1 Financial mathematics

Example 1.1. Mrs Y will need 150,000 CZK after 5 years to pay for a transfer of the "flat rights". The bank offers a term account with nominal interest rate 4.4% credited quarterly. How much money must Mrs Y save to cover the necessary amount for the transfer?

Solution:

\[ FV = 15,000; \quad n = 5; \quad m = 4; \quad i^{(m)} = 4.4%; \quad PV =? \]

\[ PV = FV \cdot \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)^{m \cdot n} = 150,000 \cdot \left( \frac{1}{1 + \frac{0.044}{4}} \right)^{4 \cdot 5} = 120,522.5 \]

Example 1.2. Mr Z bought a house for 3,600,000 CZK using mortgage loan of amount 2,500,000 CZK. He will repay 16,000 CZK monthly in arrear over 25 years. What is the nominal annual interest rate which the bank offered to Mr Z?

Solution:

\[ PV = 2,500,000; \quad P = 16,000; \quad m = 12; \quad n = 25; \quad i^{(m)} =? \]

The formula for the calculation of the present value:

\[ PV = P \cdot \sum_{t=1}^{m \cdot n} \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)^t = P \cdot m \cdot a^{(m)}_{\bar{n}} = P \cdot m \cdot \frac{1 - v^n}{i^{(m)}}, \]

where

\[ v^n = \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)^{m \cdot n}. \]

Therefore, we need to solve the equation

\[ 2,500,000 = 16,000 \cdot 12 \cdot \frac{1 - v^{25}}{i^{(12)}}. \]

The solution is then \( i^{(12)} = 0.059 = 5.9\%. \)

Be careful with the notation. The term \( a^{(m)}_{\bar{n}} \) denotes an annuity in advance payable \( m \)-thly. However, the payments are of size \( 1/m \). Thus, it was necessary to multiply \( a^{(m)}_{\bar{n}} \) by \( m \) in our case.
Example 1.3. Mrs Y makes deposits of 100 at time 0, and \( x \) at time 3. The fund grows at a force of interest

\[ \delta_t = \frac{t^2}{100}, \quad t > 0. \]

Let the amount of interest earned from time 3 to 6 is also equal to \( x \). Calculate \( x \).

Solution:

\[ C_0 = 100; \quad C_3 = x; \quad \delta_t = \frac{t^2}{100}, \quad t > 0 \]

At time 3, after making a deposit, we have

\[ FV_3 = C_0 \cdot e^{\int_0^3 \delta_t \, dt} + C_3 = 100 \cdot e^{\int_0^3 \frac{t^2}{100} \, dt} + x, \]

where

\[ \int_0^3 \frac{t^2}{100} \, dt = \frac{1}{100} \cdot \left[ \frac{t^3}{3} \right]_0^3 = \frac{27}{300} = \frac{9}{100}, \]

and therefore

\[ FV_3 = 100 \cdot e^{\frac{9}{100}} + x. \]

At time 6, there is

\[ FV_6 = FV_3 \cdot e^{\int_3^6 \delta_t \, dt} \]

in the fund, where

\[ \int_3^6 \frac{t^2}{100} \, dt = \frac{1}{100} \cdot \left[ \frac{t^3}{3} \right]_3^6 = \frac{189}{300} = \frac{63}{100}. \]

The interest gained between times 3 and 6 should be equal to \( x \) and thus it ought to hold

\[ FV_6 - FV_3 = x \Rightarrow FV_3 \cdot e^{\frac{63}{100}} - FV_3 = x \Rightarrow FV_3 \cdot \left( e^{\frac{63}{100}} - 1 \right) = x \]

\[ \Rightarrow (100e^{\frac{9}{100}} + x) \cdot \left( e^{\frac{63}{100}} - 1 \right) = x. \]

After solving this equation, we obtain \( x = 784.59 \).

Example 1.4. Mr X wants to borrow 150,000\$\. He would like to repay this loan in 2 years by periodic semiannual payments. Bank offers the nominal interest rate 6.9\%. Mr X has 45,000\$ on his account where the interest is credited monthly under nominal interest rate 2.5\%. He can save 40,000\$ every half a year from his salary.

a) Plot cash-flow.

b) Will Mr X have enough money on his account to cover the semiannual loan payments?

c) How much money should he hold at the beginning to cover the loan payments?
Solution:

\[ PV = 150,000; \ m = 2; \ i^{(m)} = 6.9\%; \ n = 2 \]
\[ A_0 = 45,000; \ p = 12; \ j^{(p)} = 2.5\%; \ S = 40,000 \]

b)

\[ PV = R \cdot \sum_{t=1}^{m\cdot n} \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)^t \Rightarrow R = \frac{PV}{\sum_{t=1}^{m\cdot n} \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)^t} = \frac{150,000}{\sum_{t=1}^{2\cdot 2} \left( \frac{1}{1 + \frac{6.9\%}{2}} \right)^t} = 40,789 \]

\[ A_0 = 45,000 \]
\[ A_{1/2} = A_0 \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} - R + S = 45,000 \left( 1 + \frac{0.025}{12} \right)^6 - 40,789 + 40,000 = 44,776 \]
\[ A_1 = A_{1/2} \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} - R + S = 44,776 \left( 1 + \frac{0.025}{12} \right)^6 - 40,789 + 40,000 = 44,549 \]
\[ A_{3/2} = A_1 \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} - R + S = 44,549 \left( 1 + \frac{0.025}{12} \right)^6 - 40,789 + 40,000 = 44,320 \]
\[ A_2 = A_{3/2} \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} - R + S = 44,320 \left( 1 + \frac{0.025}{12} \right)^6 - 40,789 + 40,000 = 44,088 \]

Mr X will have enough money to cover the loan payments.

c)

\[ A_2^* = 0 = \left[ \left( A_0 \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} + S - R \right) \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} + S - R \right] \cdot \left( 1 + \frac{j^{(p)}}{p} \right)^{p/2} + S - R \]

Our aim is to express \( A_0^* \). The previous equation can be adjusted to the following equation

\[ A_0^* = \sum_{t=1}^{4} \frac{R - S}{\left( 1 + \frac{j^{(p)}}{p} \right)^{p/2}} = \left( 40,789 - 40,000 \right) \sum_{t=1}^{4} \left( 1 + \frac{0.025}{12} \right)^{6\cdot t} = 3,059. \]

Example 1.5. Mr X sold his car for 200,000 CZK. He paid this amount to his account where the interest is credited monthly with nominal interest rate 2.8%. He decided to buy a new car 9 months after. There was a necessary advance payment of 50,000 CZK taken from the account. Then, the debt was repaid by payments 9,000 CZK monthly. How long the money on the account can cover these payments?
Solution:

\[ C_0 = 200,000; \ m = 12; \ i^{(m)} = 2.8\%; \ C_{9/12} = -50,000; \ P = 9,000; \ n =? \]

\[ C_0 \left( 1 + \frac{i^{(m)}}{m} \right)^9 + C_{9/12} = P \cdot m \cdot a^{(m)}_{\overline{m}} \]

Now we need to express the term \( a^{(m)}_{\overline{m}} \).

\[ a^{(m)}_{\overline{m}} = \frac{C_0 \left( 1 + \frac{i^{(m)}}{m} \right)^9 + C_{9/12}}{P \cdot m} \Rightarrow a^{(12)}_{\overline{12}} = \frac{200,000 \left( 1 + \frac{0.028}{12} \right)^9 - 50,000}{9,000 \cdot 12} = 1.43 \]

Assuming

\[ a^{(m)}_{\overline{m}} = \frac{1 - v^n}{i^{(m)}} = \frac{1 - \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)^{m-n}}{i^{(m)}} \]

we can express \( n \) as

\[ n = \frac{\ln \left( 1 - i^{(m)} \cdot a^{(m)}_{\overline{m}} \right)}{m \cdot \ln \left( \frac{1}{1 + \frac{i^{(m)}}{m}} \right)} = \frac{\ln (1 - 0.028 \cdot 1.43)}{12 \cdot \ln \left( \frac{1}{1 + \frac{0.028}{12}} \right)} = 1.46. \]

Since \( n \) is equal to 1.46 years, which is 17.5 months, the money can cover the payments for 17 months.

**Example 1.6.** Calculate the net present value for a bond with the nominal value 1,000 $, annual coupon rate 6\% and term to maturity 3 years. Consider a yield curve with annual spot/forward interest rates 3, 4, 5 \%.

**Solution:**

\[ N = 1,000; \ c = 6\%; \ n = 3; \ i_1 = 3\%; \ i_2 = 4\%; \ i_3 = 5\%; \ PV =? \]

\[ C = N \cdot c = 1,000 \cdot 0.06 = 60 \]

\[ PV = \frac{C}{1 + i_1} + \frac{C}{(1 + i_1)(1 + i_2)} + \frac{N + C}{(1 + i_1)(1 + i_2)(1 + i_3)} = \frac{60}{1 + 0.03} + \frac{60}{(1 + 0.03)(1 + 0.04)} + \frac{1,000 + 60}{(1 + 0.03)(1 + 0.04)(1 + 0.05)} = 1,056.69 \]
2 Demography (after the 4th lecture)

Notation
- \( T_x \) – random remaining lifetime of a person at age \( x \),
- \( tq_x = P(T_x < t) \) – probability of dying,
- \( t_p_x = 1 - tq_x \) – probability of surviving,
- \( \mu_x \) – force of mortality.

In addition to approximative assumptions presented at the lecture, another assumption, which can be supposed, is

(d) Assumption of linearity II (Balducci):
\[
1 - uq_x + u = (1 - u) \cdot q_x, \quad 0 \leq u \leq 1, \quad x \in \mathbb{N}_0
\]

Do not forget that many of calculations in this collection use the fundamental assumption \( \mathcal{L}(T_{x+t}) = \mathcal{L}(T_x - t | T_x > t) \).

Example 2.1. Assume that \( \mu_{x+t} = \mu_x \) for all \( t \in [0, 1] \). Let \( 1q_x = q_x = 0.16 \). Estimate \( t \) for which it holds \( t_p_x = 0.95 \).

Solution:

Under the assumption of constant force of mortality, it holds \( t_p_x = (p_x)^t \).
\[
tp_x = (p_x)^t = (1 - q_x)^t \Rightarrow \ln tp_x = t \cdot \ln (1 - q_x)
\]
\[
\Rightarrow t = \frac{\ln tp_x}{\ln (1 - q_x)} = \frac{\ln 0.95}{\ln (1 - 0.16)} = 0.294
\]

Example 2.2. Let \( q_x = 0.05 \) under basic level of the force of mortality \( \mu_{x+t} \). Assume that \( \mu'_{x+t} = \mu_{x+t} + c \) and estimate \( c \) for which \( q'_x = 0.07 \).

Solution:

For \( \mu_{x+t} \):
\[
1 - q_x = \exp\left(-\int_{0}^{1} \mu_{x+s} ds\right)
\]

For \( \mu'_{x+t} \):
\[
1 - q'_x = \exp\left(-\int_{0}^{1} \mu'_{x+s} ds\right) = \exp\left(-\int_{0}^{1} \mu_{x+s} + c ds\right)
\]
\[
= \exp\left(-\int_{0}^{1} \mu_{x+s} ds\right) \cdot \exp(-c) = (1 - q_x) \cdot \exp(-c)
\]
Therefore, we get
\[
e^{-c} = \frac{1 - q'_x}{1 - q_x} \Rightarrow c = -\ln\left(\frac{1 - q'_x}{1 - q_x}\right) = -\ln\left(\frac{1 - 0.07}{1 - 0.05}\right) = 0.021.
\]
Example 2.3. Let
\[ t p_x = \frac{100 - x - t}{100 - x}, \quad 0 \leq x \leq 100, \quad 0 \leq t < 100 - x. \]

Compute \( \mu_{45} \).

