1 Net Premium Reserve - Review

Example 1.1. Derive the net premium reserve for the term insurance contract for
n = 20 years of a person at age x = 40 when

1. the sum insured C = 10,000 CZK is constant over the whole contract life,
2. the sum insured is constant over the first 10 years and than increases by 5,000
CZK each year.

Assume that the premium is payed regularly over the whole contract duration.

Solution:

1) In the first case, the sum insured is assumed to be a constant. Thus,

\[ P = \frac{10,000 \cdot A_{40:x}^{1}}{\ddot{a}_{40:20}}. \]

The reserve at time \( k \) can be written as follows

\[ kV_{40} = 10,000 \cdot A_{40+k:20-k}^{1} - P \cdot \ddot{a}_{40+k:20-k}^{1} \quad k = 0, 1, ..., 19 \]

2) When assuming the adjustment of increasing the sum insured by 5,000 each year
after first 10 years, the premium and the reserve must be changed to

\[ P = \frac{10,000 \cdot A_{40:20}^{1} + 5,000 \cdot 10 \cdot IA_{40:20}^{1}}{\ddot{a}_{40:20}} \]
and
\[
V_{40} = \begin{cases} 
10,000 \cdot A_{40+k:20-|k|}^1 + 5,000 \cdot 10^{-k}(IA)_{40+k:10}^1 - P \cdot \ddot{a}_{40+k:20-|k|}, & k = 0, \ldots, 9 \\
10,000 \cdot A_{40+k:20-|k|}^1 + 5,000 \cdot \left[(k - 10) \cdot A_{40+k:20-|k|}^1 + (IA)_{40+k:20-|k|}^1\right] - P \cdot \ddot{a}_{40+k:20-|k|}, & k = 10, \ldots, 19
\end{cases}
\]

Example 1.2. Consider a person at age \(x = 35\) who has signed an insurance contract for life annuity in advance deferred to age 65 with monthly payments of CZK 5000. Moreover, assume that the premium refund agreement is active during the deferment period. Derive the formula for the net premium reserve.

Solution:

First of all, it is worth reminding (when assuming a life annuity, but the concept is similar also for the other usual types of insurance contracts) that \(\ddot{a}_x^{(m)}\) is a sign for a whole life annuity whose payments are of size \(\frac{1}{m}\) and are made \(m\)-times a year.

Since payments are made monthly, to obtain the sum insured, one has to multiply 5,000 by 12. We can write down the total loss and then derive the formula for the premium.

\[
L = \begin{cases} 
12 \cdot (K + S^{(12)}) \cdot P \cdot v^{K+S^{(12)}} - P \cdot \sum_{k=0}^{12} v^{K+S^{(12)}-1} v^{\frac{k}{12}}, & K = 0, 1, \ldots, 29 \\
5,000 \cdot \sum_{k=360}^{12} v^{\frac{K+S^{(12)}-1}{12}} - P \cdot \sum_{k=0}^{359} v^{\frac{k}{12}}, & K = 30, 31, \ldots
\end{cases}
\]

Therefore,

\[
P = \frac{5,000 \cdot 30 \cdot \ddot{a}_{35}^{(12)}}{\ddot{a}_{35.35}^{(12)} - (IA)^{12})_{35.35}},
\]

is the value of monthly premium.

Values of the net premium reserve will be examined at the ends of years.

\[
V_{35} = \begin{cases} 
60,000 \cdot 30-k(\ddot{a}_{35+k}^{(12)} + 12 \cdot P \cdot (IA)^{12}_{35+k:30-|k|}) + 12 \cdot P \cdot k \cdot A_{35+k:30-|k|}^1 - 12 \cdot P \cdot \ddot{a}_{35+k:30-|k|}^{(12)} & k = 0, 1, \ldots, 29 \\
60,000 \cdot \ddot{a}_{35+k}^{(12)}, & k = 30, 31, \ldots
\end{cases}
\]

Remark: Be careful with the standard increasing term insurance and life annuity because, when dealing with reserves, a special term must be added.

