NMFP407 Mathematics of Life Insurance 1

Collection of solved examples

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1 Demography

Notation

- T_x – random remaining lifetime of a person at age x,
- ${}_tq_x = P(T_x \leq t)$ – probability of dying,
- ${}_tp_x = 1 - {}_tq_x$ – probability of surviving,
- μ_x – force of mortality.

Do not forget that many of calculations in this collection use the fundamental assumption $\boldsymbol{\mathcal{L}(T_{x+t}) = \mathcal{L}(T_x - t | T_x > t)}$.

Example 1.1. *Let $q_x = 0.05$ under basic level of the force of mortality μ_{x+t}. Assume that $\mu'_{x+t} = \mu_{x+t} + c$ and estimate c for which $q'_x = 0.07$.*

Solution:

For μ_{x+t}:

$$1 - q_x = \exp\left(-\int_0^1 \mu_{x+s}\, ds\right)$$

For μ'_{x+t}:

$$\begin{aligned} 1 - q'_x &= \exp\left(-\int_0^1 \mu'_{x+s}\, ds\right) = \exp\left(-\int_0^1 \mu_{x+s} + c\, ds\right) \\ &= \exp\left(-\int_0^1 \mu_{x+s}\, ds\right) \cdot \exp(-c) = (1 - q_x) \cdot \exp(-c) \end{aligned}$$

Therefore, we get

$$e^{-c} = \frac{1 - q'_x}{1 - q_x} \Rightarrow c = -\ln\left(\frac{1 - q'_x}{1 - q_x}\right) = -\ln\left(\frac{1 - 0.07}{1 - 0.05}\right) = 0.021.$$

Example 1.2. *Let*

$${}_tp_x = \frac{100 - x - t}{100 - x},\ 0 \leq x \leq 100,\ 0 \leq t < 100 - x.$$

Compute μ_{45}.

Solution:

We can calculate μ_{45} using the following equation

$$\mu_{x+t} = -\frac{d}{dt} \ln\left({}_t p_x\right),$$

evaluated at $t = 45 - x$.

The desired quantity μ_{45} is thus calculated as follows:

$$\begin{aligned}\mu_{45} &= -\frac{d}{dt} \ln\left(\frac{100-x-t}{100-x}\right)\Bigg|_{t=45-x} = -\frac{d}{dt}\left[\ln(100-x-t) - \ln(100-x)\right]\Bigg|_{t=45-x} \\ &= -\frac{d}{dt}\ln(100-x-t)\Bigg|_{t=45-x} = -\frac{1}{100-x-t}\cdot(-1)\Bigg|_{t=45-x} \\ &= \frac{1}{100-x-(45-x)} = \frac{1}{55}.\end{aligned}$$

Example 1.3. *Consider a non-smoker with the force of mortality μ_x and remaining lifetime T_x and a smoker with $\mu'_x = c \cdot \mu_x$, $c > 0$ and T'_x whose lives are independent. Derive an explicit formula for the probability that the smoker will live longer than the nonsmoker, i.e.*

$$P(T'_x > T_x).$$

Solution:

First of all, the joint distribution of (T_x, T'_x) must be found. According to the fact that the lives are independent, the joint density is just the product of marginal densities

$$\begin{aligned}f(t) &= {}_t p_x \cdot \mu_{x+t}, \\ f'(s) &= {}_s p'_x \cdot \mu'_{x+s} = {}_s p'_x \cdot c \cdot \mu_{x+s} = \left({}_s p_x\right)^c \cdot c \cdot \mu_{x+s},\end{aligned}$$

where we used

$${}_s p'_x = \exp\left(-\int_0^s c\cdot\mu_{x+u}\, du\right) = \left[\exp\left(-\int_0^s \mu_{x+u}\, du\right)\right]^c = \left({}_s p_x\right)^c.$$

The joint density (s for smoker and t for non-smoker) is then

$$f(t,s) = \mu_{x+t}\cdot {}_t p_x \cdot c \cdot \mu_{x+s} \cdot \left({}_s p_x\right)^c.$$

The sought probability that the smoker will survive the non-smoker is obtained as follows:

$$\begin{aligned}
P(T_x' > T_x) &= \iint\limits_{s>t} f(t,s)\,ds\,dt = \int_0^\infty \int_t^\infty f(t,s)\,ds\,dt \\
&= \int_0^\infty \int_t^\infty f(t)\cdot f'(s)\,ds\,dt \\
&= \int_0^\infty f(t)\cdot\left(\int_t^\infty f'(s)\,ds\right)\,dt \\
&= \int_0^\infty f(t)\cdot(1-F'(t))\,\,dt = \int_0^\infty \mu_{x+t}\cdot {}_tp_x\cdot {}_tp_x'\,dt \\
&= \int_0^\infty \mu_{x+t}\cdot {}_tp_x\cdot ({}_tp_x)^c\,\,dt = \underbrace{\int_0^\infty \mu_{x+t}\cdot ({}_tp_x)^{c+1}\,\,dt}_{:=I} \\
&\overset{\text{Per Partes}}{=} \begin{bmatrix} u = ({}_tp_x)^c & v' = {}_tp_x\cdot\mu_{x+t} \\ u' = c\cdot({}_tp_x)^{c-1}\cdot\frac{d}{dt}{}_tp_x = -c\cdot({}_tp_x)^c\cdot\mu_{x+t} & v = -{}_tp_x \end{bmatrix} \\
&= \underbrace{\left[-\,({}_tp_x)^{c+1}\right]_0^\infty}_{=0-(-1)=1} - \underbrace{\int_0^\infty c\cdot\mu_{x+t}\cdot({}_tp_x)^{c+1}\,\,dt}_{:=c\cdot I}\,.
\end{aligned}$$

We are in a situation where it should hold

$$P(T_x' > T_x) = I = 1 - c\cdot I.$$

The last step is to calculate I:

$$I = 1 - c\cdot I \Rightarrow I = \frac{1}{c+1} = P(T_x' > T_x).$$

Example 1.4. *Assume that $\mu_{x+t} = \mu_x$ for all $t \in [0,1]$. Let ${}_1q_x = q_x = 0.16$. Estimate t for which it holds ${}_tp_x = 0.95$.*

Solution:

Under the assumption of constant force of mortality, it holds ${}_tp_x = (p_x)^t$.

$$\begin{aligned}
{}_tp_x = (p_x)^t = (1-q_x)^t &\Rightarrow \ln {}_tp_x = t\cdot\ln(1-q_x) \\
\Rightarrow t = \frac{\ln {}_tp_x}{\ln(1-q_x)} &= \frac{\ln 0.95}{\ln(1-0.16)} = 0.294
\end{aligned}$$

Example 1.5. *Consider $u \in [0,1]$. Show that under*

1. *the assumption of linearity, it holds*

$${}_{x+u}p_0 = (1-u)\,{}_xp_0 + u\,{}_{x+1}p_0,$$

2. *the assumption of constant force of mortality, it holds*

$$ {}_{x+u}p_0 = \left({}_xp_0\right)^{1-u} \cdot \left({}_{x+1}p_0\right)^u, $$

3. *the assumption of linearity II (Balducci ass.), it holds*

$$ \frac{1}{{}_{x+u}p_0} = \frac{1-u}{{}_xp_0} + \frac{u}{{}_{x+1}p_0}. $$

Solution:

1)

$$ \begin{aligned} {}_{x+u}p_0 &= {}_xp_0 \cdot {}_up_x = (1 - {}_xq_0) \cdot (1 - {}_uq_x) \overset{\text{Ass.}}{=} (1 - {}_xq_0) \cdot (1 - u \cdot q_x) \\ &= 1 - {}_xq_0 - u \cdot q_x + u \cdot {}_xq_0 \cdot q_x = {}_xp_0 - u \cdot (1 - p_x) \cdot {}_xp_0 \\ &= {}_xp_0 - u \cdot {}_xp_0 + u \cdot {}_xp_0 \cdot p_x = {}_xp_0 - u \cdot {}_xp_0 + u \cdot {}_{x+1}p_0 \\ &= (1-u) \cdot {}_xp_0 + u \cdot {}_{x+1}p_0 \end{aligned} $$

2)

$$ {}_up_x = \exp\left(-\int_0^u \mu_{x+y}\, dy\right) \overset{\text{Ass.}}{=} \exp\left(-\int_0^u \mu_x\, dy\right) = \exp\left(-\mu_x \cdot u\right) = (p_x)^u $$

$$ {}_{x+u}p_0 = {}_xp_0 \cdot {}_up_x = \left({}_xp_0\right)^{1+u-u} \cdot (p_x)^u = \left({}_xp_0\right)^{1-u} \cdot \left({}_{x+1}p_0\right)^u $$

3)

$$ \begin{aligned} {}_{x+1}p_0 = {}_{x+u}p_0 \cdot {}_{1-u}p_{x+u} \Rightarrow \frac{1}{{}_{x+u}p_0} &= \frac{{}_{1-u}p_{x+u}}{{}_{x+1}p_0} = \frac{1 - {}_{1-u}q_{x+u}}{{}_{x+1}p_0} \overset{\text{Ass.}}{=} \frac{1-(1-u)\cdot q_x}{{}_{x+1}p_0} \\ &= \frac{1-(1-u)\cdot(1-p_x)}{{}_{x+1}p_0} = \frac{p_x + u - u \cdot p_x}{{}_{x+1}p_0} = \frac{(1-u)\cdot p_x}{{}_{x+1}p_0} + \frac{u}{{}_{x+1}p_0} \\ &= \frac{(1-u)\cdot p_x}{p_x \cdot {}_xp_0} + \frac{u}{{}_{x+1}p_0} = \frac{1-u}{{}_xp_0} + \frac{u}{{}_{x+1}p_0} \end{aligned} $$

Example 1.6. *Consider the decomposition of T_x to curtate K_x and fractional S_x remaining lifetime. Under the Balducci assumption, derive an explicit formula for the conditional probability*

$$ P(S_x \le u | K_x = k), \ u \in [0,1]. $$

Solution:

$$ \begin{aligned} P(S_x \le u | K_x = k) &= \frac{P(S_x \le u, K_x = k)}{P(K_x = k)} = \frac{P(k < T_x \le k+u)}{P(k \le T_x < k+1)} = \frac{{}_{k+u}q_x - {}_kq_x}{{}_{k+1}q_x - {}_kq_x} \\ &= \frac{{}_kp_x \cdot {}_uq_{x+k}}{{}_kp_x \cdot q_{x+k}} = \frac{1 - {}_up_{x+k}}{q_{x+k}} \end{aligned} $$

Taking into account the following relationship

$$p_{x+k} = {}_u p_{x+k} \cdot {}_{1-u} p_{x+k+u} \Rightarrow {}_u p_{x+k} = \frac{p_{x+k}}{{}_{1-u} p_{x+k+u}},$$

we can write

$$\begin{aligned} P(S \leq u | K = k) &= \frac{1 - \frac{p_{x+k}}{{}_{1-u} p_{x+k+u}}}{q_{x+k}} = \frac{1 - \frac{1-q_{x+k}}{1-(1-u) \cdot q_{x+k}}}{q_{x+k}} = \frac{\frac{u \cdot q_{x+k}}{1-(1-u) \cdot q_{x+k}}}{q_{x+k}} \\ &= \frac{u}{1-(1-u) \cdot q_{x+k}}. \end{aligned}$$

We can notice that the expression for the conditional distribution of S_x depends on k. For this reason, the random variables S_x and K_x are not independent under the Balducci assumption.