Solution:

We can calculate \( \mu_{45} \) using the following equation
\[ \mu_{x+t} = -\frac{d}{dt} \ln (t p_x), \]
evaluated at \( t = 45 - x \).

The desired quantity \( \mu_{45} \) is thus calculated as follows:
\[
\begin{align*}
\mu_{45} &= -\frac{d}{dt} \ln \left( \frac{100 - x - t}{100 - x} \right) \bigg|_{t=45-x} = -\frac{d}{dt} \left[ \ln (100 - x - t) - \ln (100 - x) \right] \bigg|_{t=45-x} \\
&= -\frac{d}{dt} \ln (100 - x - t) \bigg|_{t=45-x} = -\frac{1}{100 - x - t} \cdot (-1) \bigg|_{t=45-x} \\
&= -\frac{1}{100 - x - (45 - x)} = \frac{1}{55}.
\end{align*}
\]

Example 2.4. Consider \( u \in (0, 1) \). Show that under

1. the assumption of linearity, it holds
\[ x + u p_0 = (1 - u) x p_0 + u x + 1 p_0, \]

2. the assumption of constant force of mortality, it holds
\[ x + u p_0 = (x p_0)^{1-u} \cdot (x+1 p_0)^u, \]

3. the assumption of linearity II (Balducci ass.), it holds
\[ \frac{1}{x + u p_0} = \frac{1 - u}{x p_0} + \frac{u}{x + 1 p_0}. \]

Solution:

1) \[ x + u p_0 = x p_0 \cdot u p_x = (1 - x q_0) \cdot (1 - u q_x) \overset{\text{Ass.}}{=} (1 - x q_0) \cdot (1 - u \cdot q_x) \]
\[ = 1 - x q_0 - u \cdot q_x + u \cdot x q_0 \cdot q_x = x p_0 - u \cdot (1 - p_x) \cdot x p_0 \]
\[ = x p_0 - u \cdot x p_0 + u \cdot x p_0 \cdot p_x = x p_0 - u \cdot x p_0 + u \cdot x + 1 p_0 \]
\[ = (1 - u) \cdot x p_0 + u \cdot x + 1 p_0 \]
We can notice that the expression for the conditional distribution of
we can write
\( x_u p_x = \exp \left( -\int_0^u \mu_{x+y} \, dy \right) \).
Solution:
\begin{equation}
x_{u+1} p_0 = x_0 p_0 \cdot u p_x = (x_0 p_0)^{1+u} \cdot (p_x)^u = (x_0 p_0)^{1-u} \cdot (x_0+1 p_0)^u
\end{equation}

3)
\begin{equation}
x_{u+1} p_0 = x_{u+1} p_0 \cdot 1-u p_{x+u} \Rightarrow \frac{1}{x_{u+1} p_0} = 1-u p_{x+u} \Rightarrow \frac{1-1-u q_{x+u}}{x_{u+1} p_0} = \frac{1}{x_{u+1} p_0} - (1-u) \cdot q_x
\end{equation}

Example 2.5. Consider the decomposition of \( T \) to curtail \( K \) and fractional \( S \) remaining lifetime. Under the Balducci assumption, derive an explicit formula for the conditional probability
\[ P(S \leq u | K = k), \; u \in (0, 1). \]

Solution:
\begin{equation}
P(S \leq u | K = k) = \frac{P(S \leq u, K = k)}{P(K = k)} = \frac{P(k < T_x \leq k + u)}{P(k \leq T_x < k + 1)}
= \frac{k u p_x \cdot q_{x+k}}{k p_x \cdot q_{x+k}} = \frac{1-u p_{x+k}}{q_{x+k}}
\end{equation}

Taking into account the following relationship
\[ p_{x+k} = u p_{x+k} \cdot 1-u p_{x+k+u} \Rightarrow u p_{x+k} = \frac{p_{x+k}}{1-u p_{x+k+u}}, \]
we can write
\begin{equation}
P(S \leq u | K = k) = \frac{1-\frac{p_{x+k}}{q_{x+k}}}{u} = \frac{1-\frac{1-q_{x+k}}{1-1-u q_{x+k}}}{q_{x+k}} = \frac{\frac{u q_{x+k}}{q_{x+k}}}{q_{x+k}}
= \frac{1-(1-u) \cdot q_{x+k}}{q_{x+k}}
\end{equation}

We can notice that the expression for the conditional distribution of \( S \) depends on \( k \). For this reason the random variables \( S \) and \( K \) are not independent under the Balducci assumption.

Example 2.6. Consider nonsmokers with the force of mortality \( \mu_x \) and remaining lifetime \( T_x \) and smokers with \( \mu'_x = c \cdot \mu_x \), \( c > 0 \) and \( T'_x \) whose lives are independent. Derive an explicit formula for the probability that the smoker will live longer than the nonsmoker, i.e.
\[ P(T'_x > T_x). \]

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Solution:

First of all, the joint distribution of \((T_x, T'_x)\) must be found. According to the fact that the lives are independent, the joint density is just the product of marginal densities

\[
  f(t) = \varrho p_x \cdot \mu_{x+t}, \\
  f'(t) = \varrho p'_x \cdot \mu'_{x+t} = \varrho p_x' \cdot c \cdot \mu_{x+t} = (\varrho p_x)^c \cdot c \cdot \mu_{x+t},
\]

where we used

\[
  \varrho p'_x = \exp \left( - \int_0^t c \cdot \mu_{x+s} \, ds \right) = \left[ \exp \left( - \int_0^t \mu_{x+s} \, ds \right) \right]^c = (\varrho p_x)^c.
\]

The joint density is then

\[
  f(t, s) = \mu_{x+t} \cdot \varrho p_x \cdot c \cdot \mu_{x+s} \cdot (s p_x)^c.
\]

The sought probability that the smoker will survive the nonsmoker is obtained as follows:

\[
  P(T'_x > T_x) = \int_{s>t} f(t, s) \, ds \, dt = \int_0^\infty f(t, s) \, ds \, dt
\]

\[
  = \int_0^\infty \mu_{x+t} \cdot \varrho p_x \cdot c \cdot \mu_{x+s} \cdot (s p_x)^c \, ds \, dt
\]

\[
  = \int_0^\infty \mu_{x+t} \cdot \varrho p_x \left( \int_t^\infty c \cdot \mu_{x+s} \cdot (s p_x)^c \, ds \right) \, dt
\]

\[
  = \int_0^\infty \mu_{x+t} \cdot \varrho p_x \cdot (1 - F'(t)) \, dt = \int_0^\infty \mu_{x+t} \cdot \varrho p_x \cdot \varrho p'_x \, dt
\]

\[
  = \int_0^\infty \mu_{x+t} \cdot \varrho p_x \cdot (\varrho p_x)^c \, dt = \left[ \int_0^\infty \mu_{x+t} \cdot (\varrho p_x)^c+1 \, dt \right]_{=0}^{:=I}
\]

Per Partes

\[
  \begin{align*}
    u &= (\varrho p_x)^c \quad v = \varrho p_x \cdot \mu_{x+t} \\
    u' &= c \cdot (\varrho p_x)^{c-1} \cdot \frac{d}{dt}(\varrho p_x) = -c \cdot (\varrho p_x)^c \cdot \mu_{x+t} \\
    v' &= -\varrho p_{x} \\
    \int_0^\infty c \cdot \mu_{x+t} \cdot (\varrho p_x)^{c+1} \, dt &= \left[ \frac{-(\varrho p_x)^{c+1}}{c+1} \right]_{0}^{\infty} = \left[ \frac{1}{c+1} \right]_{0}^{\infty} = cI
  \end{align*}
\]

We are in the situation where it should hold

\[
  P(T'_x > T_x) = I = 1 - c \cdot I.
\]

The last step is to calculate \(I\):

\[
  I = 1 - c \cdot I \Rightarrow I = \frac{1}{c+1} = P(T'_x > T_x).
\]
3 Capital Life Insurance (after the 6th lecture)

3.1 Capital life insurance with constant sum insured

Example 3.1. Consider a fund of 100 independent lifes of age \( x \) with contracted whole life insurance with sum insured \( SI \). Derive the amount which is sufficient to cover the future liabilities with 95% probability. Consider

1. whole life insurance with \( SI \) payable at the end of the year of death (assume that \( A_x \) is known),

2. whole life insurance with \( SI=100,000 \) CZK payable at the moment of death under constant force of interest \( \delta = 0.06 \) and constant force of mortality \( \mu = 0.04 \).

Solution:

a)

Let us denote:

\( H \ldots \) random variable corresponding to net present value of all future payments

\( h \ldots \) fund to cover liabilities

Our aim is to determine \( h \). Since, it should hold \( P(H \leq h) = 0.95 \), with the use of Central Limit Theorem we obtain

\[
P(H \leq h) = P\left( \frac{H - EH}{\sqrt{VarH}} \leq \frac{h - EH}{\sqrt{VarH}} \right) \xrightarrow{n \to \infty} \Phi \left( \frac{h - EH}{\sqrt{VarH}} \right) = 0.95,
\]

where \( \Phi \) is the distribution function of the standard normal distribution.

Solution of the last equation is then as follows:

\[
h = EH + u_{0.95} \cdot \sqrt{VarH},
\]

where \( u_{0.95} \) represents the 0.95 quantile of the normal distribution which is equal to 1.96.

The random variable \( H \) has the following form:

\[
H = SI \cdot \sum_{i=1}^{100} v^{K_i+1}.
\]

Therefore, the expected value of \( H \) is

\[
EH = E \left( SI \cdot \sum_{i=1}^{100} v^{K_i+1} \right) = SI \cdot \left( \sum_{i=1}^{100} E(v^{K_i+1}) \right) = 100 \cdot SI \cdot E(v^{K+1})
\]

\[= 100 \cdot SI \cdot A_x,\]
and the variance is

\[ \text{Var} H = \text{Var} \left( ST \cdot \sum_{i=1}^{100} v^{K_i+1} \right) = ST^2 \cdot \left( \sum_{i=1}^{100} \text{Var}(v^{K_i+1}) \right) = 100 \cdot ST^2 \cdot \text{Var}(v^{K+1}). \]

In the previous equations, the fact that \( K_i \)'s (representing the lifes) are independent and identically distributed as a general random variable \( K \), was used.

The only unknown quantity needed to calculate \( h \) is \( \text{Var}(v^{K+1}) \). Since,

\[ \text{Var}(v^{K+1}) = E(v^{K+1})^2 - [E(v^{K+1})]^2 = E(v^{K+1})^2 - (A_x)^2, \]

we have to calculate \( E(v^{K+1})^2 \), which is done as follows:

\[ E(v^{K+1})^2 = \sum_{k=0}^{\infty} (v^{2k+2} \cdot k p_x \cdot q_{x+k}). \]

There exists also an approximative way, how to obtain \( E(v^{K+1})^2 \). The discount factor \( v^2 \) can be rewrite as \( \left( \frac{1}{1+i} \right)^2 = \frac{1}{1+2i+2^i} \). The quantity \( i^2 \) is very small and thus \( v^2 \approx \frac{1}{1+2i} = \frac{1}{1+i} = v_*. \) The consequence of this approximation is that

\[ E(v^{K+1})^2 = \sum_{k=0}^{\infty} (v^{k+1}^* \cdot k p_x \cdot q_{x+k}) = A_{x,*}, \]

where \( A_{x,*} \) is \( A_x \) calculated using \( i_* = 2i \).

The very last step would be inserting these things into \( h = E H + u_{0.95} \cdot \sqrt{\text{Var} H}. \)

b)

Now, we will assume that the payment is paid immediately at the moment of death \( (Z = v^T) \).