Example 1.3. Use the net premium reserve for conversion of an insurance and reduction of the sum insured. Consider \(m\)-years deferred temporary life annuity in arrear with duration \(n\) years, sum insured \(C_1\) and annual net premium paid yearly over the deferment period. However, premium payment ended after \(m' < m\) years, but the contract continues with reduced sum insured \(C_2\). Derive an explicit formula for \(C_2\).
the following formulas

\[ P = \frac{C_1 \cdot m |a_x : m|}{\tilde{a}_x : m} \]

At time \( m' \), when the premium payments were stopped, the reserve is

\[ m'V_x = C_1 \cdot m-m'|a_{x+m'} : m| - P \cdot \tilde{a}_{x+m'; m-m}| \]

This reserve can be used as a single payment for the net single premium of the same insurance with the sum insured \( C_2 \) for the remaining years.

\[ m'V_x = NSP = C_2 \cdot m-m'|a_{x+m'} : m| \]

\[ C_2 = \frac{C_1 \cdot m-m'|a_{x+m'} : m| - P \cdot \tilde{a}_{x+m'; m-m}|}{m-m'|a_{x+m'} : m|} = C_1 \cdot \left( 1 - \frac{m |a_x : m| \cdot \tilde{a}_{x+m'; m-m}|}{m-m'|a_{x+m'} : m|} \right) \]

**Example 1.4.** Consider \( m \)-years deferred term insurance with duration \( n \) years and net annual premium paid over first \( m_1 < m \). Apply the general recursive formula for net premium reserve and decompose the premium into savings and risk components.

**Solution:**

Recursive formula for general net-premium reserve can be written as

\[ kV_x = \sum_{j=0}^{\infty} \left( c_{k+j+1} \cdot v^{j+1} \cdot jP_{x+k} \cdot q_{x+k+j} \right) - \sum_{j=0}^{\infty} \left( \Pi_{k+j} \cdot v^j \cdot jP_{x+k} \right) \]

The premium can be decomposed to the savings premium and the risk premium using the following formulas

\[ \Pi_k = v \cdot k+1V_x - kV_x, \]

\[ \Pi_k = (c_{k+1} - k+1V_x) \cdot v \cdot q_{x+k}. \]

Considered \( m \)-years deferred term insurance with duration \( n \) years and net annual premium paid over first \( m_1 < m \) can be assumed in terms of the values \( c_l \) and \( \Pi_l \) as a general insurance with

\[ c_1 = \ldots = c_m = 0, \quad c_{m+1} = \ldots = c_{m+n} = 1, \quad c_{m+n+1} = c_{m+n+2} = \ldots = 0, \]

\[ \Pi_0 = \ldots = \Pi_{m_1-1} = P = \frac{m \cdot A^{1}_{x+m|}}{\tilde{a}_x : m|}, \quad \Pi_{m_1} = \Pi_{m_1+1} = \ldots = 0 \]

and these values would be assumed in the recursive formula for the reserve stated above. When obtaining the values \( kV_x \), one can also calculate the savings and risk premiums.

The standard form of the reserve is as follows

\[
\begin{align*}
    kV_x &= \begin{cases}
    m-k|A_{x+k : m|} - P \cdot \tilde{a}_{x+k ; m_1-k|} & \text{if } k = 0, \ldots, m_1 - 1 \\
    m-k|A_{x+k : m|} & \text{if } k = m_1, \ldots, m - 1 \\
    A_{x+k : m+n-m-k|} & \text{if } k = m_1, \ldots, m + n - 1
    \end{cases}
\end{align*}
\]
2 Net Premium Reserve II - Continuous Model

Example 2.1. Consider the continuous (time) model for the net premium reserve calculation. Show that

\[
\mathbb{E} \left[ v^T \cdot V(T) \right] = \mathbb{E} \left[ \int_0^T \Pi^s(t) \cdot v^t \, dt \right],
\]
i.e. that the expected value of discounted net premium reserve at the moment of death is equal to the expected value of discounted savings component of the premium cumulated until the moment of death.