2 Financial Mathematics

Example 2.1. *Mrs Y will need 150,000 CZK after 5 years to pay for a transfer of the "flat rights". The bank offers a term account with nominal interest rate 4.4% credited quarterly. How much money must Mrs Y save to cover the necessary amount for the transfer?*

Solution:

$FV = 150{,}000;\ n = 5;\ m = 4;\ i^{(m)} = 4.4\%;\ PV = ?$

$$PV = FV \cdot \left(\frac{1}{1+\frac{i^{(m)}}{m}}\right)^{m \cdot n} = 150{,}000 \cdot \left(\frac{1}{1+\frac{0.044}{4}}\right)^{4 \cdot 5} = 120{,}522.5$$

Example 2.2. *Mr Z bought a house for 3,600,000 CZK using mortgage loan of amount 2,500,000 CZK. He will repay 16,000 CZK monthly in arrear over 25 years. What is the nominal annual interest rate which the bank offered to Mr Z?*

Solution:

$PV = 2{,}500{,}000;\ P = 16{,}000;\ m = 12;\ n = 25;\ i^{(m)} = ?$

The formula for the calculation of the present value:

$$PV = P \cdot \sum_{t=1}^{m \cdot n} \left(\frac{1}{1+\frac{i^{(m)}}{m}}\right)^{t} = P \cdot m \cdot a_{\overline{n}|}^{(m)} = P \cdot m \cdot \frac{1 - v^n}{i^{(m)}},$$

where

$$v^n = \left(\frac{1}{1+\frac{i^{(m)}}{m}}\right)^{m \cdot n}.$$

Therefore, we need to solve the equation

$$2{,}500{,}000 = 16{,}000 \cdot 12 \cdot \frac{1 - v^{25}}{i^{(12)}}.$$

The solution is then $i^{(12)} = 0.059 = 5.9\%$.

Be careful with the notation. The term $a_{\overline{n}|}^{(m)}$ denotes an annuity in arrear payable m-thly. However, the payments are of size $1/m$. Thus, it was necessary to multiply $a_{\overline{n}|}^{(m)}$ by m in our case.

Example 2.3. *Mrs Y makes deposits of 100 at time 0, and x at time 3. The fund grows at a force of interest*

$$\delta_t = \frac{t^2}{100},\ t > 0.$$

Let the amount of interest earned from time 3 to 6 is also equal to x. Calculate x.

Solution:

$C_0 = 100;\ C_3 = x;\ \delta_t = \frac{t^2}{100}\ t > 0$

At time 3, after making a deposit, we have

$$FV_3 = C_0 \cdot e^{\{\int_0^3 \delta_t dt\}} + C_3 = 100 \cdot e^{\{\int_0^3 \frac{t^2}{100} dt\}} + x,$$

where

$$\int_0^3 \frac{t^2}{100} dt = \frac{1}{100} \cdot \left[\frac{t^3}{3}\right]_0^3 = \frac{1}{300} \cdot (27 - 0) = \frac{27}{300} = \frac{9}{100},$$

and therefore

$$FV_3 = 100 \cdot e^{\frac{9}{100}} + x.$$

At time 6, there is

$$FV_6 = FV_3 \cdot e^{\{\int_3^6 \delta_t dt\}}$$

in the fund, where

$$\int_3^6 \frac{t^2}{100} dt = \frac{1}{100} \cdot \left[\frac{t^3}{3}\right]_3^6 = \frac{1}{300} \cdot (216 - 27) = \frac{189}{300} = \frac{63}{100}.$$

The interest gained between times 3 and 6 should be equal to x and thus it ought to hold

$$FV_6 - FV_3 = x \Rightarrow FV_3 \cdot e^{\frac{63}{100}} - FV_3 = x \Rightarrow FV_3 \cdot (e^{\frac{63}{100}} - 1) = x$$

$$\Rightarrow (100e^{\frac{9}{100}} + x) \cdot (e^{\frac{63}{100}} - 1) = x.$$

After solving this equation, we obtain $x = 784.59$.

Example 2.4. *Mr X wants to borrow 150,000\$. He would like to repay this loan in 2 years by periodic semiannual payments. Bank offers the nominal interest rate 6.9%. Mr X has 45,000\$ on his account where the interest is credited monthly under nominal interest rate 2.5%. He can save 40,000\$ every half a year from his salary.*

a) *What is the account balance after a half of year?*

b) *How much money should Mr X hold at the beginning to cover the loan payments?*

Solution:

$PV = 150{,}000;\ m = 2;\ i^{(m)} = 6.9\%;\ n = 2$
$A_0 = 45{,}000;\ p = 12;\ j^{(p)} = 2.5\%;\ S = 40{,}000$

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b)

$$PV = R \cdot \sum_{t=1}^{m \cdot n} \left(\frac{1}{1 + \frac{i^{(m)}}{m}}\right)^t \Rightarrow R = \frac{PV}{\sum_{t=1}^{m \cdot n} \left(\frac{1}{1+\frac{i^{(m)}}{m}}\right)^t} = \frac{150{,}000}{\sum_{t=1}^{2 \cdot 2} \left(\frac{1}{1+\frac{0.069}{2}}\right)^t} = 40{,}789$$

$A_0 = 45{,}000$

$$A_{1/2} = A_0 \cdot \left(1 + \frac{j^{(p)}}{p}\right)^{p/2} - R + S = 45{,}000 \left(1 + \frac{0.025}{12}\right)^{6} - 40{,}789 + 40{,}000 = 44{,}776$$

c)

$$A_2^* = 0 = \left[\left[\left[A_0^* \cdot \left(1 + \frac{j^{(p)}}{p}\right)^{p/2} + S - R\right] \cdot \left(1 + \frac{j^{(p)}}{p}\right)^{p/2} + S - R\right] \cdot \left(1 + \frac{j^{(p)}}{p}\right)^{p/2} + S - R\right] \cdot \left(1 + \frac{j^{(p)}}{p}\right)^{p/2} + S - R$$

Our aim is to express A_0^*. The previous equation can be adjusted to the following equation

$$A_0^* = \sum_{t=1}^{4} \frac{R - S}{\left(1 + \frac{j^{(p)}}{p}\right)^{\frac{p \cdot t}{2}}} = (40{,}789 - 40{,}000) \sum_{t=1}^{4} \left(\frac{1}{1 + \frac{0.025}{12}}\right)^{6 \cdot t} = 3{,}059.$$

Example 2.5. *Mr X sold his car for 200,000 CZK. He paid this amount to his account where the interest is credited monthly with nominal interest rate 2.8%. He decided to buy a new car 9 months after. There was a necessary advance payment of 50,000 CZK taken from the account. Then, the debt was repaid by payments 9,000 CZK monthly. How long the money on the account can cover these payments?*

Solution:

$C_0 = 200{,}000;\ m = 12;\ i^{(m)} = 2.8\%;\ C_{9/12} = -50{,}000;\ P = 9{,}000;\ n = ?$

$$C_0 \left(1 + \frac{i^{(m)}}{m}\right)^{9} + C_{9/12} = P \cdot m \cdot a_{\overline{n}|}^{(m)}$$

Now we need to express the term $a_{\overline{n}|}^{(m)}$.

$$a_{\overline{n}|}^{(m)} = \frac{C_0 \left(1 + \frac{i^{(m)}}{m}\right)^{9} + C_{9/12}}{P \cdot m} \Rightarrow a_{\overline{n}|}^{(12)} = \frac{200{,}000 \left(1 + \frac{0.028}{12}\right)^{9} - 50{,}000}{9{,}000 \cdot 12} \doteq 1.43$$

Assuming

$$a_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}} = \frac{1 - \left(\frac{1}{1 + \frac{i^{(m)}}{m}}\right)^{m \cdot n}}{i^{(m)}},$$

we can express n as

$$n = \frac{\ln\left(1 - i^{(m)} \cdot a_{\overline{n}|}^{(m)}\right)}{m \cdot \ln\left(\frac{1}{1 + \frac{i^{(m)}}{m}}\right)} = \frac{\ln\left(1 - 0.028 \cdot 1.43\right)}{12 \cdot \ln\left(\frac{1}{1 + \frac{0.028}{12}}\right)} = 1.46.$$

Since n is equal to 1.46 years, which is 17.5 months, the money can cover the payments for 17 months.

Example 2.6. *Calculate the net present value for a bond with the nominal value 1,000 \$, annual coupon rate 6% and term to maturity 3 years. Consider a yield curve with annual spot/forward interest rates 3, 4, 5 %.*

Solution:

$N = 1{,}000;\ c = 6\%;\ n = 3;\ i_1 = 3\%;\ i_2 = 4\%;\ i_3 = 5\%;\ PV = ?$

$$C = N \cdot c = 1{,}000 \cdot 0.06 = 60$$

$$\begin{aligned} PV &= \frac{C}{1+i_1} + \frac{C}{(1+i_1)(1+i_2)} + \frac{N+C}{(1+i_1)(1+i_2)(1+i_3)} \\ &= \frac{60}{1+0.03} + \frac{60}{(1+0.03)(1+0.04)} + \frac{1{,}000+60}{(1+0.03)(1+0.04)(1+0.05)} \\ &= 1{,}056.69 \end{aligned}$$

3 Capital Life Insurance

3.1 Capital life insurance with constant sum insured

Example 3.1. *Consider a fund of 100 independent lives of age x with contracted whole life insurance with sum insured SI. Derive the amount which is sufficient to cover the future liabilities with 95 % probability. Consider*

1. *whole life insurance with SI payable at the end of the year of death (assume that A_x is known),*
2. *whole life insurance with SI=100,000 CZK payable at the moment of death under constant force of interest $\delta = 0.06$ and constant force of mortality $\mu = 0.04$.*

Solution:

a)

Let us denote:
H ... random variable corresponding to net present value of all future payments
h ... fund to cover liabilities

Our aim is to determine h. Since, it should hold $P(H \leq h) = 0.95$, with the use of Central Limit Theorem we obtain

$$P(H \leq h) = P\left(\frac{H - \mathbb{E}H}{\sqrt{\mathrm{Var}H}} \leq \frac{h - \mathbb{E}H}{\sqrt{\mathrm{Var}H}}\right) \xrightarrow{n \to \infty} \Phi\left(\frac{h - \mathbb{E}H}{\sqrt{\mathrm{Var}H}}\right) = 0.95,$$

where Φ is the distribution function of the standard normal distribution.

Solution of the last equation is then as follows:

$$h = \mathbb{E}H + u_{0.95} \cdot \sqrt{\mathrm{Var}H},$$

where $u_{0.95}$ represents the 0.95 quantile of the normal distribution which is equal to 1.64.