The relationship for the fund \( h \) is the same as in the previous case. However, the random variable \( H \) has the following form:

\[ H = 10^5 \cdot \sum_{i=1}^{100} v^T_i. \]

We will continue with calculations of \( EH \) and \( \text{Var} H \).

\[ EH = 10^5 \cdot 100 \cdot E(v^T) = 10^7 \cdot E(v^T), \]

\[ E(v^T) = \int_0^{\infty} v^t \cdot t p_x \cdot \mu_{x+t} \, dt. \]

Now, we will use the assumption of constant force of mortality \( \mu_x = \mu \).

\[ t p_x = e^{\{-\int_0^t \mu_{x+y} \, dy\}} = e^{\{-\int_0^t \mu \, dy\}} = e^{-t \mu}, \]
If we substitute the calculated values into the equation for \( H \), we get
\[
E(H) = 10^7 \cdot E(v^T) = 10^7 \cdot 0.4 = 4 \cdot 10^6.
\]
Therefore, the variance of \( H \) is
\[
\text{Var}(H) = (10^8)^2 \cdot 100 \cdot \text{Var}(v^T) = 10^{12} \cdot (E(v^T)^2 - [E(v^T)]^2),
\]
\[
E(v^T)^2 = E(v^{2T}) = \int_0^\infty v^2 \cdot q \cdot x \cdot \mu x + t \cdot dt = \int_0^\infty (1 + i)^{-2} \cdot e^{-t \mu} \cdot \mu dt
\]
\[
= \mu \cdot \frac{1}{2 \cdot \delta + \mu} = 0.04 \cdot \frac{1}{2 \cdot 0.06 + 0.04} = 0.04 \cdot \frac{16}{0.16} = 0.25.
\]
Therefore, the variance of \( H \) is
\[
\text{Var}(H) = 10^{12} \cdot (E(v^T)^2 - [E(v^T)]^2) = 10^{12} \cdot (0.25 - 0.4^2) = 10^{12} \cdot 0.09 = 9 \cdot 10^{10}.
\]
If we substitute the calculated values into the equation for \( h \), we get
\[
h = E(H) + 1.96 \cdot \sqrt{\text{Var}(H)} = 4 \cdot 10^6 + 1.96 \cdot \sqrt{(9 \cdot 10^{10})} = 4 \cdot 10^6 + 5.88 \cdot 10^6 = 4,588,000.
\]

Example 3.2. Prove the following relations between commutation functions:
1. \( C_x = vD_x - D_{x+1} \),
2. \( M_x = vN_x - N_{x+1} \),
3. \( R_x = vS_x - S_{x+1} \),
4. \( M_x = D_x - dN_x \),
5. \( R_x = N_x - dS_x \).

Solution:

1) \[
C_x = d_x \cdot v^{x+1} = (l_x - l_{x+1}) \cdot v^{x+1} = l_x \cdot v^x \cdot v - l_{x+1} \cdot v^{x+1} = v \cdot D_x - D_{x+1}
\]

2) \[
M_x = \sum_{k=0}^\infty C_{x+k} = \sum_{k=0}^\infty (d_{x+k} \cdot v^{x+k+1}) = \sum_{k=0}^\infty [(l_{x+k} - l_{x+k+1}) \cdot v^{x+k+1}]
\]
\[
= v \cdot \sum_{k=0}^\infty (l_{x+k} \cdot v^{x+k}) - \sum_{k=0}^\infty (l_{x+k+1} \cdot v^{x+k+1}) = v \cdot \sum_{k=0}^\infty D_{x+k} - \sum_{k=0}^\infty D_{x+k+1}
\]
\[
= v \cdot N_x - N_{x+1}
\]
Example 3.3. Using the commutation functions, derive explicit formulas for the net single premiums of the following capital life insurances with sum insured equal to one:

1. whole life insurance (net single premium is denoted by \( A_x \)),
2. term insurance with duration \( n \) years (\( A^1_{1:n} \)),
3. pure endowment with duration \( n \) years (\( A^{1\:n}_x \)),
4. endowments with duration \( n \) years (\( A^1_{x:n} \)),
5. \( m \)-years deferred whole life insurance (\( m\:|A_x \)),
6. \( m \)-years deferred term insurance with duration \( n \) years (\( m\:|A^1_{1:n} \)),
7. \( m \)-years deferred pure endowment with duration \( n \) years (\( m\:|A^{1\:n}_x \)),
8. \( m \)-years deferred endowments with duration \( n \) years (\( m\:|A^1_{x:n} \)).

Solution:

1) \( Z = v^{k+1}, \quad K = 0, 1, 2, \ldots \)

\[ A_x = \sum_{k=0}^{\infty} (v^{k+1} \cdot k p_x \cdot q_{x+k}) = \sum_{k=0}^{\infty} \left( v^{k+1} \cdot \frac{l_{x+k}}{l_x} \cdot \frac{d_{x+k}}{l_{x+k}} \right) = \sum_{k=0}^{\infty} \frac{d_{x+k} \cdot v^{x+k+1}}{l_x \cdot v_x} \]

\[ = \sum_{k=0}^{\infty} \frac{C_{x+k}}{D_x} = \frac{M_x}{D_x} \]

2) \( Z = \begin{cases} 
  v^{K+1}, & K = 0, 1, \ldots, n - 1 \\
  0, & K = n, n + 1, \ldots 
\end{cases} \)
\[ A_{x: m} = \sum_{k=0}^{n-1} \left( v^{k+1} \cdot k \cdot p_x \cdot q_{x+k} \right) = \sum_{k=0}^{n-1} \left( v^{k+1} \cdot \frac{d_{x+k}}{l_x} \right) = \sum_{k=0}^{n-1} \frac{C_{x+k}}{D_x} = \sum_{k=0}^{\infty} \frac{C_{x+k}}{D_x} - \sum_{k=n}^{\infty} \frac{C_{x+k}}{D_x} \]

\[
\frac{M_x}{D_x} - \sum_{k=0}^{\infty} C_{x+n+k} = \frac{M_x}{D_x} - \frac{M_{x+n}}{D_x} = \frac{M_x - M_{x+n}}{D_x}
\]

3) \[ Z = \begin{cases} 0, & K = 0, 1, \ldots, n - 1 \\ v^n, & K = n, n + 1, \ldots \end{cases} \]

\[ A_{x: n} = v^n \cdot n \cdot p_x = v^n \cdot \frac{l_{x+n}}{l_x} = \frac{v^{x+n} \cdot l_{x+n}}{v^x \cdot l_x} = \frac{D_{x+n}}{D_x} \]

4) \[ Z = \begin{cases} v^{K+1}, & K = 0, 1, \ldots, n - 1 \\ v^n, & K = n, n + 1, \ldots \end{cases} \]

\[ A_{x: n} = A_{x: n} + A_{x: n} = \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \]

5) \[ Z = \begin{cases} 0, & K = 0, 1, \ldots, m - 1 \\ v^{K+1}, & K = m, m + 1, \ldots \end{cases} \]

\[ m | A_x = \sum_{k=m}^{\infty} \left( v^{k+1} \cdot k \cdot p_x \cdot q_{x+k} \right) = \sum_{k=0}^{\infty} \left( v^{k+m+1} \cdot k \cdot p_{x+m} \cdot q_{x+m+k} \right) = v^m \cdot m \cdot p_x \cdot A_{x+m} = v^m \cdot \frac{l_{x+m}}{l_x} \cdot A_{x+m} = \frac{D_{x+m}}{D_x} \cdot \frac{M_{x+m}}{D_{x+m}} = \frac{M_{x+m}}{D_x} \]

Remark: The relationship for the deferment \( m | A_x = v^m \cdot m \cdot p_x \cdot A_{x+m} \) holds generally even for other types of insurance.

6) \[ Z = \begin{cases} 0, & K = 0, 1, \ldots, m - 1 \\ v^{K+1}, & K = m, m + 1, \ldots, m + n - 1 \\ 0, & K = m + n, m + n + 1, \ldots \end{cases} \]

\[ m | A_{x: m} = v^m \cdot m \cdot p_x \cdot A_{x+m: m} = \frac{v^{x+m} \cdot l_{x+m}}{v^x \cdot l_x} \cdot A_{x+m: m} = \frac{D_{x+m}}{D_x} \cdot \frac{M_{x+m} - M_{x+m+n}}{D_{x+m}} = \frac{M_{x+m} - M_{x+m+n}}{D_x} \]

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Example 3.4. Prove the following recursive formula

\[ A_x = v q_x + v p_x A_{x+1}. \]

Solution:

\[
A_x = \sum_{k=0}^{\infty} \left( v^{k+1} \cdot k p_x \cdot q_{x+k} \right) = v \cdot 0 p_x \cdot q_x + \sum_{k=1}^{\infty} \left( v^{k+1} \cdot k p_x \cdot q_{x+k} \right)
\]

\[ = v \cdot q_x + v \cdot \sum_{k=1}^{\infty} \left( v^k \cdot k p_x \cdot q_{x+k} \right) = v \cdot q_x + v \cdot \sum_{k=0}^{\infty} \left( v^{k+1} \cdot k+1 p_x \cdot q_{x+k+1} \right) \]

\[ = v \cdot q_x + v \cdot \sum_{k=0}^{\infty} \left( v^{k+1} \cdot p_x \cdot k p_{x+1} \cdot q_{x+k+1} \right) \]

\[ = v \cdot q_x + v \cdot p_x \cdot \sum_{k=0}^{\infty} \left( v^{k+1} \cdot k p_{x+1} \cdot q_{x+k+1} \right) = v \cdot q_x + v \cdot p_x \cdot A_{x+1} \]

Example 3.5. Consider TIR \( i = 3\% \), commutation functions \( D_{76} = 400 \), \( D_{77} = 360 \), and net single premium for the whole life insurance \( A_{76} = 0.8 \). Derive \( A_{77} \).
Solution:

We can use the recursive formula from the previous example:

\[ A_x = v \cdot q_x + v \cdot p_x \cdot A_{x+1} \Rightarrow A_{x+1} = \frac{A_x - v \cdot q_x}{v \cdot p_x}, \]

where \( v = \frac{1}{1+i} \).

Next step would be to express \( p_x \) and \( q_x \) using the commutation functions:

\[ p_x = \frac{l_{x+1}}{l_x} = \frac{v \cdot x^{x+1}}{v \cdot D_x}, \]

\[ q_x = 1 - p_x = 1 - \frac{D_{x+1}}{v \cdot D_x} = \frac{v \cdot D_x - D_{x+1}}{v \cdot D_x}. \]

Thus, \( A_{x+1} \) can be calculated as

\[ A_{x+1} = \frac{A_x - \frac{v \cdot D_x - D_{x+1}}{v \cdot D_x}}{\frac{D_{x+1}}{D_x}} = \frac{A_{x-1} \cdot D_x - v \cdot D_x + D_{x+1}}{D_{x+1}} \]

which corresponds numerically to

\[ A_{77} = \frac{A_{76} \cdot D_{76} - v \cdot D_{76} + D_{77}}{D_{77}} = \frac{A_{76} \cdot D_{76} - D_{76} + D_{77}}{D_{77}} = 0.8 \cdot 400 - \frac{400}{1+0.03} + 360 = 0.8101. \]

### 3.2 Capital life insurance with variable sum insured

**Example 3.6.** Prove that the net single premium for the whole life insurance with variable sum insured can be expressed as

\[ NSP = c_1 \cdot A_x + (c_2 - c_1) \cdot 1|A_{x-1} + (c_3 - c_2) \cdot 2|A_x + \ldots. \]

