □

Corollary 5.6. *Suppose κ is an inaccessible cardinal. The following are equivalent:*

(1) *κ is weakly compact,*
(2) *each (κ, κ)-connected graph has a K_κ minor.*

For a more succinct characterization of Kurepa trees we define a so called *Kurepa minor family* in a graph.

Definition 5.7. Suppose κ and λ are infinite cardinals. Let G be a graph and $\{W_\alpha \mid \alpha < \lambda\}$ a collection of K_κ minors of G. We say that $\{W_\alpha \mid \alpha < \lambda\}$ forms a *κ-Kurepa minor family* of size λ if for each α and β a set of size less than κ separates W_α from W_β in G.

Theorem 5.8. *Suppose κ is a regular cardinal. The existence of a κ-Kurepa tree is equivalent to the existence of a (κ, κ)-connected graph of size κ, which has a κ-Kurepa minor family of size at least κ^+.*

Proof. If T is a κ-Kurepa tree then the comparability graph of T clearly has all the necessary properties.

On the other hand given such a graph the tree T_G is a κ-Kurepa tree. From the construction we obtain that the height of T_G is at most κ. The size of the levels is less than κ by Proposition 5.3.

As for the branches, we get from Proposition 3.6 that each K_κ minor defines a κ-branch in T_G. It is enough to observe that since G has a κ-Kurepa minor family of size κ^+ we get that each pair of branches coming from this family is eventually different. Since each pair of minors is separated by a set, X, of size less than κ we obtain that there is some $\alpha < \kappa$ such that the set of nodes t with the property that V_t intersects X lie in $(T_G)_{<\alpha}$, this implies that the branches defined from the minors are indeed different. □

6. The κ-Hadwiger conjecture

Theorem 3.4 was used to prove that $\mathfrak{hc} = \mathfrak{st}$ but it also shows us where to look for models of the Hadwiger conjecture on higher cardinals.

Definition 6.1. The κ-Hadwiger conjecture states that every graph of size κ whose chromatic number is κ has a K_κ minor.

The case when $\kappa = \omega_1$ consistently holds, see Theorem 1.2, and by our result is equivalent to $\mathfrak{st} > \omega_1$. For κ limit the conjecture always fails as shown in Proposition 2.2. The generalized continuum hypothesis implies that the κ-Hadwiger conjecture fails for every infinite κ, this follows from Theorem 1.1.

Using the technique of Laver and Shelah (see the closing remarks in [6]) we get a model where each tree with no κ^+-branch of size κ^+ is κ-special, except possibly at successors of singular cardinals. By Theorem 3.4 this also models the κ^+-Hadwiger conjecture.

Theorem 6.2. *Suppose κ is a regular cardinal. It is consistent that the κ^+-Hadwiger conjecture holds.*

7. Closing remarks

As far as the author is concerned the construction of Brochet and Diestel [1, Theorem 4.2.] is not very well known in the set theory community and the question is how far can we push graph properties onto trees and vice versa. We have seen that chromaticity, having a large minor, independent sets and notions of connectedness all translate to properties of the tree. On the other hand the simple construction of a comparability graph from a tree reveals some graph properties arising from trees.

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