The random variable H has the following form:

$$H = SI \cdot \sum_{i=1}^{100} v^{K_i+1}.$$

Therefore, the expected value of H is

$$\mathbb{E}H = \mathbb{E}\left(SI \cdot \sum_{i=1}^{100} v^{K_i+1}\right) = SI \cdot \left(\sum_{i=1}^{100} \mathbb{E}(v^{K_i+1})\right) = 100 \cdot SI \cdot \mathbb{E}(v^{K+1})$$
$$= 100 \cdot SI \cdot A_x,$$

and the variance is

$$\mathrm{Var}H = \mathrm{Var}\left(SI \cdot \sum_{i=1}^{100} v^{K_i+1}\right) = SI^2 \cdot \left(\sum_{i=1}^{100} \mathrm{Var}(v^{K_i+1})\right) = 100 \cdot SI^2 \cdot \mathrm{Var}(v^{K+1}).$$

In the previous equations, the fact that K_is (representing the lifes) are independent and identically distributed as a general random variable K, was used.

The only unknown quantity needed to calculate h is $\mathrm{Var}(v^{K+1})$. Since,

$$\mathrm{Var}(v^{K+1}) = \mathbb{E}(v^{K+1})^2 - \left[\mathbb{E}(v^{K+1})\right]^2 = \mathbb{E}(v^{K+1})^2 - (A_x)^2,$$

we have to calculate $\mathbb{E}(v^{K+1})^2$, which is done as follows:

$$\mathbb{E}(v^{K+1})^2 = \sum_{k=0}^{\infty}(v^{2k+2} \cdot {}_kp_x \cdot q_{x+k}).$$

There exists also an approximative way, how to obtain $\mathrm{E}(v^{K+1})^2$. The discount factor v^2 can be rewrite as $(\frac{1}{1+i})^2 = \frac{1}{1+2i+i^2}$. The quantity i^2 is very small and thus $v^2 \approx \frac{1}{1+2i} = \frac{1}{1+i_*} = v_*$. The consequence of this approximation is that

$$\mathbb{E}(v^{K+1})^2 = \sum_{k=0}^{\infty}(v_*^{k+1} \cdot {}_kp_x \cdot q_{x+k}) = A_{x,*},$$

where $A_{x,*}$ is A_x calculated using $i_* = 2i$.

The very last step would be inserting these things into $h = \mathbb{E}H + u_{0.95} \cdot \sqrt{\mathrm{Var}H}$.

b)

Now, we will assume that the payment is paid immediately at the moment of death ($Z = v^T$).

The relationship for the fund h is the same as in the previous case. However, the random variable H has the following form:

$$H = 10^5 \cdot \sum_{i=1}^{100} v^{T_i}.$$

We will continue with calculations of $\mathbb{E}H$ and $\mathrm{Var}H$.

$$\mathbb{E}H = 10^5 \cdot 100 \cdot \mathbb{E}(v^T) = 10^7 \cdot \mathbb{E}(v^T),$$
$$\mathbb{E}(v^T) = \int_0^{\infty} v^t \cdot {}_tp_x \cdot \mu_{x+t}\, dt.$$

Now, we will use the assumption of constant force of mortality $\mu_x = \mu$.

$${}_tp_x = e^{\{-\int_0^t \mu_{x+y}\, dy\}} = e^{\{-\int_0^t \mu\, dy\}} = e^{-t\cdot\mu},$$

$$\mathbb{E}(v^T) = \int_0^\infty (1+i)^{-t} \cdot e^{-t\cdot\mu} \cdot \mu \, dt = \int_0^\infty e^{-\delta\cdot t} \cdot e^{-t\cdot\mu} \cdot \mu \, dt = \mu \cdot \int_0^\infty e^{-t\cdot(\delta+\mu)} \, dt$$
$$= \mu \cdot \left[-\frac{1}{\delta+\mu} \cdot e^{-t\cdot(\delta+\mu)} \right]_0^\infty = 0.04 \cdot (-\frac{1}{0.06+0.04}) \cdot (0-1) = \frac{0.04}{0.1} = 0.4.$$

Therefore, the expected value of H is

$$\mathbb{E}H = 10^7 \cdot \mathbb{E}(v^T) = 10^7 \cdot 0.4 = 4 \cdot 10^6.$$

Similarly with the variance:

$$\mathrm{Var}H = (10^5)^2 \cdot 100 \cdot \mathrm{Var}(v^T) = 10^{12} \cdot \left(\mathbb{E}(v^T)^2 - [\mathbb{E}(v^T)]^2 \right),$$

$$\mathbb{E}(v^T)^2 = \mathbb{E}(v^{2\cdot T}) = \int_0^\infty v^{2\cdot t} \cdot {}_tp_x \cdot \mu_{x+t} \, dt = \int_0^\infty (1+i)^{-2\cdot t} \cdot e^{-t\cdot\mu} \cdot \mu \, dt$$
$$= \int_0^\infty e^{-2\cdot\delta\cdot t} \cdot e^{-t\cdot\mu} \cdot \mu \, dt = \int_0^\infty e^{-t\cdot(2\cdot\delta+\mu)} \cdot \mu \, dt = \mu \cdot \left[-\frac{1}{2\cdot\delta+\mu} \cdot e^{-t\cdot(2\cdot\delta+\mu)} \right]_0^\infty$$
$$= \mu \cdot \frac{1}{2\cdot\delta+\mu} = 0.04 \cdot \frac{1}{2\cdot 0.06+0.04} = \frac{0.04}{0.16} = 0.25.$$

Therefore, the variance of H is

$$\mathrm{Var}H = 10^{12} \cdot \left(\mathbb{E}(v^T)^2 - [\mathbb{E}(v^T)]^2 \right) = 10^{12} \cdot (0.25 - 0.4^2) = 10^{12} \cdot 0.09 = 9 \cdot 10^{10}.$$

If we substitute the calculated values into the equation for h, we get

$$h = \mathbb{E}H + 1.64 \cdot \sqrt{\mathrm{Var}H} = 4 \cdot 10^6 + 1.64 \cdot \sqrt{(9 \cdot 10^{10})} = 4 \cdot 10^6 + 4.92 \cdot 10^5 = 4,492,000.$$

Example 3.2. *Prove the following recursive formula*

$$A_x = v \, q_x + v \, p_x \, A_{x+1}.$$

Solution:

$$A_x = \sum_{k=0}^\infty \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k} \right) = v \cdot {}_0p_x \cdot q_x + \sum_{k=1}^\infty \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k} \right)$$
$$= v \cdot q_x + v \cdot \sum_{k=1}^\infty \left(v^k \cdot {}_kp_x \cdot q_{x+k} \right) = v \cdot q_x + v \cdot \sum_{k=0}^\infty \left(v^{k+1} \cdot {}_{k+1}p_x \cdot q_{x+k+1} \right)$$
$$= v \cdot q_x + v \cdot \sum_{k=0}^\infty \left(v^{k+1} \cdot p_x \cdot {}_kp_{x+1} \cdot q_{x+k+1} \right)$$
$$= v \cdot q_x + v \cdot p_x \cdot \sum_{k=0}^\infty \left(v^{k+1} \cdot {}_kp_{x+1} \cdot q_{x+k+1} \right) = v \cdot q_x + v \cdot p_x \cdot A_{x+1}$$

Example 3.3. *Prove the following relations between commutation functions:*

1. $C_x = vD_x - D_{x+1}$,
2. $M_x = vN_x - N_{x+1}$,
3. $R_x = vS_x - S_{x+1}$,
4. $M_x = D_x - dN_x$,
5. $R_x = N_x - dS_x$.

Solution:

1)

$$C_x = d_x \cdot v^{x+1} = (l_x - l_{x+1}) \cdot v^{x+1} = l_x \cdot v^x \cdot v - l_{x+1} \cdot v^{x+1} = v \cdot D_x - D_{x+1}$$

2)

$$M_x = \sum_{k=0}^{\infty} C_{x+k} = \sum_{k=0}^{\infty} (v \cdot D_{x+k} - D_{x+k+1}) = v \cdot \sum_{k=0}^{\infty} D_{x+k} - \sum_{k=0}^{\infty} D_{x+k+1} = v \cdot N_x - N_{x+1}$$

3)

$$R_x = \sum_{k=0}^{\infty} M_{x+k} = \sum_{k=0}^{\infty} (v \cdot N_{x+k} - N_{x+k+1}) = v \cdot \sum_{k=0}^{\infty} N_{x+k} - \sum_{k=0}^{\infty} N_{x+k+1} = v \cdot S_x - S_{x+1}$$

4)

$$M_x = v \cdot N_x - N_{x+1} = v \cdot N_x - (N_x - D_x) = D_x - (1 - v) \cdot N_x = D_x - d \cdot N_x$$

5)

$$R_x = v \cdot S_x - S_{x+1} = v \cdot S_x - (S_x - N_x) = N_x - (1 - v) \cdot S_x = N_x - d \cdot S_x$$

Example 3.4. *Using the commutation functions, derive explicit formulas for the net single premiums of the following capital life insurances with sum insured equal to one:*

1. *whole life insurance (net single premium is denoted by A_x),*
2. *term insurance with duration n years ($A^1_{x:\overline{n}|}$),*
3. *pure endowment with duration n years ($A_{x:\overset{1}{\overline{n}|}}$),*
4. *endowment with duration n years ($A_{x:\overline{n}|}$),*
5. *m-years deferred whole life insurance ($_{m|}A_x$),*
6. *m-years deferred term insurance with duration n years ($_{m|}A^1_{x:\overline{n}|}$),*

7. *m-years deferred pure endowment with duration n years* ($_{m|}A_{x:\overline{n|}}^{\ \ 1}$),

8. *m-years deferred endowments with duration n years* ($_{m|}A_{x:\overline{n|}}$).

Solution:

1)

$$\boxed{Z = v^{K+1}, \qquad K = 0, 1, 2, \ldots}$$

$$A_x = \sum_{k=0}^{\infty} \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right) = \sum_{k=0}^{\infty} \left(v^{k+1} \cdot \frac{l_{x+k}}{l_x} \cdot \frac{d_{x+k}}{l_{x+k}}\right) = \sum_{k=0}^{\infty} \frac{d_{x+k} \cdot v^{x+k+1}}{l_x \cdot v^x}$$
$$= \frac{\sum_{k=0}^{\infty} C_{x+k}}{D_x} = \frac{M_x}{D_x}$$

2)

$$\boxed{Z = \begin{cases} v^{K+1}, & K = 0, 1, \ldots, n-1 \\ 0, & K = n, n+1, \ldots \end{cases}}$$

$$A_{x:\overline{n|}}^{1} = \sum_{k=0}^{n-1} \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right) = \sum_{k=0}^{n-1} \left(v^{k+1} \cdot \frac{d_{x+k}}{l_x}\right) = \sum_{k=0}^{n-1} \frac{C_{x+k}}{D_x} = \sum_{k=0}^{\infty} \frac{C_{x+k}}{D_x} - \sum_{k=n}^{\infty} \frac{C_{x+k}}{D_x}$$
$$= \frac{M_x}{D_x} - \frac{\sum_{k=0}^{\infty} C_{x+n+k}}{D_x} = \frac{M_x}{D_x} - \frac{M_{x+n}}{D_x} = \frac{M_x - M_{x+n}}{D_x}$$

3)

$$\boxed{Z = \begin{cases} 0, & K = 0, 1, \ldots, n-1 \\ v^n, & K = n, n+1, \ldots \end{cases}}$$

$$A_{x:\overline{n|}}^{\ \ 1} = v^n \cdot {}_np_x = v^n \cdot \frac{l_{x+n}}{l_x} = \frac{v^{x+n} \cdot l_{x+n}}{v^x \cdot l_x} = \frac{D_{x+n}}{D_x}$$