**Solution:**

\[ Z = c_{K+1} \cdot v^{K+1}, \quad K = 0, 1, 2, \ldots \]

\[ EZ = \sum_{k=0}^{\infty} \left( c_{k+1} \cdot v^{k+1} \cdot kP_x \cdot q_{x+k} \right) = c_1 \cdot \left[ \sum_{k=0}^{\infty} \left( v^{k+1} \cdot kP_x \cdot q_{x+k} \right) - \sum_{k=1}^{\infty} \left( v^{k+1} \cdot kP_x \cdot q_{x+k} \right) \right] + c_2 \cdot \left[ \sum_{k=1}^{\infty} \left( v^{k+1} \cdot kP_x \cdot q_{x+k} \right) - \sum_{k=2}^{\infty} \left( v^{k+1} \cdot kP_x \cdot q_{x+k} \right) \right] + \ldots \]

\[ = c_1 \cdot \left( A_x - 1|A_x \right) + c_2 \cdot \left( 1|A_{x-2} - 2|A_x \right) + \ldots = c_1 \cdot A_x + (c_2 - c_1) \cdot 1|A_x + \ldots. \]
Example 3.7. Prove that the net single premium for the term insurance with variable sum insured, i.e. \( c_{k+1} = 0 \) for \( k \geq n \), can be expressed as

\[
NSP = c_n A_{x;\eta}^1 + (c_{n-1} - c_n) A_{x;n-\eta}^1 + \cdots + (c_1 - c_2) A_{x;\eta}^1
\]

Solution:

\[
Z = \begin{cases} 
    c_{K+1} \cdot v_{K+1}^k, & K = 0, 1, \ldots, n - 1 \\
    0, & K = n, n + 1, \ldots
\end{cases}
\]

\[
EZ = \sum_{k=0}^{n-1} \left( c_{k+1} \cdot v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) = c_n \cdot \left[ \sum_{k=0}^{n-1} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) - \sum_{k=0}^{n-2} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) \right] \\
+ c_{n-1} \cdot \left[ \sum_{k=0}^{n-2} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) - \sum_{k=0}^{n-3} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) \right] \\
\vdots \\
+ c_2 \cdot \left[ \sum_{k=0}^{1} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) - \sum_{k=0}^{0} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) \right] + c_1 \cdot \left[ \sum_{k=0}^{0} \left( v_{k+1}^k \cdot kP_x \cdot q_{x+k} \right) \right] \\
= c_n \cdot (A_{x;\eta}^1 - A_{x;n-\eta}^1) + c_{n-1} \cdot (A_{x;n-\eta}^1 - A_{x;n-2\eta}^1) + \cdots + c_2 \cdot (A_{x,2\eta}^1 - A_{x,\eta}^1) + c_1 \cdot A_{x,\eta}^1 \\
= c_n \cdot A_{x;\eta}^1 + (c_{n-1} - c_n) \cdot A_{x;n-\eta}^1 + \cdots + (c_1 - c_2) \cdot A_{x,\eta}^1.
\]

Example 3.8. Consider a whole life insurance with variable sum insured, whose value is given according to the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c_{k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( \geq 9 )</td>
<td>7</td>
</tr>
</tbody>
</table>

Find an explicit formula for the net single premium.

Solution:

We can use the alternative expression of net single premium from Example 3.7.

\[
NSP = 10 \cdot A_x - 1 \cdot g|A_x - 1 \cdot 5|A_x - 1 \cdot 9|A_x = \frac{10 \cdot M_x - M_{x+2} - M_{x+5} - M_{x+9}}{D_x}
\]

Example 3.9. Using the commutation functions, derive explicit formulas for the net single premiums of the following capital life insurances:

1. standard increasing whole life insurance (net single premium is denoted by \((IA)_x\)),
2. standard increasing term insurance with duration \( n \) years \((IA)_x^1:n\eta\),
3. standard increasing endowments with duration \( n \) years \((IA)_x^1:n\eta\).
4. \( m \)-years deferred standard increasing whole life insurance \( m|IA)_x \),

5. standard increasing term insurance with duration \( n \) years \( (DA)_{x,n} \).

**Solution:**

1) 

\[
Z = (K + 1) \cdot v^{K+1}, \quad K = 0, 1, 2, \ldots
\]

\[
(IA)_x = \sum_{k=0}^{\infty} [(k + 1) \cdot v^{k+1} \cdot k p_x \cdot q_{x+k}] = \sum_{k=0}^{\infty} [(k + 1) \cdot v^{k+1} \cdot \frac{d_{x+k}}{l_x}]
\]

\[
= \sum_{k=0}^{\infty} [(k + 1) \cdot \frac{d_{x+k} \cdot v^{x+k+1}}{l_x \cdot v^{x}}] = \sum_{k=0}^{\infty} [(k + 1) \cdot \frac{C_{x+k}}{D_x}]
\]

\[
= \frac{C_x + 2 \cdot C_{x+1} + 3 \cdot C_{x+2} + \ldots}{D_x} = \sum_{k=0}^{\infty} C_{x+k} + \sum_{k=0}^{\infty} C_{x+1+k} + \ldots
\]

\[
= \frac{M_x + M_{x+1} + \ldots}{D_x} = \frac{R_x}{D_x}
\]

2) 

\[
Z = \begin{cases} 
(K + 1) \cdot v^{K+1}, & K = 0, 1, \ldots, n-1 \\
0, & K = n, n+1, \ldots
\end{cases}
\]

\[
(IA)_{x,n}^1 = \sum_{k=0}^{n-1} [(k + 1) \cdot v^{k+1} \cdot k p_x \cdot q_{x+k}]
\]

\[
= \sum_{k=0}^{n-1} [(k + 1) \cdot v^{k+1} \cdot k p_x \cdot q_{x+k}] - \sum_{k=0}^{n-1} [(k + 1) \cdot v^{k+1} \cdot k p_x \cdot q_{x+k}]
\]

\[
= (IA)_x - \sum_{k=0}^{n-1} [(k + n + 1) \cdot v^{k+n+1} \cdot k p_x \cdot q_{x+k+n}]
\]

\[
= (IA)_x - \sum_{k=0}^{n-1} [(k + n + 1) \cdot \frac{d_{x+k+n} \cdot v^{x+k+n+1}}{l_x \cdot v^{x}}]
\]

\[
= (IA)_x - \sum_{k=0}^{n-1} [(k + n + 1) \cdot \frac{C_{x+n+k}}{D_x}]
\]

\[
= \frac{R_x}{D_x} - \frac{(n+1) \cdot C_{x+n} + (n+2) \cdot C_{x+n+1} + (n+3) \cdot C_{x+n+2} + \ldots}{D_x}
\]

\[
= \frac{R_x}{D_x} - \frac{(n+1) \cdot M_{x+n} + M_{x+n+1} + M_{x+n+2} + \ldots}{D_x}
\]

\[
= \frac{R_x}{D_x} - \frac{n \cdot M_{x+n} + R_{x+n}}{D_x} = \frac{R_x - n \cdot M_{x+n} - R_{x+n}}{D_x}
\]

**Remark:** Alternatively, the following relationship could be used:

\[
(IA)_{x,n}^1 = A_x + 1|A_x + 2|A_x + \ldots + n-1|A_x - n \cdot n|A_x
\]

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3) \[ Z = \begin{cases} (K + 1) \cdot v^{K+1}, & K = 0, 1, \ldots, n - 1 \\ n \cdot v^n, & K = n, n + 1, \ldots \end{cases} \]

\[ (IA)_{x:1} = (IA)_{x:1} + (IA)_{x:1} + n \cdot A_{x:1} \]

\[ = R_x - n \cdot M_{x+n} - R_{x+n} + n \cdot D_{x+n} \]

\[ \frac{D_x}{D_x} \]

4) \[ Z = \begin{cases} 0, & K = 0, 1, \ldots, m - 1 \\ (K - m + 1) \cdot v^{K+1}, & K = m, m + 1, \ldots \end{cases} \]

\[ m_1(A)_{x} = \sum_{k=0}^{\infty} [(k + 1) \cdot v^{k+1} \cdot k \cdot p_x \cdot q_{x+k}] = \sum_{k=0}^{\infty} [(k + 1) \cdot v^{k+m+1} \cdot k \cdot p_x \cdot q_{x+k+m}] \]

\[ = \sum_{k=0}^{\infty} \left[ (k + 1) \cdot \frac{d_{x+k+m} \cdot v^{x+k+m+1}}{l_x \cdot v^x} \right] = \sum_{k=0}^{\infty} \left[ (k + 1) \cdot C_{x+k+m} \right] \]

\[ = \frac{C_{x+m} + 2 \cdot C_{x+m+1} + 3 \cdot C_{x+m+2} + \ldots}{D_x} = \frac{M_{x+m} + M_{x+m+1} + M_{x+m+2} + \ldots}{D_x} = \frac{R_{x+m}}{D_x} \]

5) \[ Z = \begin{cases} (n - K) \cdot v^{K+1}, & K = 0, 1, \ldots, n - 1 \\ 0, & K = n, n + 1, \ldots \end{cases} \]

\[ (DA)_{x:1} = \sum_{k=0}^{n-1} [(n - k) \cdot v^{k+1} \cdot k \cdot p_x \cdot q_{x+k}] = \sum_{k=0}^{n-1} \left[ (n - k) \cdot \frac{d_{x+k} \cdot v^{x+k+1}}{l_x \cdot v^x} \right] \]

\[ = \sum_{k=0}^{n-1} \left[ (n - k) \cdot C_{x+k} \right] = \frac{n \cdot C_x + (n - 1) \cdot C_{x+1} + \ldots + C_{x+n-1}}{D_x} \]

\[ = \frac{n \cdot M_x - M_{x+1} - \ldots - M_{x+n}}{D_x} = \frac{n \cdot M_x - R_{x+1} + R_{x+n+1}}{D_x} \]

3.3 Capital life insurance payable at moment of death and at the end of m-th part of year of death

Example 3.10. Consider a whole life insurance with variable SI when the sum payable is incremented m times a year, by 1/m each time. We assume that the sum insured is payable

1. at the end of the m-th part of the year in which death occurs \( (I^{(m)}A^{(m)})_x \),
2. immediately on death \((I^{(m)}\bar{A})_x\).

Derive the net single premium under the assumption of linearity.

Solution:

Under the assumption of linearity it holds:

\[ K \perp S, \quad S \sim U(0, 1), \]

where \( K = |T| \), \( T = K + S \) and \( U(0, 1) \) is the uniform distribution on \((0, 1)\) and also

\[ K \perp S^{(m)}, \quad S^{(m)} \sim U^D(\frac{1}{m}, \ldots, \frac{m}{m}), \]

where \( S^{(m)} = \lfloor \frac{m-S+1}{m} \rfloor, \lfloor \cdot \rfloor \) is the floor function and \( U^D(\frac{1}{m}, \ldots, \frac{m}{m}) \) is the discrete uniform distribution.