Solution:

We assume the overall loss as

\[
L = c(T) \cdot v^T - \int_0^T \Pi(t) \cdot v^t \, dt.
\]

Using the principle of equivalence \( \mathbb{E}L = 0 \), we can prove the given relation as follows:

\[
0 = \mathbb{E}L = \mathbb{E} \left[ c(T) \cdot v^T \right] - \mathbb{E} \left[ \int_0^T \Pi(t) \cdot v^t \, dt \right]
= \mathbb{E} \left[ c(T) \cdot v^T \right] - \mathbb{E} \left[ \int_0^\infty \Pi(t) \cdot v^t \cdot I[T > t] \, dt \right]
= \int_0^\infty c(t) \cdot v^t \cdot t^p x \cdot \mu x \cdot dt - \int_0^\infty \Pi(t) \cdot v^t \cdot t^p x \, dt
= \int_0^\infty \left[ V(t) \cdot \mu x + \delta \cdot V(t) \right] \cdot v^t \cdot t^p x \, dt
= \int_0^\infty V(t) \cdot v^t \cdot t^p x \cdot \mu x + \delta \cdot V(t) \cdot v^t \cdot t^p x \, dt
= \mathbb{E} \left[ v^T \cdot V(T) \right] - \mathbb{E} \left[ \int_0^T \Pi^s(t) \cdot v^t \, dt \right].
\]

We used the decomposition of the premium rate \( \Pi(t) \) into the saving component and the risk component, which in the continuous model can be written as

\[
\Pi(t) = \underbrace{V'(t)}_{\Pi^r(t)} - \delta \cdot V(t) + \underbrace{(c(t) - V(t)) \cdot \mu x + t}_{\Pi^s(t)}.
\]

Example 2.2. Consider the continuous (time) model for the net premium reserve calculation where the technical gain at time \( t \) can be decomposed into savings and risk component as

\[
G(t, t + dt) = G^s(t, t + dt) + G^r(t, t + dt),
\]
\[
G^s(t, t + dt) = (\delta(t) - \delta) \cdot V(t) \, dt,
\]
\[
G^r(t, t + dt) = \begin{cases} -(c(t) - V(t)), & t \leq T < t + dt, \\ \Pi'(t) \, dt, & T \geq t + dt. \end{cases}
\]
Show that the technical loss can be approximated by minus risk component of the technical gain, i.e.

\[ L(t, t + dt) \approx -G^r(t, t + dt). \]

Solution:

Firstly, we have to define the technical loss. It is an analogy to the technical gain taken with negative sign but without real force of interest. Only the technical interest rate (TIR) is considered.

Therefore, the technical loss can be written as

\[ L(t, t + dt) = \begin{cases} c(t) - V(t), & t \leq T < t + dt, \\ \frac{V(t + dt)}{1 + \delta dt} - V(t) - \frac{\Pi(t) dt}{1 + \delta dt}, & T \geq t + dt. \end{cases} \]

In the first case, when \( t \leq T < t + dt \), it is obvious that the technical loss \( L(t, t + dt) \) can not only be approximated by \( -G^r(t, t + dt) \), but there is even an equivalence.

In the second case, when \( T \geq t + dt \), we can rewrite \( L(t, t + dt) \) as

\[
L(t, t + dt) = \frac{1}{1 + \delta dt} \left( V(t, t + dt) - V(t) - \delta \cdot V(t) dt - \Pi(t) dt \right) \\
= \frac{1}{1 + \delta dt} \left( V'(t) - \delta \cdot V(t) dt - \Pi(t) dt \right) \\
= \frac{1}{1 + \delta dt} \left( \Pi'(t) dt - \Pi(t) dt \right) = -\frac{\Pi'(t) dt}{1 + \delta dt}.
\]

Finally, if we assume \( \delta dt \approx 0 \) then \( L(t, t + dt) \approx -\Pi'(t) dt = -G^r(t, t + dt) \).