4)

$$\boxed{Z = \begin{cases} v^{K+1}, & K = 0, 1, \ldots, n-1 \\ v^n, & K = n, n+1, \ldots \end{cases}}$$

$$A_{x:\overline{n|}} = A_{x:\overline{n|}}^{1} + A_{x:\overline{n|}}^{\ \ 1} = \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

5)

$$\boxed{Z = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^{K+1}, & K = m, m+1, \ldots \end{cases}}$$

$$\begin{aligned}
{}_{m|}A_x &= \sum_{k=m}^{\infty} \left(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \right) = \sum_{k=0}^{\infty} \left(v^{k+m+1} \cdot {}_{k+m} p_x \cdot q_{x+m+k} \right) \\
&= v^m \cdot {}_m p_x \cdot \sum_{k=0}^{\infty} \left(v^{k+1} \cdot {}_k p_{x+m} \cdot q_{x+m+k} \right) = v^m \cdot {}_m p_x \cdot A_{x+m} \\
&= v^m \cdot \frac{l_{x+m}}{l_x} \cdot A_{x+m} = \frac{v^{x+m} \cdot l_{x+m}}{v^x \cdot l_x} \cdot A_{x+m} = \frac{D_{x+m}}{D_x} \cdot \frac{M_{x+m}}{D_{x+m}} = \frac{M_{x+m}}{D_x}
\end{aligned}$$

Remark: The relationship for the deferment ${}_{m|}A_x = v^m \cdot {}_m p_x \cdot A_{x+m}$ holds generally even for other types of insurance.

6)

$$\boxed{Z = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^{K+1}, & K = m, m+1, \ldots, m+n-1 \\ 0, & K = m+n, m+n+1, \ldots \end{cases}}$$

$$\begin{aligned}
{}_{m|}A^1_{x:\overline{n|}} &= v^m \cdot {}_m p_x \cdot A^1_{x+m:\overline{n|}} = \frac{v^{x+m} \cdot l_{x+m}}{v^x \cdot l_x} \cdot A^1_{x+m:\overline{n|}} \\
&= \frac{D_{x+m}}{D_x} \cdot \frac{M_{x+m} - M_{x+m+n}}{D_{x+m}} = \frac{M_{x+m} - M_{x+m+n}}{D_x}
\end{aligned}$$

7)

$$\boxed{Z = \begin{cases} 0, & K = 0, 1, \ldots, m+n-1 \\ v^{m+n}, & K = m+n, m+n+1, \ldots \end{cases}}$$

$${}_{m|}A_{x:\overline{n|}}^{\;\;1} = v^{m+n} \cdot {}_{m+n} p_x = \frac{v^{x+m+n} \cdot l_{x+m+n}}{v^x \cdot l_x} = \frac{D_{x+m+n}}{D_x}$$

8)

$$\boxed{Z = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^{K+1}, & K = m, m+1, \ldots, m+n-1 \\ v^{m+n}, & K = m+n, m+n+1, \ldots \end{cases}}$$

$$\begin{aligned}
{}_{m|}A_{x:\overline{n|}} &= {}_{m|}A^1_{x:\overline{n|}} + {}_{m|}A_{x:\overline{n|}}^{\;\;1} = \frac{M_{x+m} - M_{x+m+n}}{D_x} + \frac{D_{x+m+n}}{D_x} \\
&= \frac{M_{x+m} - M_{x+m+n} + D_{x+m+n}}{D_x}
\end{aligned}$$

Example 3.5. *Consider TIR $i = 3\%$, commutation functions $D_{76} = 400$, $D_{77} = 360$, and net single premium for the whole life insurance $A_{76} = 0.8$. Derive A_{77}.*

Solution:

We can use the recursive formula from the previous example:

$$A_x = v \cdot q_x + v \cdot p_x \cdot A_{x+1} \Rightarrow A_{x+1} = \frac{A_x - v \cdot q_x}{v \cdot p_x},$$

where $v = \frac{1}{1+i}$.

Next step would be to express p_x and q_x using the commutation functions:

$$p_x = \frac{l_{x+1}}{l_x} = \frac{l_{x+1} \cdot v^{x+1}}{l_x \cdot v^{x+1}} = \frac{D_{x+1}}{v \cdot D_x},$$

$$q_x = 1 - p_x = 1 - \frac{D_{x+1}}{v \cdot D_x} = \frac{v \cdot D_x - D_{x+1}}{v \cdot D_x}.$$

Thus, A_{x+1} can be calculated as

$$A_{x+1} = \frac{A_x - v \cdot \frac{v \cdot D_x - D_{x+1}}{v \cdot D_x}}{v \cdot \frac{D_{x+1}}{v \cdot D_x}} = \frac{\frac{A_x \cdot D_x - v \cdot D_x + D_{x+1}}{D_x}}{\frac{D_{x+1}}{D_x}} = \frac{A_x \cdot D_x - v \cdot D_x + D_{x+1}}{D_{x+1}}$$

which corresponds numerically to

$$\begin{aligned} A_{77} &= \frac{A_{76} \cdot D_{76} - v \cdot D_{76} + D_{77}}{D_{77}} = \frac{A_{76} \cdot D_{76} - \frac{D_{76}}{1+i} + D_{77}}{D_{77}} \\ &= \frac{0.8 \cdot 400 - \frac{400}{1+0.03} + 360}{360} = 0.8101. \end{aligned}$$

3.2 Capital life insurance with variable sum insured

Example 3.6. *Prove that the net single premium for the whole life insurance with variable sum insured can be expressed as*

$$NSP = c_1\, A_x + (c_2 - c_1)\, {}_{1|}A_x + (c_3 - c_2)\, {}_{2|}A_x + \dots.$$

Solution:

$$\boxed{Z = c_{K+1} \cdot v^{K+1}, \qquad K = 0, 1, 2, \dots}$$

$$\begin{aligned} \mathbb{E}Z = \sum_{k=0}^{\infty} \left(c_{k+1} \cdot v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right) &= c_1 \cdot \left[\sum_{k=0}^{\infty} \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right) - \sum_{k=1}^{\infty} \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right)\right] \\ &+ c_2 \cdot \left[\sum_{k=1}^{\infty} \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right) - \sum_{k=2}^{\infty} \left(v^{k+1} \cdot {}_kp_x \cdot q_{x+k}\right)\right] + \dots \\ &= c_1 \cdot (A_x - {}_{1|}A_x) + c_2 \cdot ({}_{1|}A_x - {}_{2|}A_x) + \dots = c_1 \cdot A_x + (c_2 - c_1) \cdot {}_{1|}A_x + \dots. \end{aligned}$$

Example 3.7. *Prove that the net single premium for the term insurance with variable sum insured, i.e. $c_{k+1} = 0$ for $k \geq n$, can be expressed as*

$$NSP = c_n \, A^1_{x:\overline{n}|} + (c_{n-1} - c_n) \, A^1_{x:\overline{n-1}|} + \cdots + (c_1 - c_2) \, A^1_{x:\overline{1}|}.$$

Solution:

$$\boxed{Z = \begin{cases} c_{K+1} \cdot v^{K+1}, & K = 0, 1, \ldots, n-1 \\ 0, & K = n, n+1, \ldots \end{cases}}$$

$$\begin{aligned}
\mathbb{E}Z = \sum_{k=0}^{n-1} \big(c_{k+1} \cdot v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) &= c_n \cdot \left[\sum_{k=0}^{n-1} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) - \sum_{k=0}^{n-2} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) \right] \\
&+ c_{n-1} \cdot \left[\sum_{k=0}^{n-2} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) - \sum_{k=0}^{n-3} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) \right] \\
&\ \vdots \\
&+ c_2 \cdot \left[\sum_{k=0}^{1} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) - \sum_{k=0}^{0} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) \right] + c_1 \cdot \sum_{k=0}^{0} \big(v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \big) \\
&= c_n \cdot (A^1_{x:\overline{n}|} - A^1_{x:\overline{n-1}|}) + c_{n-1} \cdot (A^1_{x:\overline{n-1}|} - A^1_{x:\overline{n-2}|}) + \ldots + c_2 \cdot (A^1_{x:\overline{2}|} - A^1_{x:\overline{1}|}) + c_1 \cdot A^1_{x:\overline{1}|} \\
&= c_n \cdot A^1_{x:\overline{n}|} + (c_{n-1} - c_n) \cdot A^1_{x:\overline{n-1}|} + \ldots + (c_1 - c_2) \cdot A^1_{x:\overline{1}|}.
\end{aligned}$$

Example 3.8. *Consider a whole life insurance with variable sum insured, whose value is given according to the following table:*

k	0	1	2	3	4	5	6	7	8	≥ 9
c_{k+1}	10	10	9	9	9	8	8	8	8	7

Find an explicit formula for the net single premium.

Solution:

We can use the alternative expression of net single premium from Example 3.7.

$$NSP = 10 \cdot A_x - 1 \cdot {}_{2|}A_x - 1 \cdot {}_{5|}A_x - 1 \cdot {}_{9|}A_x = \frac{10 \cdot M_x - M_{x+2} - M_{x+5} - M_{x+9}}{D_x}$$

Example 3.9. *Using the commutation functions, derive explicit formulas for the net single premiums of the following capital life insurances:*

1. *standard increasing whole life insurance (net single premium is denoted by $(IA)_x$),*
2. *standard increasing term insurance with duration n years $(IA)^1_{x:\overline{n}|}$,*
3. *standard increasing endowments with duration n years $(IA)_{x:\overline{n}|}$,*

4. *m-years deferred standard increasing whole life insurance* ${}_{m|}(IA)_x$,
5. *standard decreasing term insurance with duration n years* $(DA)^1_{x:\overline{n}|}$.