1)

Assume

\[ Z^{(m)} = (K + S^{(m)}) \cdot v^{K+S^{(m)}}. \]

Then

\[ (I^{(m)}A^{(m)})_x = EZ^{(m)} = E \left( (K + S^{(m)}) \cdot v^{K+S^{(m)}} \right) \]

\[ = E \left( (K + 1) \cdot v^{K+1} \right) \cdot E \left[ v^{S^{(m)}-1} \right] \]

\[ = (IA)_x \cdot E \left[ v^{-(1-S^{(m)})} \right] \]

\[ = (IA)_x \cdot E \left[ (1 - S^{(m)}) \cdot v^{-(1-S^{(m)})} \right] \]

We have to calculate the remaining terms (1) and (2) in the previous equation.

\[ E \left[ v^{-(1-S^{(m)})} \right] = E \left[ (1 + i)^{1-S^{(m)}} \right] = (1 + i) \cdot E \left[ (1 + i)^{S^{(m)}} \right] \]

\[ = (1 + i) \cdot \sum_{k=1}^{m} \left[ (1 + i)^{-\frac{k}{m}} \cdot \frac{1}{m} \right] = \frac{1+i}{m} \cdot \left[ (1 + i)^{-\frac{1}{m}} \right] \cdot \frac{1 - \left( (1 + i)^{-\frac{1}{m}} \right)^m}{1 - (1 + i)^{-\frac{1}{m}}} \]

\[ = \frac{1+i}{m} \cdot \left[ (1 + i)^{-\frac{1}{m}} \right] \cdot \frac{1 - (1 + i)^{-1}}{1 - (1 + i)^{-\frac{1}{m}}} = \frac{i}{i^{(m)}} \]

For the calculation of the term (2) we will use the following relationship.

\[ \sum_{k=1}^{m} (k \cdot a^{k-1}) = \left( \sum_{k=1}^{m} a^k \right) ' = \left( a \cdot \frac{1-a^m}{1-a} \right) ' = \frac{1 - (m+1) \cdot a^m + m \cdot a^{m+1}}{(1-a)^2} \]
Using this relationship for \( a = v^{\frac{1}{m}} \) we obtain

\[
E \left[ S^{(m)} \cdot v^{-(1-S^{(m)})} \right] = \frac{1}{m} \cdot \sum_{k=1}^{m} \left[ \frac{k}{m} \cdot v^{\frac{k}{m} - 1} \right] = \frac{1}{m^2} \cdot v^{\frac{1}{m} - 1} \sum_{k=1}^{m} \left[ k \cdot (v^{\frac{1}{m}})^{k-1} \right]
\]

\[
= \frac{1}{m^2} \cdot v^{\frac{1}{m} - 1} \cdot \frac{1 - (m + 1) \cdot v + m \cdot v^{\frac{m+1}{m}}}{(1 - v^{\frac{1}{m}})^2}
\]

\[
= v^{\frac{1}{m}} \cdot \frac{(1 + i) - (m + 1) + m \cdot v^{\frac{1}{m}}}{m^2 \cdot (1 - v^{\frac{1}{m}})^2}
\]

\[
= v^{\frac{1}{m}} \cdot \frac{i - m \cdot (1 - v^{\frac{1}{m}})}{m \cdot (1 - v^{\frac{1}{m}})^2}.
\]

Since \( m \cdot (1 - v^{\frac{1}{m}}) \), we have

\[
E \left[ S^{(m)} \cdot v^{-(1-S^{(m)})} \right] = v^{\frac{1}{m}} \cdot \frac{i - d^{(m)}}{[d^{(m)}]^2}.
\]

Furthermore, it holds

\[
\frac{v^{\frac{1}{m}}}{d^{(m)}} = \frac{1}{i^{(m)}}.
\]

Therefore

\[
E \left[ S^{(m)} \cdot v^{-(1-S^{(m)})} \right] = \frac{i - d^{(m)}}{i^{(m)}} \cdot d^{(m)}.
\]

Now we are able to complete the required form for \( (I^{(m)}A^{(m)})_x \) as

\[
(I^{(m)}A^{(m)})_x = \frac{i}{i^{(m)}} \cdot (IA)_x - \frac{i}{d^{(m)}} \cdot A_x + \frac{i - d^{(m)}}{i^{(m)}} \cdot d^{(m)} \cdot A_x.
\]

2)

Assume

\[
Z = (K + S^{(m)}) \cdot v^T.
\]

Then

\[
(I^{(m)}\overline{A})_x = EZ = E \left[ (K + S^{(m)}) \cdot v^T \right] = E \left[ ((K + 1) + S^{(m)} - 1) \cdot v^T \right]
\]

\[
= E \left[ (K + 1) \cdot v^T \right] + E \left[ S^{(m)} \cdot v^T \right] - Ev^T
\]

\[
= (I\overline{A})_x + E \left[ S^{(m)} \cdot v^{(K+1)-(1-S)} \right] - \overline{A}_x = (I\overline{A})_x + E \left[ S^{(m)} \cdot v^{-(1-S)} \right] \cdot A_x - \overline{A}_x.
\]

Similarly as in the first part, where

\[
E \left[ S^{(m)} \cdot v^{-(1-S^{(m)})} \right] = \frac{i - d^{(m)}}{i^{(m)} \cdot d^{(m)}},
\]

it can be shown that in this case it holds

\[
E \left[ S^{(m)} \cdot v^{-(1-S)} \right] = \frac{i - d^{(m)}}{\delta \cdot d^{(m)}}.
\]
Since it also holds
\[ \overline{A}_x = \frac{i}{\delta} \cdot A_x, \]
and
\[ (IA)_x = \frac{i}{\delta}(IA)_x, \]
the final form for \((I^{(m)}A)_x\) is
\[ (I^{(m)}A)_x = \frac{i}{\delta}(IA)_x + \frac{i - d^{(m)}}{\delta} \cdot A_x - \frac{i}{\delta} \cdot A_x. \]

**Example 3.11.** Consider a whole life insurance with continuously increasing sum insured, i.e. \(c(t) = t\), which is payable immediately on death. Derive the net single premium \((TA)_x\) under the assumption of linearity.

**Solution:**
Assume \(Z = T \cdot v^T\).
Then under the assumption of linearity
\[ (TA)_x = EZ = \sum_{k=0}^{\infty} \{ E[Z|K = k] \cdot P(K = k) \} \]
\[ = \sum_{k=0}^{\infty} \{ E[v^{K+1} \cdot v^{S-1} \cdot (K+S)|K = k] \cdot P(K = k) \} \]
\[ = \sum_{k=0}^{\infty} \{ v^{k+1} \cdot E[(k+S) \cdot v^{S-1}] \cdot P(K = k) \}. \]

\[ E[(k+S) \cdot v^{S-1}] = \int_{0}^{1} (k+s) \cdot v^{s-1} \, ds = k \cdot \int_{0}^{1} v^{s-1} \, ds + \int_{0}^{1} s \cdot v^{s-1} \, ds, \]
where
\[ \int_{0}^{1} v^{s-1} \, ds = \frac{1}{v} \int_{0}^{1} e^{s\ln(v)} \, ds = \frac{1}{v} \left[ \frac{1}{\ln(v)} \cdot e^{s\ln(v)} \right]_{0}^{1} = \frac{v - 1}{v \cdot \ln(v)} = -\frac{(v - 1)}{v \cdot \delta}, \]
and
\[ \int_{0}^{1} s \cdot v^{s-1} \, ds = \frac{1}{v} - \frac{v}{v \cdot \ln(v)} \cdot \left[ \frac{1}{\ln(v)} - \frac{1}{v \cdot \ln(v)} \right] = \frac{1}{\ln(v)} - \frac{1}{|\ln(v)|^2} + \frac{1}{v \cdot |\ln(v)|^2}, \]

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Together, we get

\[
E[(k + S) \cdot v^{S-1}] = -\frac{k \cdot (v - 1)}{\delta} - \frac{1}{\delta} - \frac{1}{\delta^2} + \frac{1}{v \cdot \delta^2} = -\frac{k}{\delta} + \frac{k}{\delta} - \frac{1}{\delta} - \frac{1}{\delta^2} + \frac{1}{v \cdot \delta^2} \\
= -\frac{k + 1}{\delta} - \frac{k \cdot (1 + i)}{\delta} - \frac{1}{\delta} + \frac{1 + i}{\delta^2} = \frac{-k - 1 + k + k \cdot i}{\delta} + \frac{-1 + 1 + i}{\delta^2} \\
= -1 + k \cdot i + \frac{i}{\delta^2}.
\]

Therefore

\[
(IA)_x = \sum_{k=0}^{\infty} \left[ v^{k+1} \cdot \left( -\frac{1 + k \cdot i}{\delta} + \frac{i}{\delta^2} \right) \cdot P(K = k) \right] \\
= -\frac{1}{\delta} \cdot E(v^{K+1}) + \frac{i}{\delta} \cdot E(K \cdot v^{K+1}) + \frac{i}{\delta^2} \cdot E(v^{K+1}) \\
= -\frac{1}{\delta} \cdot E(v^{K+1}) + \frac{i}{\delta} \cdot E[(K + 1) \cdot v^{K+1}] - \frac{i}{\delta} \cdot E(v^{K+1}) + \frac{i}{\delta^2} \cdot E(v^{K+1}) \\
= -\frac{1}{\delta} \cdot A_x + \frac{i}{\delta} \cdot (IA)_x - \frac{i}{\delta} \cdot A_x + \frac{i}{\delta^2} \cdot A_x = \frac{i \cdot (IA)_x - (1 + i) \cdot A_x}{\delta} + \frac{i}{\delta^2} \cdot A_x.
\]
4 Life Annuities (after the 7th lecture)

Example 4.1. Using the commutation functions, derive explicit formulas for the net single premiums of the following life annuities with sum insured equal to one:

1. whole life annuity in advance (net single premium is denoted by \( \ddot{a}_x \)),
2. whole life annuity in arrear \( a_x \),
3. temporary life annuity in advance with duration \( n \) years \( \ddot{a}_x^n \),
4. temporary life annuity in arrear with duration \( n \) years \( a_x^n \),
5. \( m \)-years deferred whole life annuity in advance \( m\ddot{a}_x \),
6. \( m \)-years deferred whole life annuity in arrear \( ma_x \),
7. \( m \)-years deferred temporary life annuity in advance with duration \( n \) years \( m\ddot{a}_x^n \),
8. \( m \)-years temporary life annuity in arrear with duration \( n \) years \( ma_x^n \).

Solution:

1) \[ Y = 1 + v + v^2 + \cdots + v^K, \quad K = 0, 1, 2, \ldots \]

\[ \ddot{a}_x = E Y = \sum_{k=0}^{K} \sum_{l=0}^{k} [v^l \cdot P(K = k)] = \sum_{k=0}^{\infty} [v^k \cdot P(K \geq k)] = \sum_{k=0}^{\infty} (v^k \cdot kP_x) \]

\[ = \sum_{k=0}^{\infty} \left( \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) = \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} = \frac{N_x}{D_x} \]

2) \[ Y = v + v^2 + \cdots + v^K, \quad K = 1, 2, \ldots \]

\[ a_x = \ddot{a}_x - 1 = \frac{N_x}{D_x} - 1 = \frac{N_x - D_x}{D_x} = \frac{N_{x+1}}{D_x} \]

3) \[ Y = \begin{cases} 1 + v + \cdots + v^K, & K = 0, 1, \ldots, n-1 \\ 1 + v + \cdots + v^{n-1}, & K = n, n+1, \ldots \end{cases} \]

\[ \ddot{a}_{x, m} = \sum_{k=0}^{n-1} (v^k \cdot kP_x) = \sum_{k=0}^{n-1} \frac{D_{x+k}}{D_x} = \frac{N_x - N_{x+n}}{D_x} \]

4) \[ Y = \begin{cases} v + \cdots + v^K, & K = 1, 2, \ldots, n \\ v + \cdots + v^n, & K = n+1, n+2, \ldots \end{cases} \]
\[ a_{x:n} = \sum_{k=1}^{n} (v^k \cdot k p_x) = \sum_{k=1}^{n} \frac{D_{x+k}}{D_x} = \sum_{k=0}^{n-1} \frac{D_{x+1+k}}{D_x} = \frac{N_{x+1} - N_{x+1+n}}{D_x} \]

5) \[ Y = \begin{cases} 
0, & K = 0, 1, \ldots, m - 1 \\
 v^m + \ldots + v^K, & K = m, m + 1, \ldots 
\end{cases} \]

\[ m|a_x = \sum_{k=m}^{\infty} (v^k \cdot k p_x) = \sum_{k=m}^{\infty} \left( \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) = 1 \cdot \sum_{k=m}^{\infty} D_{x+k} = \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} D_{x+m+k} \]

\[ = \frac{N_{x+m}}{D_x} \]

6) \[ Y = \begin{cases} 
0, & K = 0, 1, \ldots, m \\
v^{m+1} + \ldots + v^K, & K = m + 1, m + 2, \ldots 
\end{cases} \]

\[ m|a_x = \sum_{k=m+1}^{\infty} (v^k \cdot k p_x) = \sum_{k=m+1}^{\infty} \left( \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) = 1 \cdot \sum_{k=m+1}^{\infty} D_{x+k} \]

\[ = \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} D_{x+m+1+k} = \frac{N_{x+m+1}}{D_x} \]