Solution:

1)

$$\boxed{Z = (K+1)\cdot v^{K+1}, \qquad K = 0,1,2,\dots}$$

$$\begin{aligned}
(IA)_x &= \sum_{k=0}^{\infty}\left[(k+1)\cdot v^{k+1}\cdot {}_kp_x\cdot q_{x+k}\right] = \sum_{k=0}^{\infty}\left[(k+1)\cdot v^{k+1}\cdot \frac{d_{x+k}}{l_x}\right]\\
&= \sum_{k=0}^{\infty}\left[(k+1)\cdot \frac{d_{x+k}\cdot v^{x+k+1}}{l_x\cdot v^x}\right] = \frac{\sum_{k=0}^{\infty}\left[(k+1)\cdot C_{x+k}\right]}{D_x}\\
&= \frac{C_x + 2\cdot C_{x+1} + 3\cdot C_{x+2}+\dots}{D_x} = \frac{\sum_{k=0}^{\infty} C_{x+k} + \sum_{k=0}^{\infty} C_{x+1+k}+\dots}{D_x}\\
&= \frac{M_x + M_{x+1}+\dots}{D_x} = \frac{R_x}{D_x}
\end{aligned}$$

2)

$$\boxed{Z = \begin{cases}(K+1)\cdot v^{K+1}, & K = 0,1,\dots,n-1\\ 0, & K = n, n+1,\dots\end{cases}}$$

$$\begin{aligned}
(IA)^1_{x:\overline{n}|} &= \sum_{k=0}^{n-1}\left[(k+1)\cdot v^{k+1}\cdot {}_kp_x\cdot q_{x+k}\right]\\
&= \sum_{k=0}^{\infty}\left[(k+1)\cdot v^{k+1}\cdot {}_kp_x\cdot q_{x+k}\right] - \sum_{k=n}^{\infty}\left[(k+1)\cdot v^{k+1}\cdot {}_kp_x\cdot q_{x+k}\right]\\
&= (IA)_x - \sum_{k=0}^{\infty}\left[(k+n+1)\cdot v^{k+n+1}\cdot {}_{k+n}p_x\cdot q_{x+k+n}\right]\\
&= (IA)_x - \sum_{k=0}^{\infty}\left[(k+n+1)\cdot \frac{d_{x+k+n}\cdot v^{x+k+n+1}}{l_x\cdot v^x}\right]\\
&= (IA)_x - \frac{\sum_{k=0}^{\infty}\left[(k+n+1)\cdot C_{x+n+k}\right]}{D_x}\\
&= \frac{R_x}{D_x} - \frac{(n+1)\cdot C_{x+n} + (n+2)\cdot C_{x+n+1} + (n+3)\cdot C_{x+n+2}+\dots}{D_x}\\
&= \frac{R_x}{D_x} - \frac{(n+1)\cdot M_{x+n} + M_{x+n+1} + M_{x+n+2}+\dots}{D_x}\\
&= \frac{R_x}{D_x} - \frac{n\cdot M_{x+n} + R_{x+n}}{D_x} = \frac{R_x - n\cdot M_{x+n} - R_{x+n}}{D_x}
\end{aligned}$$

Remark: Alternatively, the following relationship could be used:

$$(IA)^1_{x:\overline{n}|} = A_x + {}_{1|}A_x + {}_{2|}A_x + \dots + {}_{n-1|}A_x - n\cdot {}_{n|}A_x$$

3)

$$\boxed{Z = \begin{cases} (K+1)\cdot v^{K+1}, & K = 0,1,\ldots,n-1 \\ n\cdot v^n, & K = n, n+1,\ldots \end{cases}}$$

$$\begin{aligned}
(IA)_{x:\overline{n}|} &= (IA)^1_{x:\overline{n}|} + (IA)_{x:\overline{n}|}^{\ \ 1} = (IA)^1_{x:\overline{n}|} + n\cdot A_{x:\overline{n}|}^{\ \ 1} \\
&= \frac{R_x - n\cdot M_{x+n} - R_{x+n}}{D_x} + n\cdot\frac{D_{x+n}}{D_x} = \frac{R_x - n\cdot M_{x+n} - R_{x+n} + n\cdot D_{x+n}}{D_x}
\end{aligned}$$

4)

$$\boxed{Z = \begin{cases} 0, & K = 0,1,\ldots,m-1 \\ (K-m+1)\cdot v^{K+1}, & K = m, m+1,\ldots \end{cases}}$$

$$\begin{aligned}
{}_{m|}(IA)_x &= \sum_{k=m}^{\infty}\left[(k-m+1)\cdot v^{k+1}\cdot {}_kp_x\cdot q_{x+k}\right] = \sum_{k=0}^{\infty}\left[(k+1)\cdot v^{k+m+1}\cdot {}_{k+m}p_x\cdot q_{x+k+m}\right] \\
&= \sum_{k=0}^{\infty}\left[(k+1)\cdot\frac{d_{x+k+m}\cdot v^{x+k+m+1}}{l_x\cdot v^x}\right] = \frac{\sum_{k=0}^{\infty}\left[(k+1)\cdot C_{x+m+k}\right]}{D_x} \\
&= \frac{C_{x+m} + 2\cdot C_{x+m+1} + 3\cdot C_{x+m+2} + \ldots}{D_x} = \frac{M_{x+m} + M_{x+m+1} + M_{x+m+2} + \ldots}{D_x} \\
&= \frac{R_{x+m}}{D_x}
\end{aligned}$$

5)

$$\boxed{Z = \begin{cases} (n-K)\cdot v^{K+1}, & K = 0,1,\ldots,n-1 \\ 0, & K = n, n+1,\ldots \end{cases}}$$

$$\begin{aligned}
(DA)^1_{x:\overline{n}|} &= \sum_{k=0}^{n-1}\left[(n-k)\cdot v^{k+1}\cdot {}_kp_x\cdot q_{x+k}\right] = \sum_{k=0}^{n-1}\left[(n-k)\cdot\frac{d_{x+k}\cdot v^{x+k+1}}{l_x\cdot v^x}\right] \\
&= \frac{\sum_{k=0}^{n-1}\left[(n-k)\cdot C_{x+k}\right]}{D_x} = \frac{n\cdot C_x + (n-1)\cdot C_{x+1} + \ldots + C_{x+n-1}}{D_x} \\
&= \frac{n\cdot M_x - M_{x+1} - \ldots - M_{x+n}}{D_x} = \frac{n\cdot M_x - R_{x+1} + R_{x+n+1}}{D_x}
\end{aligned}$$

3.3 Capital life insurance payable at the moment of death and at the end of *m*-th part of the year of death

Example 3.10. *Consider a whole life insurance with variable SI when the sum insured is incremented m times a year, by $1/m$ each time. We assume that the sum insured is payable*

1. *at the end of the m-th part of the year in which death occurs* $(I^{(m)}A^{(m)})_x$,

2. immediately at the moment of death $(I^{(m)}\overline{A})_x$.

Derive the net single premium under the assumption of linearity.

Solution:

Under the assumption of linearity it holds:

$$K \perp\!\!\!\perp S, \qquad S \sim U(0,1),$$

where $K = \lfloor T \rfloor$, $T = K + S$ and $U(0,1)$ is the uniform distribution on $(0,1)$ and also

$$K \perp\!\!\!\perp S^{(m)}, \qquad S^{(m)} \sim U^D(\frac{1}{m}, \dots \frac{m}{m}),$$

where $S^{(m)} = \frac{\lfloor m \cdot S + 1 \rfloor}{m}$, $\lfloor \cdot \rfloor$ is the floor function and $U^D(\frac{1}{m}, \dots \frac{m}{m})$ is the discrete uniform distribution.

1)

Assume

$$Z^{(m)} = (K + S^{(m)}) \cdot v^{K+S^{(m)}}.$$

Then

$$\begin{aligned}
(I^{(m)}A^{(m)})_x &= \mathbb{E}Z^{(m)} = \mathbb{E}\left[(K + S^{(m)}) \cdot v^{K+S^{(m)}}\right] \\
&= \mathbb{E}\left[((K+1) + (S^{(m)} - 1)) \cdot v^{(K+1)+(S^{(m)}-1)}\right] \\
&= \underbrace{\mathbb{E}\left[(K+1) \cdot v^{K+1}\right]}_{(IA)_x} \cdot \mathbb{E}\left[v^{S^{(m)}-1}\right] + \mathbb{E}\left[(S^{(m)} - 1) \cdot v^{S^{(m)}-1}\right] \cdot \underbrace{\mathbb{E}\left[v^{K+1}\right]}_{A_x} \\
&= (IA)_x \cdot \mathbb{E}\left[v^{-(1-S^{(m)})}\right] - A_x \cdot \mathbb{E}\left[(1 - S^{(m)}) \cdot v^{-(1-S^{(m)})}\right] \\
&= (IA)_x \cdot \underbrace{\mathbb{E}\left[v^{-(1-S^{(m)})}\right]}_{(1)} - A_x \cdot \underbrace{\mathbb{E}\left[v^{-(1-S^{(m)})}\right]}_{(1)} + A_x \cdot \underbrace{\mathbb{E}\left[S^{(m)} \cdot v^{-(1-S^{(m)})}\right]}_{(2)}.
\end{aligned}$$

We have to calculate the remaining terms (1) and (2) in the previous equation.

$$\begin{aligned}
\mathbb{E}\left[v^{-(1-S^{(m)})}\right] &= \mathbb{E}\left[(1+i)^{1-S^{(m)}}\right] = (1+i) \cdot \mathbb{E}\left[(1+i)^{-S^{(m)}}\right] \\
&= (1+i) \cdot \sum_{k=1}^{m} \left[(1+i)^{-\frac{k}{m}} \cdot \frac{1}{m}\right] = \frac{1+i}{m} \cdot \left[(1+i)^{-\frac{1}{m}}\right] \cdot \frac{1 - \left[(1+i)^{-\frac{1}{m}}\right]^m}{1 - (1+i)^{-\frac{1}{m}}} \\
&= \frac{1+i}{m} \cdot \frac{1}{(1+i)^{\frac{1}{m}}} \cdot \frac{1 - (1+i)^{-1}}{1 - (1+i)^{-\frac{1}{m}}} = \frac{i}{m \cdot \left[(1+i)^{\frac{1}{m}} - 1\right]} = \frac{i}{i^{(m)}}
\end{aligned}$$

For the calculation of the term (2) we will use the following relationship.

$$\sum_{k=1}^{m} \left(k \cdot a^{k-1}\right) = \left(\sum_{k=1}^{m} a^k\right)' = \left(a \cdot \frac{1 - a^m}{1 - a}\right)' = \frac{1 - (m+1) \cdot a^m + m \cdot a^{m+1}}{(1-a)^2}$$

Using this relationship for $a = v^{\frac{1}{m}}$ we obtain

$$\begin{aligned}
\mathbb{E}\left[S^{(m)} \cdot v^{-(1-S^{(m)})}\right] &= \frac{1}{m} \cdot \sum_{k=1}^{m} \left[\frac{k}{m} \cdot v^{\frac{k}{m}-1}\right] = \frac{1}{m^2} \cdot v^{\frac{1}{m}-1} \sum_{k=1}^{m} \left[k \cdot \left(v^{\frac{1}{m}}\right)^{k-1}\right] \\
&= \frac{1}{m^2} \cdot v^{\frac{1}{m}-1} \cdot \frac{1-(m+1)\cdot v + m \cdot v^{\frac{m+1}{m}}}{\left(1-v^{\frac{1}{m}}\right)^2} \\
&= \frac{v^{\frac{1}{m}}}{m^2} \cdot \frac{(1+i)-(m+1)+m\cdot v^{\frac{1}{m}}}{\left(1-v^{\frac{1}{m}}\right)^2} \\
&= v^{\frac{1}{m}} \cdot \frac{i - m \cdot \left(1-v^{\frac{1}{m}}\right)}{\left[m \cdot \left(1-v^{\frac{1}{m}}\right)\right]^2}.
\end{aligned}$$

Since $m \cdot \left(1-v^{\frac{1}{m}}\right) = d^{(m)}$, we have

$$\mathbb{E}\left[S^{(m)} \cdot v^{-(1-S^{(m)})}\right] = v^{\frac{1}{m}} \cdot \frac{i-d^{(m)}}{\left[d^{(m)}\right]^2}.$$

Furthermore, it holds

$$\frac{v^{\frac{1}{m}}}{d^{(m)}} = \frac{1}{i^{(m)}}.$$

Therefore

$$\mathbb{E}\left[S^{(m)} \cdot v^{-(1-S^{(m)})}\right] = \frac{i-d^{(m)}}{i^{(m)} \cdot d^{(m)}}.$$