7) \[ Y = \begin{cases} 
0, & K = 0, 1, \ldots, m - 1 \\
v^m + \ldots + v^K, & K = m, m + n - 1 \\
v^m + \ldots + v^{m+n-1}, & K = m + n, m + n + 1, \ldots 
\end{cases} \]

\[ m|a_{x:n} = \sum_{k=m}^{m+n-1} (v^k \cdot k p_x) = \sum_{k=m}^{m+n-1} \left( \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) = 1 \cdot \sum_{k=m}^{m+n-1} D_{x+k} \]

\[ = \frac{1}{D_x} \cdot \left[ \sum_{k=m}^{\infty} D_{x+k} - \sum_{k=0}^{\infty} D_{x+k} \right] = \frac{1}{D_x} \cdot \left[ \sum_{k=0}^{\infty} D_{x+m+k} - \sum_{k=0}^{\infty} D_{x+m+n+k} \right] \]

\[ = \frac{N_{x+m} - N_{x+m+n}}{D_x} \]

8) \[ Y = \begin{cases} 
0, & K = 0, 1, \ldots, m \\
v^{m+1} + \ldots + v^K, & K = m + 1, \ldots, m + n \\
v^{m+1} + \ldots + v^{m+n}, & K = m + n + 1, m + n + 2, \ldots 
\end{cases} \]
Example 4.2. Prove the following recursive formula

\[ \ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}. \]

Solution:

\[
\ddot{a}_x = \sum_{k=0}^{\infty} (v^k \cdot kp_x) = 1 + \sum_{k=1}^{\infty} (v^k \cdot kp_x) = 1 + \sum_{k=0}^{\infty} (v^{k+1} \cdot k+1p_x) \\
= 1 + v \sum_{k=0}^{\infty} (v^k \cdot p_x \cdot k+1p_{x+1}) = 1 + v \cdot p_x \cdot \sum_{k=0}^{\infty} (v^k \cdot k+1p_{x+1}) = 1 + v \cdot p_x \cdot \ddot{a}_{x+1}
\]

Example 4.3. Consider \( e_x = E[K] \). Show that

1. \( A_x > v^{e_x+1} \),
2. \( \ddot{a}_x < \ddot{a}_{e_x+1} \).

Solution:

1) \( A_x = E[v^{K+1}] \)

If we denote \( f(z) = v^{z+1} \) then

\( v^{e_x+1} = f(E[K]) \)

and

\( A_x = E[f(K)] \).

Since \( f(z) \) is a convex function \( [f''(z) > 0] \), due to Jensen inequality it holds

\( E[f(K)] > f(E[K]) \).

Therefore

\( A_x > v^{e_x+1} \).
2)  
\[ \ddot{a}_x = \frac{1 - A_x}{d} = \frac{1 - A_x}{1 - v} = \frac{1 - \text{E}[v^{K+1}]}{1 - v} \]
\[ \ddot{a}_{x+1} = \ddot{a}_{E[K]+1} = \frac{1 - v^{E(K)+1}}{1 - v} \]

If we denote \( g(z) = \frac{1 - v^{z+1}}{1 - v} \) then
\[ \ddot{a}_{x+1} = g[\text{E}(K)] \]
and
\[ \ddot{a}_x = \text{E}[g(K)]. \]

Since \( g(z) \) is a concave function \( [g''(z) < 0] \), due to Jensen inequality it holds
\[ \text{E}[g(K)] < g[\text{E}(K)]. \]

Therefore
\[ \ddot{a}_x < \ddot{a}_{x+1} \]

**Example 4.4.** Find relations between net single premiums under basic force of mortality \( \mu_x \) and the one increased by a constant \( \mu'_x = \mu_x + c, \ c > 0 \) for the following life insurances: pure endowment, life annuity in advance and whole life insurance.

**Solution:**

Insurances with an increased force of mortality will be denoted by an asterisk.

1)  
\[ (A_x^{1\cdot \text{m}})^* = v^n \cdot n p_x^* = v^n \cdot \exp\{- \int_0^n (\mu_{x+s} + c) \, ds\} = v^n \cdot e^{-c \cdot n p_x} = e^{-c \cdot n} \cdot A_x^{1\cdot \text{m}} \]

Since \( c > 0 \), then \( e^{-c \cdot n} < 1 \) and it holds
\[ (A_x^{1\cdot \text{m}})^* < A_x^{1\cdot \text{m}} \]

Interpretation: If people die more (faster), fewer people will be paid, the insurance company will save and premiums may be lower.

2)  
\[ (\ddot{a}_x)^* = \sum_{k=0}^{\infty} (v^k \cdot k p_x^*) = \sum_{k=0}^{\infty} (v^k \cdot k p_x \cdot e^{-c \cdot k}) = \sum_{k=0}^{\infty} [(v \cdot e^{-c})^k \cdot k p_x] \]

Again, \( e^{-c} < 1 \) and therefore
\[ (v \cdot e^{-c})^k < v^k. \]

Because of that, the following inequality holds
\[ (\ddot{a}_x)^* = \sum_{k=0}^{\infty} [(v \cdot e^{-c})^k \cdot k p_x] < \sum_{k=0}^{\infty} [v^k \cdot k p_x] = \ddot{a}_x. \]
Interpretation: Similarly as in 1). The only difference is that now fewer payments related to one insured are considered.

3) 

\[(A_x)^* = 1 - d \cdot (\bar{a}_x)^*\]

Now we can use the previous result and get

\[(A_x)^* = 1 - d \cdot (\bar{a}_x)^* > 1 - d \cdot \bar{a}_x = A_x.\]

Interpretation: As people die earlier, claims are paid also earlier and premiums must therefore be higher.

**Example 4.5.** Using the commutation functions, derive explicit formulas for the net single premiums of:

1. standard increasing whole life annuity in advance \((I\bar{a})_x\),
2. standard increasing temporary life annuity in advance with duration \(n\) years \((I\bar{a})_{x\,\!n}\),
3. \(m\)-years deferred standard increasing whole life annuity in advance \(m|(I\bar{a})_x\),
4. \(m\)-years deferred standard increasing temporary life annuity in advance with duration \(n\) years \(m|(I\bar{a})_{x\,\!n}\).

**Solution:**

1) \[Y = 1 + 2 \cdot v + \cdots + (K + 1) \cdot v^K, \quad K = 0, 1, 2, \ldots\]

\[
(I\bar{a})_x = \sum_{k=0}^{\infty} \left[ (k + 1) \cdot v^k \cdot kP_x \right] = \sum_{k=0}^{\infty} \left[ (k + 1) \cdot \frac{\ell_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right] = \sum_{k=0}^{\infty} \frac{(k + 1) \cdot D_{x+k}}{D_x} \\
= \frac{D_x + 2 \cdot D_{x+1} + 3 \cdot D_{x+2} + \cdots}{D_x} = \frac{N_x + N_{x+1} + N_{x+2} + \cdots}{D_x} = \frac{S_x}{D_x}
\]

2) \[Y = \begin{cases} 
1 + 2 \cdot v + \cdots + (K + 1) \cdot v^K, & K = 0, \ldots, n - 1 \\
1 + 2 \cdot v + \cdots + n \cdot v^{n-1}, & K = n, n + 1, \ldots
\end{cases}\]
\[(I\ddot{a})_{x, \eta} = \sum_{k=0}^{n-1} [(k+1) \cdot v^k \cdot k p_x] = \sum_{k=0}^{n-1} \left( \frac{(k+1) \cdot l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) = \sum_{k=0}^{n-1} \left( \frac{(k+1) \cdot D_{x+k}}{D_x} \right) \]

\[= \frac{1}{D_x} \cdot \left[ \sum_{k=0}^{\infty} (k+1) \cdot D_{x+k} - \sum_{k=n}^{\infty} [(k+1) \cdot D_{x+k}] \right] \]

\[(I\ddot{a})_x - \frac{1}{D_x} \cdot \sum_{k=n}^{\infty} [(k+1) \cdot D_{x+k}] = (I\ddot{a})_x - \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} [(k+n+1) \cdot D_{x+n+k}] \]

\[(I\ddot{a})_x - (n+1) \cdot D_{x+n} + (n+2) \cdot D_{x+n+1} + \cdots = \frac{S_x}{D_x} - \frac{n \cdot N_{x+n} + S_{x+n}}{D_x} \]

\[= \frac{S_x - S_{x+n} - n \cdot N_{x+n}}{D_x} \]

\[Y = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^m + \cdots + (K-m+1) \cdot v^K, & K = m, m+1, \ldots \end{cases} \]

\[m_1(I\ddot{a})_x = \sum_{k=m}^{\infty} [(k-m+1) \cdot v^k \cdot k p_x] = \sum_{k=m}^{\infty} \left( \frac{(k-m+1) \cdot l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) \]

\[= \frac{1}{D_x} \cdot \sum_{k=m}^{\infty} [(k-m+1) \cdot D_{x+k}] = \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} [(k+1) \cdot D_{x+m+k}] \]

\[= \frac{D_{x+m} + 2 \cdot D_{x+m+1} + 3 \cdot D_{x+m+2} + \cdots}{D_x} = \frac{N_{x+m} + N_{x+m+1} + N_{x+m+2} + \cdots}{D_x} \]

\[= \frac{S_{x+m}}{D_x} \]

\[Y = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^m + \cdots + (K-m+1) \cdot v^K, & K = m, \ldots, m+n-1 \\ v^m + \cdots + n \cdot v^{m+n-1}, & K = m+n, m+n+1, \ldots \end{cases} \]

\[m_1(I\ddot{a})_{x, \eta} = \sum_{k=m}^{m+n-1} [(k-m+1) \cdot v^k \cdot k p_x] = \sum_{k=m}^{m+n-1} \left( \frac{(k-m+1) \cdot l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x} \right) \]

\[= \frac{1}{D_x} \cdot \sum_{k=m}^{m+n-1} [(k-m+1) \cdot D_{x+k}] = \frac{1}{D_x} \cdot \sum_{k=0}^{n-1} [(k+1) \cdot D_{x+m+k}] \]

\[= \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} [(k+1) \cdot D_{x+m+k}] - \sum_{k=n}^{\infty} [(k+1) \cdot D_{x+m+k}] \]

\[= m_1(I\ddot{a})_x - \frac{(n+1) \cdot D_{x+m+n} + (n+2) \cdot D_{x+m+n+1} + \cdots}{D_x} \]

\[= \frac{S_{x+m}}{D_x} - \frac{n \cdot N_{x+m+n} + S_{x+m+n}}{D_x} = \frac{S_{x+m} - S_{x+m+n} - n \cdot N_{x+m+n}}{D_x} \]

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Example 4.6. Prove the following relation between the net single premiums:

\[(IA)_x = \ddot{a}_x - d (I\ddot{a})_x.\]

Solution:

There are two ways how to prove this relation.

1) Using commutation functions

Transforming the relation into the form using the commutation functions, we want to prove that

\[\frac{R_x}{D_x} = \frac{N_x}{D_x} - d \cdot \frac{S_x}{D_x}.\]

This can be shown by

\[\frac{N_x}{D_x} - d \cdot \frac{S_x}{D_x} = \frac{N_x - (1 - v) \cdot S_x}{D_x} = \frac{v \cdot S_x - S_{x+1}}{D_x} = \frac{R_x}{D_x},\]

where we used the result from Example 3.2 (3.).

2) Trick

\[\sum_{k=0}^{K} v^k \cdot (k + 1) = \sum_{k=0}^{K} v^k \sum_{l=0}^{k} 1 = \sum_{k=0}^{K} \sum_{l=0}^{k} v^k = \sum_{l=0}^{K} \left( v^l \cdot \frac{1 - v^{K+1-l}}{1 - v} \right)\]

\[= \frac{1}{d} \sum_{l=0}^{K} (v^l - v^{K+1}) = \frac{1}{d} \left[ \frac{1 - v^{K+1}}{1 - v} - (K + 1) \cdot v^{K+1} \right]\]

and after applying the expectation on both sides of the last equation, we obtain

\[(I\ddot{a})_x = \frac{1}{d} \cdot [\ddot{a}_x - (IA)_x],\]

which is after an adjustment the same as the desired form.