Now we are able to complete the required form for $(I^{(m)}A^{(m)})_x$ as

$$(I^{(m)}A^{(m)})_x = \frac{i}{i^{(m)}} \cdot (IA)_x - \frac{i}{i^{(m)}} \cdot A_x + \frac{i-d^{(m)}}{i^{(m)} \cdot d^{(m)}} \cdot A_x.$$

2)

Assume

$$Z = (K + S^{(m)}) \cdot v^T.$$

Then

$$\begin{aligned}
(I^{(m)}\overline{A})_x &= \mathbb{E}Z = \mathbb{E}\left[(K+S^{(m)}) \cdot v^T\right] = \mathbb{E}\left[\left((K+1)+S^{(m)}-1\right) \cdot v^T\right] \\
&= \mathbb{E}\left[(K+1) \cdot v^T\right] + \mathbb{E}\left[S^{(m)} \cdot v^T\right] - \mathbb{E}v^T \\
&= (I\overline{A})_x + \mathbb{E}\left[S^{(m)} \cdot v^{(K+1)-(1-S)}\right] - \overline{A}_x = (I\overline{A})_x + \mathbb{E}\left[S^{(m)} \cdot v^{-(1-S)}\right] \cdot A_x - \overline{A}_x.
\end{aligned}$$

Similarly as in the first part, where

$$\mathbb{E}\left[S^{(m)} \cdot v^{-(1-S^{(m)})}\right] = \frac{i-d^{(m)}}{i^{(m)} \cdot d^{(m)}},$$

it can be shown that in this case it holds

$$\mathbb{E}\left[S^{(m)} \cdot v^{-(1-S)}\right] = \frac{i-d^{(m)}}{\delta \cdot d^{(m)}}.$$

Since it also holds

$$\overline{A}_x = \frac{i}{\delta} \cdot A_x,$$

and

$$(I\overline{A})_x = \frac{i}{\delta}(IA)_x,$$

the final form for $(I^{(m)}\overline{A})_x$ is

$$(I^{(m)}\overline{A})_x = \frac{i}{\delta}(IA)_x + \frac{i - d^{(m)}}{\delta \cdot d^{(m)}} \cdot A_x - \frac{i}{\delta} \cdot A_x.$$

Example 3.11. *Consider a whole life insurance with continuously increasing sum insured, i.e. $c(t) = t$, which is payable immediately on death. Derive the net single premium $(\overline{IA})_x$ under the assumption of linearity.*

Solution:

Assume

$$Z = T \cdot v^T.$$

Then under the assumption of linearity

$$\begin{aligned}
(\overline{IA})_x = \mathbb{E}Z &= \sum_{k=0}^{\infty} \left\{ \mathbb{E}\big[Z|K=k\big] \cdot P\left(K=k\right) \right\} \\
&= \sum_{k=0}^{\infty} \left\{ \mathbb{E}\big[v^{K+1} \cdot v^{S-1} \cdot (K+S)|K=k\big] \cdot P\left(K=k\right) \right\} \\
&= \sum_{k=0}^{\infty} \left\{ v^{k+1} \cdot \mathbb{E}\big[(k+S) \cdot v^{S-1}\big] \cdot P\left(K=k\right) \right\}.
\end{aligned}$$

$$\mathbb{E}\big[(k+S) \cdot v^{S-1}\big] = \int_0^1 (k+s) \cdot v^{s-1}\, ds = k \cdot \int_0^1 v^{s-1}\, ds + \int_0^1 s \cdot v^{s-1}\, ds,$$

where

$$\int_0^1 v^{s-1}\, ds = \frac{1}{v} \cdot \int_0^1 e^{s \cdot \ln(v)}\, ds = \frac{1}{v} \cdot \left[\frac{1}{\ln(v)} \cdot e^{s \cdot \ln(v)} \right]_0^1 = \frac{v-1}{v \cdot \ln(v)} = \frac{i}{\delta},$$

and

$$\begin{aligned}
\int_0^1 s \cdot v^{s-1}\, ds &\overset{\text{Per Partes}}{=} \begin{bmatrix} u = s & v' = v^{s-1} \\ u' = 1 & v = \frac{1}{v \cdot \ln(v)} \cdot v^s \end{bmatrix} = \left[\frac{s \cdot v^s}{v \cdot \ln(v)} \right]_0^1 - \int_0^1 \frac{1}{v \cdot \ln(v)} \cdot v^s\, ds \\
&= \frac{1}{\ln(v)} - \frac{1}{\ln(v)} \cdot \int_0^1 v^{s-1}\, ds = -\frac{1}{\delta} + \frac{1}{\delta} \cdot \frac{i}{\delta} = -\frac{1}{\delta} + \frac{i}{\delta^2}.
\end{aligned}$$

Together, we get

$$\mathbb{E}\big[(k+S) \cdot v^{S-1}\big] = k \cdot \frac{i}{\delta} - \frac{1}{\delta} + \frac{i}{\delta^2} = \frac{k \cdot i - 1}{\delta} + \frac{i}{\delta^2}.$$

Therefore

$$\begin{aligned}
(\overline{IA})_x &= \sum_{k=0}^{\infty}\left[v^{k+1}\cdot\left(\frac{k\cdot i-1}{\delta}+\frac{i}{\delta^2}\right)\cdot P\left(K=k\right)\right] \\
&= \frac{i}{\delta}\cdot\mathbb{E}\left(K\cdot v^{K+1}\right)-\frac{1}{\delta}\cdot\mathbb{E}\left(v^{K+1}\right)+\frac{i}{\delta^2}\cdot\mathbb{E}\left(v^{K+1}\right) \\
&= \frac{i}{\delta}\cdot\mathbb{E}\left[(K+1)\cdot v^{K+1}\right]-\frac{i}{\delta}\cdot\mathbb{E}\left(v^{K+1}\right)-\frac{1}{\delta}\cdot\mathbb{E}\left(v^{K+1}\right)+\frac{i}{\delta^2}\cdot\mathbb{E}\left(v^{K+1}\right) \\
&= \frac{i}{\delta}\cdot(IA)_x-\frac{i}{\delta}\cdot A_x-\frac{1}{\delta}\cdot A_x+\frac{i}{\delta^2}\cdot A_x=\frac{i\cdot(IA)_x-(1+i)\cdot A_x}{\delta}+\frac{i}{\delta^2}\cdot A_x.
\end{aligned}$$

4 Life Annuities

Example 4.1. *Using the commutation functions, derive explicit formulas for the net single premiums of the following life annuities with sum insured equal to one:*

1. *whole life annuity in advance (net single premium is denoted by $\ddot{a}_x$),*
2. *whole life annuity in arrear a_x,*
3. *temporary life annuity in advance with duration n years $\ddot{a}_{x\overline{n}|}$,*
4. *temporary life annuity in arrear with duration n years $a_{x\overline{n}|}$,*
5. *m-years deferred whole life annuity in advance ${}_{m|}\ddot{a}_x$,*
6. *m-years deferred whole life annuity in arrear ${}_{m|}a_x$,*
7. *m-years deferred temporary life annuity in advance with duration n years ${}_{m|}\ddot{a}_{x\overline{n}|}$,*
8. *m-years temporary life annuity in arrear with duration n years ${}_{m|}a_{x\overline{n}|}$.*

Solution:

1)

$$\boxed{Y = 1 + v + v^2 + \cdots + v^K, \qquad K = 0, 1, 2, \ldots}$$

$$\ddot{a}_x = \mathbb{E}Y = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left[v^l \cdot P\left(K = k\right)\right] = \sum_{k=0}^{\infty} \left[v^k \cdot P(K \geq k)\right] = \sum_{k=0}^{\infty} \left(v^k \cdot {}_k p_x\right)$$
$$= \sum_{k=0}^{\infty} \left(\frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right) = \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} = \frac{N_x}{D_x}$$

2)

$$\boxed{Y = v + v^2 + \cdots + v^K, \qquad K = 1, 2, \ldots}$$

$$a_x = \ddot{a}_x - 1 = \frac{N_x}{D_x} - 1 = \frac{N_x - D_x}{D_x} = \frac{N_{x+1}}{D_x}$$

3)

$$\boxed{Y = \begin{cases} 1 + v + \cdots + v^K, & K = 0, 1, \ldots, n-1 \\ 1 + v + \cdots + v^{n-1}, & K = n, n+1, \ldots \end{cases}}$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(v^k \cdot {}_k p_x\right) = \sum_{k=0}^{n-1} \frac{D_{x+k}}{D_x} = \frac{N_x - N_{x+n}}{D_x}$$

4)

$$\boxed{Y = \begin{cases} 0, & K = 0 \\ v + \cdots + v^K, & K = 1, 2 \ldots, n \\ v + \cdots + v^n, & K = n+1, n+2, \ldots \end{cases}}$$

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} \left(v^k \cdot {}_k p_x\right) = \sum_{k=1}^{n} \frac{D_{x+k}}{D_x} = \sum_{k=0}^{n-1} \frac{D_{x+1+k}}{D_x} = \frac{N_{x+1} - N_{x+1+n}}{D_x}$$

5)

$$\boxed{Y = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^m + \cdots + v^K, & K = m, m+1, \ldots \end{cases}}$$

$$\begin{aligned} {}_{m|}\ddot{a}_x &= \sum_{k=m}^{\infty} \left(v^k \cdot {}_k p_x\right) = \sum_{k=m}^{\infty} \left(\frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right) = \frac{1}{D_x} \cdot \sum_{k=m}^{\infty} D_{x+k} = \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} D_{x+m+k} \\ &= \frac{N_{x+m}}{D_x} \end{aligned}$$

6)

$$\boxed{Y = \begin{cases} 0, & K = 0, 1, \ldots, m \\ v^{m+1} + \cdots + v^K, & K = m+1, m+2, \ldots \end{cases}}$$

$$\begin{aligned} {}_{m|}a_x &= \sum_{k=m+1}^{\infty} \left(v^k \cdot {}_k p_x\right) = \sum_{k=m+1}^{\infty} \left(\frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right) = \frac{1}{D_x} \cdot \sum_{k=m+1}^{\infty} D_{x+k} \\ &= \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} D_{x+m+1+k} = \frac{N_{x+m+1}}{D_x} \end{aligned}$$

7)

$$\boxed{Y = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^m + \cdots + v^K, & K = m, \ldots, m+n-1 \\ v^m + \cdots + v^{m+n-1}, & K = m+n, m+n+1, \ldots \end{cases}}$$

$$\begin{aligned} {}_{m|}\ddot{a}_{x:\overline{n}|} &= \sum_{k=m}^{m+n-1} \left(v^k \cdot {}_k p_x\right) = \sum_{k=m}^{m+n-1} \left(\frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right) = \frac{1}{D_x} \cdot \sum_{k=m}^{m+n-1} D_{x+k} \\ &= \frac{1}{D_x} \cdot \left[\sum_{k=m}^{\infty} D_{x+k} - \sum_{k=m+n}^{\infty} D_{x+k}\right] = \frac{1}{D_x} \cdot \left[\sum_{k=0}^{\infty} D_{x+m+k} - \sum_{k=0}^{\infty} D_{x+m+n+k}\right] \\ &= \frac{N_{x+m} - N_{x+m+n}}{D_x} \end{aligned}$$

8)

$$\boxed{Y = \begin{cases} 0, & K = 0, 1, \dots, m \\ v^{m+1} + \dots + v^K, & K = m+1, \dots, m+n \\ v^{m+1} + \dots + v^{m+n}, & K = m+n+1, m+n+2, \dots \end{cases}}$$

$$\begin{aligned} {}_{m|}a_{x:\overline{n}|} &= \sum_{k=m+1}^{m+n} \left(v^k \cdot {}_k p_x\right) = \sum_{k=m+1}^{m+n} \left(\frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right) = \frac{1}{D_x} \cdot \sum_{k=m+1}^{m+n} D_{x+k} \\ &= \frac{1}{D_x} \cdot \left[\sum_{k=m+1}^{\infty} D_{x+k} - \sum_{k=m+n+1}^{\infty} D_{x+k}\right] = \frac{1}{D_x} \cdot \left[\sum_{k=0}^{\infty} D_{x+m+1+k} - \sum_{k=0}^{\infty} D_{x+m+n+1+k}\right] \\ &= \frac{N_{x+m+1} - N_{x+m+n+1}}{D_x} \end{aligned}$$

Example 4.2. *Prove the following recursive formula*

$$\ddot{a}_x = 1 + v\, p_x\, \ddot{a}_{x+1}.$$

Solution:

$$\begin{aligned} \ddot{a}_x &= \sum_{k=0}^{\infty} \left(v^k \cdot {}_k p_x\right) = 1 + \sum_{k=1}^{\infty} \left(v^k \cdot {}_k p_x\right) = 1 + \sum_{k=0}^{\infty} \left(v^{k+1} \cdot {}_{k+1} p_x\right) \\ &= 1 + v \cdot \sum_{k=0}^{\infty} \left(v^k \cdot p_x \cdot {}_k p_{x+1}\right) = 1 + v \cdot p_x \cdot \sum_{k=0}^{\infty} \left(v^k \cdot {}_k p_{x+1}\right) = 1 + v \cdot p_x \cdot \ddot{a}_{x+1} \end{aligned}$$

Example 4.3. *Consider* $e_x = \mathbb{E}[K]$. *Show that*

1. $A_x > v^{e_x+1}$,
2. $\ddot{a}_x < \ddot{a}_{\overline{e_x+1}|}$.

Solution:

1)

$$A_x = \mathbb{E}[v^{K+1}]$$

If we denote $f(z) = v^{z+1}$ then

$$v^{e_x+1} = f[\mathbb{E}(K)]$$

and

$$A_x = \mathbb{E}[f(K)].$$

Since $f(z)$ is a strictly convex function $\left[f''(z) > 0\right]$, due to Jensen inequality it holds

$$\mathbb{E}[f(K)] > f[\mathbb{E}(K)].$$

Therefore

$$A_x > v^{e_x+1}.$$

2)

$$\ddot{a}_x = \frac{1 - A_x}{d} = \frac{1 - A_x}{1 - v} = \frac{1 - \mathbb{E}\left[v^{K+1}\right]}{1 - v}$$

$$\ddot{a}_{\overline{e_x+1}|} = \ddot{a}_{\overline{\mathbb{E}(K)+1}|} = \frac{1 - v^{\mathbb{E}(K)+1}}{1 - v}$$

If we denote $g(z) = \frac{1-v^{z+1}}{1-v}$ then

$$\ddot{a}_{\overline{e_x+1}|} = g[\mathbb{E}(K)]$$

and

$$\ddot{a}_x = \mathbb{E}[g(K)].$$

For $g(z)$ we can calculate

$$\begin{aligned} g(z) &= \frac{1 - v^{z+1}}{1 - v} = \frac{1}{1 - v} - \frac{v}{1 - v} \cdot v^z, \\ g^{'}(z) &= -\frac{v}{1 - v} \cdot v^z \cdot \ln(v), \\ g^{''}(z) &= -\frac{v}{1 - v} \cdot v^z \cdot (\ln(v))^2. \end{aligned}$$

The first term is negative since it is equal to $-\frac{1}{i}$. The remaining two terms are positive and, hence, $g^{''}(z) < 0$.

Since $g(z)$ is a concave function $\left[g^{''}(z) < 0\right]$, due to Jensen inequality it holds

$$\mathbb{E}[g(K)] < g[\mathbb{E}(K)].$$

Therefore

$$\ddot{a}_x < \ddot{a}_{\overline{e_x+1}|}.$$

Remark: Relation $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{1-v}$ *is possible to use even when* n *is not an integer. Let's assume* $n = k + s$ *where* k *is an integer and* s *is not. It can be shown that the formula holds and it corresponds to the situation when the first* k *payments are paid regularly and are of amount one, and at time* n *there is an additional payment with a different amount.*

Example 4.4. *Using the commutation functions, derive explicit formulas for the net single premiums of:*

1. *standard increasing whole life annuity in advance* $(I\ddot{a})_x$*,*
2. *standard increasing temporary life annuity in advance with duration* n *years* $(I\ddot{a})_{x\overline{n}|}$*,*
3. m*-years deferred standard increasing whole life annuity in advance* ${}_{m|}(I\ddot{a})_x$*.*

Solution:

1)

$$\boxed{Y = 1 + 2 \cdot v + \cdots + (K+1) \cdot v^K, \qquad K = 0, 1, 2, \ldots}$$

$$\begin{aligned}
(I\ddot{a})_x &= \sum_{k=0}^{\infty} \left[(k+1) \cdot v^k \cdot {}_kp_x\right] = \sum_{k=0}^{\infty} \left[(k+1) \cdot \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right] = \sum_{k=0}^{\infty} \frac{(k+1) \cdot D_{x+k}}{D_x} \\
&= \frac{D_x + 2 \cdot D_{x+1} + 3 \cdot D_{x+2} + \cdots}{D_x} = \frac{N_x + N_{x+1} + N_{x+2} + \cdots}{D_x} = \frac{S_x}{D_x}
\end{aligned}$$

2)

$$\boxed{Y = \begin{cases} 1 + 2 \cdot v + \cdots + (K+1) \cdot v^K, & K = 0, 1, \ldots, n-1 \\ 1 + 2 \cdot v + \cdots + n \cdot v^{n-1}, & K = n, n+1, \ldots \end{cases}}$$

$$\begin{aligned}
(I\ddot{a})_{x:\overline{n}|} &= \sum_{k=0}^{n-1} \left[(k+1) \cdot v^k \cdot {}_kp_x\right] = \sum_{k=0}^{n-1} \left[(k+1) \cdot \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right] = \sum_{k=0}^{n-1} \frac{(k+1) \cdot D_{x+k}}{D_x} \\
&= \frac{1}{D_x} \cdot \left[\sum_{k=0}^{\infty} (k+1) \cdot D_{x+k} - \sum_{k=n}^{\infty} \left[(k+1) \cdot D_{x+k}\right]\right] \\
&= (I\ddot{a})_x - \frac{1}{D_x} \cdot \sum_{k=n}^{\infty} \left[(k+1) \cdot D_{x+k}\right] = (I\ddot{a})_x - \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} \left[(k+n+1) \cdot D_{x+n+k}\right] \\
&= (I\ddot{a})_x - \frac{(n+1) \cdot D_{x+n} + (n+2) \cdot D_{x+n+1} + \cdots}{D_x} = \frac{S_x}{D_x} - \frac{n \cdot N_{x+n} + S_{x+n}}{D_x} \\
&= \frac{S_x - S_{x+n} - n \cdot N_{x+n}}{D_x}
\end{aligned}$$

3)

$$\boxed{Y = \begin{cases} 0, & K = 0, 1, \ldots, m-1 \\ v^m + \cdots + (K-m+1) \cdot v^K, & K = m, m+1, \ldots \end{cases}}$$

$$\begin{aligned}
{}_{m|}(I\ddot{a})_x &= \sum_{k=m}^{\infty} \left[(k-m+1) \cdot v^k \cdot {}_kp_x\right] = \sum_{k=m}^{\infty} \left[(k-m+1) \cdot \frac{l_{x+k} \cdot v^{x+k}}{l_x \cdot v^x}\right] \\
&= \frac{1}{D_x} \cdot \sum_{k=m}^{\infty} \left[(k-m+1) \cdot D_{x+k}\right] = \frac{1}{D_x} \cdot \sum_{k=0}^{\infty} \left[(k+1) \cdot D_{x+m+k}\right] \\
&= \frac{D_{x+m} + 2 \cdot D_{x+m+1} + 3 \cdot D_{x+m+2} + \cdots}{D_x} = \frac{N_{x+m} + N_{x+m+1} + N_{x+m+2} + \cdots}{D_x} \\
&= \frac{S_{x+m}}{D_x}
\end{aligned}$$

Example 4.5. *Prove the following relation between the net single premiums:*

$$(IA)_x = \ddot{a}_x - d\,(I\ddot{a})_x.$$

Solution:

There are two ways how to prove this relation.

1) Using commutation functions

Transforming the relation into the form using the commutation functions, we want to prove that

$$\frac{R_x}{D_x} = \frac{N_x}{D_x} - d \cdot \frac{S_x}{D_x}.$$

This can be directly seen from the result from **Example 3.3 (5.)**.

2) Trick

$$\sum_{k=0}^{K} v^k \cdot (k+1) = \sum_{k=0}^{K} v^k \sum_{l=0}^{k} 1 = \sum_{l=0}^{K} \sum_{k=l}^{K} v^k = \sum_{l=0}^{K} \left(v^l \cdot \frac{1 - v^{K+1-l}}{1-v} \right)$$
$$= \frac{1}{d} \sum_{l=0}^{K} \left(v^l - v^{K+1} \right) = \frac{1}{d} \cdot \left[\frac{1 - v^{K+1}}{1-v} - (K+1) \cdot v^{K+1} \right]$$

and after applying the expectation on both sides of the last equation, we obtain

$$(I\ddot{a})_x = \frac{1}{d} \cdot \left[\ddot{a}_x - (IA)_x \right],$$

which is after an adjustment the same as the desired form.