Example 4.7. Consider standard increasing whole life annuity payable \(m\)-times a year in advance where the payments are incremented once a year, by \(1/m\) each time. Derive an explicit formula for the net single premium using the commutation functions.

Solution:

In this case we assume the following payments

<table>
<thead>
<tr>
<th>Time:</th>
<th>0</th>
<th>(\frac{1}{m})</th>
<th>(\frac{1}{m})</th>
<th>(\ldots)</th>
<th>1</th>
<th>(1 + \frac{1}{m})</th>
<th>(\ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>(\frac{1}{m})</td>
<td>(\frac{1}{m})</td>
<td>(\frac{1}{m})</td>
<td>(\ldots)</td>
<td>(\frac{2}{m})</td>
<td>(\frac{2}{m})</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>
\[
(I\ddot{a})^{(m)}_x = \sum_{k=0}^{\infty} k|\ddot{a}^{(m)}_{x+k}| = \sum_{k=0}^{\infty} \left[ v^k \cdot k p_x \cdot \ddot{a}^{(m)}_{x+k} \right]
\]

Now we can use the approximation from the lecture

\[\ddot{a}^{(m)}_x \approx \ddot{a}_x - \frac{m-1}{2 \cdot m}.\]

Using this approximation we get

\[
(I\ddot{a})^{(m)}_x \approx \sum_{k=0}^{\infty} \left[ v^k \cdot k p_x \cdot \ddot{a}_{x+k} - \frac{m-1}{2 \cdot m} \cdot \sum_{k=0}^{\infty} v^k \cdot k p_x \right] = \sum_{k=0}^{\infty} \left[ \frac{D_{x+k}}{D_x} \cdot N_{x+k} \right] - \frac{m-1}{2 \cdot m} \cdot \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x}
\]

\[
= \sum_{k=0}^{\infty} \frac{N_{x+k}}{D_x} - \frac{m-1}{2 \cdot m} \cdot \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} = S_x - \frac{m-1}{2 \cdot m} \cdot N_x
\]

Alternatively, we can derive the formula without this approximation. In that case we have

\[
(I\ddot{a})^{(m)}_x = \sum_{k=0}^{\infty} k|\ddot{a}^{(m)}_x| = \sum_{k=0}^{\infty} \left[ v^k \cdot k p_x \cdot \ddot{a}_x \right] - \beta(m) \cdot \sum_{k=0}^{\infty} v^k \cdot k p_x
\]

\[
= \alpha(m) \cdot \sum_{k=0}^{\infty} v^k \cdot k p_x \cdot \ddot{a}_x - \beta(m) \cdot \sum_{k=0}^{\infty} v^k \cdot k p_x = \alpha(m) \cdot (I\ddot{a})_x - \beta(m) \cdot \ddot{a}_x,
\]

where

\[
\alpha(m) = \frac{d \cdot i}{d^{(m)} \cdot i^{(m)}}, \quad \beta(m) = \frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}}.
\]

Therefore

\[
(I\ddot{a})^{(m)}_x = \frac{d \cdot i}{d^{(m)} \cdot i^{(m)}} \cdot (I\ddot{a})_x - \frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}} \cdot \ddot{a}_x = \frac{d \cdot i}{d^{(m)} \cdot i^{(m)}} \cdot S_x - \frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}} \cdot N_x \cdot \frac{D_x}{D_x}.
\]
5 Annual Net Premium (after the 9th lecture)

Example 5.1. Using the actuarial symbols and commutation functions, derive explicit formulas for the annual net premiums paid during the deferment period of the following life insurances and annuities (with sum insured equal to one):

1. \(m\)-years deferred whole life insurance,
2. \(m\)-years deferred pure endowment with duration \(n\) years,
3. \(m\)-years deferred whole life annuity in advance,
4. \(m\)-years deferred temporary life annuity in arrear with duration \(n\) years,

Solution:

1) \[ L = \begin{cases} 
0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m - 1 \\
\nu^{K+1} - P \cdot \sum_{k=0}^{m-1} v^k, & K = m, m + 1, \ldots 
\end{cases} \]

\[ EL = 0 = m[A_{x: \overline{m}}] - P \cdot \hat{\overline{a}}_{x: \overline{m}} \]

\[ \downarrow \]

\[ P = \frac{m[A_{x: \overline{m}}]}{\hat{\overline{a}}_{x: \overline{m}}} \]

\[ \downarrow \]

\[ P = \frac{M_{x+m}}{D_x} = \frac{M_{x+m}}{N_x - N_{x+m}} \]

2) \[ L = \begin{cases} 
0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m - 1 \\
0 - P \cdot \sum_{k=0}^{m-1} v^k, & K = m, m + 1, \ldots, m + n - 1 \\
\nu^{m+n-1} - P \cdot \sum_{k=0}^{m-1} v^k, & K = m + n, m + n + 1, \ldots 
\end{cases} \]

\[ EL = 0 = m[A_{x: \overline{m}}]^{1} - P \cdot \hat{\overline{a}}_{x: \overline{m}} \]

\[ \downarrow \]

\[ P = \frac{m[A_{x: \overline{m}}]}{\hat{\overline{a}}_{x: \overline{m}}} \]

\[ \downarrow \]

\[ P = \frac{D_{x+m+n}}{D_x} = \frac{D_{x+m+n}}{N_x - N_{x+m}} \]

3) \[ L = \begin{cases} 
0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m - 1 \\
\sum_{k=m}^{K} v^k - P \cdot \sum_{k=0}^{m-1} v^k, & K = m, m + 1, \ldots 
\end{cases} \]
\[ EL = 0 = m\ddot{a}_x - P \cdot \dot{a}_{x:m} \]

\[ P = \frac{m\ddot{a}_x}{\dot{a}_{x:m}} \]

\[ P = \frac{N_{x+m}}{N_x - N_{x+m}} \]

4) \[
L = \begin{cases} 
0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m - 1 \\
0 - P \cdot \sum_{k=0}^{m-1} v^k, & K = m \\
\sum_{k=m+1}^{K} v^k - P \cdot \sum_{k=0}^{m-1} v^k, & K = m + 1, \ldots, m + n \\
\sum_{k=m+1}^{K} v^k - P \cdot \sum_{k=0}^{m-1} v^k, & K = m + n + 1, m + n + 2, \ldots 
\end{cases}
\]

\[ EL = 0 = m\dot{a}_{x:m} - P \cdot \ddot{a}_{x:m} \]

\[ P = \frac{m\dot{a}_{x:m}}{\ddot{a}_{x:m}} \]

\[ P = \frac{N_{x+m+1} - N_{x+m+1}}{N_x - N_{x+m}} \]

Example 5.2. Consider pure endowment with duration \( n \) years where the annual net premium is paid during the whole insurance duration. Moreover, the premium refund agreement is active. i.e. in the case of death of the insured person the premium paid until the death is paid to a beneficiary at the end of the year. Derive the total loss and the annual net premium. Compare the premiums for the insurance contracts with and without the premium refund.

Solution:

First, we have to calculate the premium for the insurance without the refund.

\[ L_1 = \begin{cases} 
0 - P_1 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, n - 1 \\
v^n - P_1 \cdot \sum_{k=0}^{n-1} v^k, & K = n, n + 1, \ldots 
\end{cases} \]

\[ EL_1 = 0 = A_{x:1} - P_1 \cdot \dot{a}_{x:1} \]

\[ P_1 = \frac{A_{x:1}}{\ddot{a}_{x:1}} \]
Now, when the premium refund is added, we obtain

\[
L_2 = \begin{cases} 
  P_2 \cdot (K + 1) \cdot v^{K+1} - P_2 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, n - 1 \\
  \nu^n - P_2 \cdot \sum_{k=0}^{n-1} v^k, & K = n, n + 1, \ldots
\end{cases}
\]

\[
EL_2 = 0 = A_{x:1\overline{m}} + P_2 \cdot (IA)_{1\overline{m}} - P_2 \cdot \ddot{a}_{x:1\overline{m}}
\]

\[
P_2 = \frac{A_{x:1\overline{m}}}{\ddot{a}_{x:1\overline{m}} - (IA)_{1\overline{m}}}
\]

When comparing obtained premiums, it is obvious that when the premium refund is included, the corresponding premium must be higher, which is achieved by subtracting the term \((IA)_{1\overline{m}}^0\) in the denominator.

**Example 5.3.** Consider \(m\)-years deferred whole life annuity in advance where the annual net premium is paid during the deferment period. Moreover, the premium refund agreement is active. i.e. in the case of death of the insured person during the deferment period the premium paid until the death is paid to a beneficiary at the end of the year. Derive the total loss and the annual net premium.

Consider also the case when the annual net premium is paid over \(m' < m\) years, but the premium refund is active over the whole deferment period.

**Solution:**

We can start with the case, when the annual premium is paid over \(m\) years.

\[
L_1 = \begin{cases} 
  P_1 \cdot (K + 1) \cdot v^{K+1} - P_1 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m - 1 \\
  \sum_{k=m}^{K} v^k - P_1 \cdot \sum_{k=0}^{m-1} v^k, & K = m, m + 1, \ldots
\end{cases}
\]

\[
EL_1 = 0 = m \ddot{a}_x + P_1 \cdot (IA)_{1\overline{m'}} - P_1 \cdot \ddot{a}_{x:1\overline{m'}}
\]

\[
P_1 = \frac{m \ddot{a}_x}{\ddot{a}_{x:1\overline{m'}} - (IA)_{1\overline{m'}}}
\]

When the premium is paid only over \(m'\) years, the total loss has the following form:

\[
L_2 = \begin{cases} 
  P_2 \cdot (K + 1) \cdot v^{K+1} - P_2 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m' - 1 \\
  P_2 \cdot m' \cdot v^{K+1} - P_2 \cdot \sum_{k=0}^{m'-1} v^k, & K = m', m' + 1 \ldots, m - 1 \\
  \sum_{k=m}^{K} v^k - P_2 \cdot \sum_{k=0}^{m'-1} v^k, & K = m, m + 1, \ldots
\end{cases}
\]

\[
EL_2 = 0 = m \ddot{a}_x + P_2 \cdot (IA)_{1\overline{m'}} + P_2 \cdot m' \cdot A_{1\overline{m'}:m'\overline{m'-1}} - P_2 \cdot \ddot{a}_{x:1\overline{m'}}
\]

\[
P_2 = \frac{m \ddot{a}_x}{\ddot{a}_{x:1\overline{m'}} - (IA)_{1\overline{m'}} - m' \cdot A_{1\overline{m':m'-1}}}
\]
Example 6.1. Derive the net premium reserve for the pure endowment contract for $n$ years when premium is paid

1. at once at the beginning as the net single premium,
2. at the beginning of each year over the whole insurance duration as the annual net premium,
3. at the beginning of each year over first $n' < n$ years as the annual net premium.

Solution:

1) \[ \text{NSP} = A_{x: \overline{1}} \]

\[ kV_x = \begin{cases} 0, & k = 0 \\ A_{x+k:n-k}, & k = 1, 2, \ldots, n - 1 \end{cases} \]

Remark: Time 0 is assumed before the payment of the net single premium. After the premium payment, it holds $0 + V_x = \text{NSP} = A_{x: \overline{1}}$ for the reserve.

2) \[ P = \frac{A_{x: \overline{1}}}{\ddot{a}_{x: \overline{1}}} \]

\[ kV_x = A_{x+k:n-k} - P \cdot \ddot{a}_{x+k:n-k}, \quad k = 0, 1, \ldots, n - 1 \]

Remark: If we assume $k = 0$ then $0V_x = 0$ (this corresponds to the principle of equivalence). For $k = n$, the reserve would be $nV_x = 1$ (the insurer must pay the sum insured to the beneficiary).

3) \[ P = \frac{A_{x: \overline{n'}}}{\ddot{a}_{x: \overline{n'}}} \]

\[ kV_x = \begin{cases} A_{x+k:n-k} - P \cdot \ddot{a}_{x+k:n-k}, & k = 0, 1, \ldots, n' - 1 \\ A_{x+k:n-k}, & k = n', n' + 1, \ldots, n - 1 \end{cases} \]

Example 6.2. Derive the net premium reserve for whole life insurance with variable sum insured:

\[ c_1 = 50, \quad c_2 = 55, \ldots, c_{10} = 95, \quad c_{11} = 100, \quad c_{12} = 100, \ldots \]

Consider standard increasing premium paid yearly over the whole insurance duration, i.e. $P, 2P, 3P, \ldots$. 
Solution:

First of all, we should derive the total loss and the formula for the annual net premium.

\[
L = \begin{cases} 
45 \cdot v^{K+1} + 5 \cdot (K + 1) \cdot v^{K+1} - P \cdot \sum_{k=0}^{K} [(k + 1) \cdot v^k], & K = 0, 1, \ldots, 10 \\
100 \cdot v^{K+1} - P \cdot \sum_{k=0}^{K} [(k + 1) \cdot v^k], & K = 11, 12, \ldots
\end{cases}
\]

\[
EL = 0 = 45 \cdot A_x + 5 \cdot (IA)^1_{x:11} + 55 \cdot 11|A_x - P \cdot (I\ddot{a})_x
\]

\[
P = \frac{45 \cdot A_x + 5 \cdot (IA)^1_{x:11} + 55 \cdot 11|A_x}{(I\ddot{a})_x}
\]

Now, we can proceed to the derivation of the formula for the reserve.

\[
kV_x = \begin{cases} 
45 \cdot A + 5 \cdot (IA)^1_{x+k:10} + 5 \cdot k \cdot A_{x+k:9} \\
+ 55 \cdot 11\cdot k\cdot A_{x+k} - P \cdot (I\ddot{a})_{x+k} - P \cdot k \cdot \ddot{a}_{x+k}, & k = 0, 1, \ldots, 10 \\
100 \cdot A_{x+k} - P \cdot (I\ddot{a})_{x+k} - P \cdot k \cdot \ddot{a}_{x+k}, & k = 11, 12, \ldots
\end{cases}
\]

Remark: Be careful with the standard increasing term insurance and life annuity because, when dealing with reserves, a special term must be added. For example, when assuming the term insurance, \((IA)^1_{x+k:10} \) starts again with payment 1, but at time \( k \) we already need to begin with payment \( k + 1 \).

**Example 6.3.** Derive the net premium reserve for the insurance with premium refund introduced in Example 5.2.

**Solution:**

\[
kV_x = A_{x+k:n-1} + P \cdot (IA)^1_{x+k:n-1} + P \cdot k \cdot A_{x+k:n-1} \\
- P \cdot \ddot{a}_{x+k:n-1} \]

where the premium \( P \) was calculated in Example 5.2.

**Example 6.4.** Derive the net premium reserve for the insurance with premium refund introduced in Example 5.3.

**Solution:**

We can assume again two cases. In the first case, the premium is paid during the whole deferment period.

\[
kV_x = \begin{cases} 
\quad m-k\ddot{a}_{x+k} + P_1 \cdot (IA)^1_{x+k:m-1} + P_1 \cdot k \cdot A_{x+k:m-1} \\
- P_1 \cdot \ddot{a}_{x+k:m-1} \]
\quad k = 0, 1, \ldots, m - 1 \\
\ddot{a}_{x+k}, \quad k = m, m + 1, \ldots
\end{cases}
\]
where the premium $P_1$ was calculated in Example 5.3.

Similarly, we can derive the reserve for the second case, in which the premium is paid only over $m'$ years.

\[
kV_x = \begin{cases} 
  m-k|\bar{a}_{x+k} + P_2 \cdot (IA)_{x+k:m-k} \cdot A_{x+k:m-k} - P_2 \cdot k \cdot A_{x+k:m-k} | & k = 0, 1, \ldots, m'-1 \\
  m-k|\bar{a}_{x+k} + P_2 \cdot m' \cdot m-k|A_{x+k:m-m'} - P_2 \cdot \bar{a}_{x+k:m-m'} | & k = m', m'+1, \ldots, m - 1 \\
  \bar{a}_{x+k} & k = m, m+1, \ldots
\end{cases}
\]

where also the premium $P_2$ was calculated in Example 5.3.

**Example 6.5.** Use the net premium reserve for conversion of an insurance and reduction of the sum insured.

Consider

1. endowment with duration $n$ years with sum insured $C_1$ and annual net premium paid yearly over the whole contract duration. However, premium payment ended after $n' < n$ years, but the contract continues with reduced sum insured $C_2$. Derive a formula for $C_2$.

2. $m$-years deferred standard increasing whole life annuity in advance with sum insured $C_1$ and annual net premium paid yearly over the deferment period. However, premium payment ended after $m' < m$ years, but the contract continues with reduced sum insured $C_2$. Derive a formula for $C_2$.

**Solution:**

1) Premium calculated for sum insured $C_1$ has the following form

\[ P = \frac{C_1 \cdot A_{x:n}}{\bar{a}_{x:n}}. \]

At time $n'$, when the premium payments were stopped, the value of the reserve is

\[ n'V_x = C_1 \cdot A_{x+n';n-n'} - P \cdot \bar{a}_{x+n';n-n'}. \]

This reserve turns to the net single premium for an endowment with sum insured $C_2$ for the remaining $n-n'$ years.

\[ n'V_x = NSP = C_2 \cdot A_{x+n';n-n'} \]

\[ \Downarrow \]

\[ C_2 = \frac{C_1 \cdot A_{x+n';n-n'} - P \cdot \bar{a}_{x+n';n-n'}}{A_{x+n';n-n'}} = C_1 \cdot \left( 1 - \frac{A_{x:n}}{\bar{a}_{x:n}} \cdot \frac{\bar{a}_{x+n';n-n'}}{A_{x+n';n-n'}} \right) \]

\[ ^1 \text{The current value of the net premium reserve belongs to the insured person and can be used to modify the insurance policy.} \]
Premium calculated at the beginning of the contract is

\[ P = \frac{C_1 \cdot m(I\ddot{a})_x}{\ddot{a}_{x:m}}. \]

At time \( m' \), when the premium payments were stopped, the value of the reserve is

\[ m'V_x = C_1 \cdot m-m'(I\ddot{a})_{x+m'} - P \cdot \ddot{a}_{x+m':m-m'}. \]

We can use this reserve as the net single premium for this annuity with sum insured \( C_2 \).

\[ m'V_x = NSP = C_2 \cdot m-m'(I\ddot{a})_{x+m'} \]

\[ \Downarrow \]

\[ C_2 = \frac{C_1 \cdot m-m'(I\ddot{a})_{x+m'} - P \cdot \ddot{a}_{x+m':m-m'}}{m-m'(I\ddot{a})_{x+m'}} = C_1 \cdot \left( 1 - \frac{m(I\ddot{a})_x}{\ddot{a}_{x:m}} \cdot \frac{\ddot{a}_{x+m':m-m'}}{m-m'(I\ddot{a})_{x+m'}} \right) \]

**Example 6.6.** Apply the general recursive formula for net premium reserve and decompose the premium into savings and risk components. Consider

1. endowment with duration \( n \) years and net annual premium paid over the whole contract duration,

2. \( m \)-years temporary life annuity in arrear with duration \( n \) year and annual net premium paid over the deferment period,

3. \( m \)-years deferred whole life annuity in advance with net single premium paid at once at the beginning.

**Solution:**

Recursive formula for general net-premium reserve can be written as

\[ kV_x = \sum_{j=0}^{\infty} \left( c_{k+j+1} \cdot v^{j+1} \cdot jP_{x+k} \cdot q_{x+k+1} \right) - \sum_{j=0}^{\infty} \left( \Pi_{k+j} \cdot v^j \cdot jP_{x+k} \right). \]

The premium can be decomposed to the savings premium and the risk premium using the following formulas

\[ \Pi^S_k = v \cdot k_{1}V_x - kV_x, \]

\[ \Pi^R_k = (c_{k+1} - k_{1}V_x) \cdot v \cdot q_{x+k}. \]
The endowment can be assumed in terms of the values \( c_l \) and \( \Pi_l \) as a general insurance with
\[
c_1 = \cdots = c_n = 1, \quad c_{n+1} = c_{n+2} = \cdots = 0
\]
\[
\Pi_0 = \cdots = \Pi_{n-1} = P = \frac{A_{x:m}}{\overline{a}_{x:m}}, \quad \Pi_n = -1, \quad \Pi_{n+1} = \Pi_{n+2} = \cdots = 0
\]
Therefore, the reserve can be written as
\[
kV_x = \sum_{j=0}^{n-k-1} (v^j \cdot j p_{x+k} \cdot q_{x+k+j}) - \frac{A_{x:m}}{\overline{a}_{x:m}} \sum_{j=0}^{n-k-1} (v^j \cdot j p_{x+k}) + v^{n-k} n-k p_{x+k}, \quad k = 0, \ldots, n - 1
\]
and for \( k = n \) the reserve would be \( nV_x = 1 \).

The savings and risk premiums could be calculated now with the use of the formulas stated above.

2) The temporary life annuity in arrear with duration \( n \) years deferred by \( m \) years with annual premium paid over the deferment period can be assumed in terms of the values \( c_l \) and \( \Pi_l \) as a general insurance with
\[
c_1 = c_2 = \cdots = 0
\]
\[
\Pi_0 = \cdots = \Pi_{m-1} = P = \frac{m!a_x}{\overline{a}_{x:m}}, \quad \Pi_m = 0, \quad \Pi_{m+1} = \cdots = \Pi_{m+n} = -1
\]
Thus, the reserve can be written as
\[
kV_x = \sum_{j=0}^{m-k-1} (v^j \cdot j p_{x+k}) + \sum_{j=m-k+1}^{m+n-k} (v^j \cdot j p_{x+k}).
\]
Since, for some ks these sums can be equal to zero, we can split this formula into
\[
kV_x = \begin{cases} 
- \frac{m!a_x}{\overline{a}_{x:m}} \sum_{j=0}^{m-k-1} (v^j \cdot j p_{x+k}) + \sum_{j=m-k+1}^{m+n-k} (v^j \cdot j p_{x+k}), & k = 0, \ldots, m - 1 \\
\sum_{j=\max\{0,m-k+1\}}^{m+n-k} (v^j \cdot j p_{x+k}), & k = m, \ldots, m + n
\end{cases}
\]
and the savings and risk premiums could be calculated.

3) The whole life annuity in advance deferred by \( m \) years with net single premium paid at the beginning can be assumed in terms of the values \( c_l \) and \( \Pi_l \) as a general
insurance with

\[ c_1 = c_2 = \cdots = 0 \]

\[ \Pi_0 = NSP = m! \bar{a}_x, \quad \Pi_1 = \cdots = \Pi_{m-1} = 0, \quad \Pi_m = \Pi_{m+1} = \cdots = -1 \]

Hence, the reserve can be written as

\[
kV_x = \begin{cases} 
-m! \bar{a}_x + \sum_{j=m}^{\infty} (v^j \cdot j p_x), & k = 0 \\
\sum_{j=\max\{0, m-k\}}^{\infty} (v^j \cdot j p_{x+k}), & k = 1, 2, \ldots
\end{cases}
\]

and the savings and risk premiums could be calculated.

This equation should be used for deriving the formula for savings premium. The risk premium can be then explained using the basic net annual premium and the computed savings premium.