Example 4.6. *Consider standard increasing whole life annuity payable m-times a year in advance where the payments are incremented once a year, by $1/m$ each time. Derive an explicit formula for the net single premium using the commutation functions.*

Solution:

In this case we assume the following payments

Time:	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m-1}{m}$	1	$1+\frac{1}{m}$	$\cdots$
Payment	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{2}{m}$	$\frac{2}{m}$	$\cdots$

$$(I\ddot{a})_x^{(m)} = \sum_{k=0}^{\infty} {}_{k|}\ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x \cdot \ddot{a}_{x+k}^{(m)} \right]$$

Now we can use the approximation from the lecture

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2 \cdot m}.$$

Using this approximation we get

$$
\begin{aligned}
(I\ddot{a})_x^{(m)} &\approx \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x \cdot \ddot{a}_{x+k}\right] - \frac{m-1}{2 \cdot m} \cdot \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x\right] \\
&= \sum_{k=0}^{\infty} \left[\frac{D_{x+k}}{D_x} \cdot \frac{N_{x+k}}{D_{x+k}}\right] - \frac{m-1}{2 \cdot m} \cdot \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} \\
&= \sum_{k=0}^{\infty} \frac{N_{x+k}}{D_x} - \frac{m-1}{2 \cdot m} \cdot \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} = \frac{S_x}{D_x} - \frac{m-1}{2 \cdot m} \cdot \frac{N_x}{D_x}
\end{aligned}
$$

Alternatively, we can derive the formula without this approximation. In that case we have

$$
\begin{aligned}
(I\ddot{a})_x^{(m)} &= \sum_{k=0}^{\infty} {}_{k|}\ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x \cdot \ddot{a}_{x+k}^{(m)}\right] = \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x \cdot (\alpha(m) \cdot \ddot{a}_{x+k} - \beta(m))\right] \\
&= \alpha(m) \cdot \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x \cdot \ddot{a}_{x+k}\right] - \beta(m) \cdot \sum_{k=0}^{\infty} \left[v^k \cdot {}_kp_x\right] \\
&= \alpha(m) \cdot (I\ddot{a})_x - \beta(m) \cdot \ddot{a}_x,
\end{aligned}
$$

where

$$
\alpha(m) = \frac{d \cdot i}{d^{(m)} \cdot i^{(m)}}, \qquad \beta(m) = \frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}}.
$$

Therefore

$$
(I\ddot{a})_x^{(m)} = \frac{d \cdot i}{d^{(m)} \cdot i^{(m)}} \cdot (I\ddot{a})_x - \frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}} \cdot \ddot{a}_x = \frac{d \cdot i}{d^{(m)} \cdot i^{(m)}} \cdot \frac{S_x}{D_x} - \frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}} \cdot \frac{N_x}{D_x}.
$$

Let's compare results when using and not using the approximation. Assume that $i = 1\%$. Then $i^{(m)} = 0.99545\%$, $d = 0.99009\%$ and $d^{(m)} = 0.99462\%$.

Therefore,

$$
\begin{aligned}
&\frac{d \cdot i}{d^{(m)} \cdot i^{(m)}} = 1.00001 \text{ which can be compared to } 1, \\
&\frac{i - i^{(m)}}{d^{(m)} \cdot i^{(m)}} = 0.45998 \text{ which can be compared to } \frac{m-1}{2 \cdot m} = 0.45833.
\end{aligned}
$$

5 Annual Net Premium

Example 5.1. *Using the actuarial symbols and commutation functions, derive explicit formulas for the annual net premiums paid during the deferment period of the following life insurances and annuities (with sum insured equal to one):*

1. *m-years deferred whole life insurance,*
2. *m-years deferred pure endowment with duration n years,*
3. *m-years deferred whole life annuity in advance,*
4. *m-years deferred temporary life annuity in arrear with duration n years,*

Solution:

1)

$$\boxed{L = \begin{cases} 0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m-1 \\ v^{K+1} - P \cdot \sum_{k=0}^{m-1} v^k, & K = m, m+1, \ldots \end{cases}}$$

$$\mathbb{E}L = 0 = {}_{m|}A_x - P \cdot \ddot{a}_{x:\overline{m}|}$$
$$\Downarrow$$
$$P = \frac{{}_{m|}A_x}{\ddot{a}_{x:\overline{m}|}}$$
$$\Downarrow$$
$$P = \frac{\frac{M_{x+m}}{D_x}}{\frac{N_x - N_{x+m}}{D_x}} = \frac{M_{x+m}}{N_x - N_{x+m}}$$

2)

$$\boxed{L = \begin{cases} 0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m-1 \\ 0 - P \cdot \sum_{k=0}^{m-1} v^k, & K = m, m+1, \ldots, m+n-1 \\ v^{m+n} - P \cdot \sum_{k=0}^{m-1} v^k, & K = m+n, m+n+1, \ldots \end{cases}}$$

$$\mathbb{E}L = 0 = {}_{m|}A_{x:\overset{1}{\overline{n}|}} - P \cdot \ddot{a}_{x:\overline{m}|}$$
$$\Downarrow$$
$$P = \frac{{}_{m|}A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{m}|}}$$
$$\Downarrow$$
$$P = \frac{\frac{D_{x+m+n}}{D_x}}{\frac{N_x - N_{x+m}}{D_x}} = \frac{D_{x+m+n}}{N_x - N_{x+m}}$$

3)

$$\boxed{L = \begin{cases} 0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m-1 \\ \sum_{k=m}^{K} v^k - P \cdot \sum_{k=0}^{m-1} v^k, & K = m, m+1, \ldots \end{cases}}$$

$$\mathbb{E}L = 0 = {}_{m|}\ddot{a}_x - P \cdot \ddot{a}_{x:\overline{m}|}$$
$$\Downarrow$$
$$P = \frac{{}_{m|}\ddot{a}_x}{\ddot{a}_{x:\overline{m}|}}$$
$$\Downarrow$$
$$P = \frac{\frac{N_{x+m}}{D_x}}{\frac{N_x - N_{x+m}}{D_x}} = \frac{N_{x+m}}{N_x - N_{x+m}}$$

4)

$$\boxed{L = \begin{cases} 0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \dots, m-1 \\ 0 - P \cdot \sum_{k=0}^{m-1} v^k, & K = m \\ \sum_{k=m+1}^{K} v^k - P \cdot \sum_{k=0}^{m-1} v^k, & K = m+1, \dots, m+n \\ \sum_{k=m+1}^{m+n} v^k - P \cdot \sum_{k=0}^{m-1} v^k, & K = m+n+1, m+n+2, \dots \end{cases}}$$

$$\mathbb{E}L = 0 = {}_{m|}a_{x:\overline{n}|} - P \cdot \ddot{a}_{x:\overline{m}|}$$
$$\Downarrow$$
$$P = \frac{{}_{m|}a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}$$
$$\Downarrow$$
$$P = \frac{\frac{N_{x+m+1} - N_{x+m+n+1}}{D_x}}{\frac{N_x - N_{x+m}}{D_x}} = \frac{N_{x+m+1} - N_{x+m+n+1}}{N_x - N_{x+m}}$$

Example 5.2. *Consider pure endowment with duration n years where the annual net premium is paid during the whole insurance duration. Moreover, the premium refund agreement is active. i.e. in the case of death of the insured person the premium paid until the death is paid to a beneficiary at the end of the year. Derive the total loss and the annual net premium. Compare the premiums for the insurance contracts with and without the premium refund.*

Solution:

First, we have to calculate the premium for the insurance without the refund.

$$\boxed{L_1 = \begin{cases} 0 - P_1 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \dots, n-1 \\ v^n - P_1 \cdot \sum_{k=0}^{n-1} v^k, & K = n, n+1, \dots \end{cases}}$$

$$\mathbb{E}L_1 = 0 = A_{x:\overline{n}|}^{\ \ 1} - P_1 \cdot \ddot{a}_{x:\overline{n}|}$$
$$\Downarrow$$
$$P_1 = \frac{A_{x:\overline{n}|}^{\ \ 1}}{\ddot{a}_{x:\overline{n}|}}$$

Now, when the premium refund is added, we obtain

$$\boxed{L_2 = \begin{cases} P_2 \cdot (K+1) \cdot v^{K+1} - P_2 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, n-1 \\ v^n - P_2 \cdot \sum_{k=0}^{n-1} v^k, & K = n, n+1, \ldots \end{cases}}$$

$$\mathbb{E}L_2 = 0 = A_{x:\overset{1}{\overline{n}|}} + P_2 \cdot (IA)^1_{x:\overline{n}|} - P_2 \cdot \ddot{a}_{x:\overline{n}|}$$
$$\Downarrow$$
$$P_2 = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|}}$$

When comparing obtained premiums, it is obvious that when the premium refund is included, the corresponding premium must be higher, which is achieved by subtracting the term $(IA)^1_{x:\overline{n}|}$ in the denominator.

Example 5.3. *Consider m-years deferred whole life annuity in advance where the annual net premium is paid during the deferment period. Moreover, the premium refund agreement is active. i.e. in the case of death of the insured person during the deferment period the premium paid until the death is paid to a beneficiary at the end of the year. Derive the total loss and the annual net premium.*
Consider also the case when the annual net premium is paid over $m' < m$ years, but the premium refund is active over the whole deferment period.

Solution:

We can start with the case, when the annual premium is paid over m years.

$$\boxed{L_1 = \begin{cases} P_1 \cdot (K+1) \cdot v^{K+1} - P_1 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m-1 \\ \sum_{k=m}^{K} v^k - P_1 \cdot \sum_{k=0}^{m-1} v^k, & K = m, m+1, \ldots \end{cases}}$$

$$\mathbb{E}L_1 = 0 = {}_{m|}\ddot{a}_x + P_1 \cdot (IA)^1_{x:\overline{m}|} - P_1 \cdot \ddot{a}_{x:\overline{m}|}$$
$$\Downarrow$$
$$P_1 = \frac{{}_{m|}\ddot{a}_x}{\ddot{a}_{x:\overline{m}|} - (IA)^1_{x:\overline{m}|}}$$

When the premium is paid only over m' years, the total loss has the following form:

$$\boxed{L_2 = \begin{cases} P_2 \cdot (K+1) \cdot v^{K+1} - P_2 \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m'-1 \\ P_2 \cdot m' \cdot v^{K+1} - P_2 \cdot \sum_{k=0}^{m'-1} v^k, & K = m', m'+1 \ldots, m-1 \\ \sum_{k=m}^{K} v^k - P_2 \cdot \sum_{k=0}^{m'-1} v^k, & K = m, m+1, \ldots \end{cases}}$$

$$\mathbb{E}L_2 = 0 = {}_{m|}\ddot{a}_x + P_2 \cdot (IA)^1_{x:\overline{m'}|} + P_2 \cdot m' \cdot {}_{m'|}A^1_{x:\overline{m-m'}|} - P_2 \cdot \ddot{a}_{x:\overline{m'}|}$$
$$\Downarrow$$
$$P_2 = \frac{{}_{m|}\ddot{a}_x}{\ddot{a}_{x:\overline{m'}|} - (IA)^1_{x:\overline{m'}|} - m' \cdot {}_{m'|}A^1_{x:\overline{m-m'}|}}$$

6 Net Premium Reserve

Example 6.1. *Derive the net premium reserve for the pure endowment contract for n years when premium is paid*

1. *at once at the beginning as the net single premium,*
2. *at the beginning of each year over the whole insurance duration as the annual net premium,*
3. *at the beginning of each year over first $n' < n$ years as the annual net premium.*

Solution:

1)

$$NSP = A_{x:\overset{1}{\overline{n}|}}$$

$$\boxed{{}_kV_x = \begin{cases} 0, & k = 0 \\ A_{x+k:\overset{1}{\overline{n-k}|}}, & k = 1, 2, \dots, n-1 \end{cases}}$$

Remark: Time 0 is assumed before the payment of the net single premium. After the premium payment, it holds ${}_{0^+}V_x = NSP = A_{x:\overset{1}{\overline{n}|}}$ for the reserve.

2)

$$P = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}$$

$$\boxed{{}_kV_x = A_{x+k:\overset{1}{\overline{n-k}|}} - P \cdot \ddot{a}_{x+k:\overline{n-k}|}, \qquad k = 0, 1, ..., n-1}$$

Remark: If we assume $k = 0$ then ${}_0V_x = 0$ (this corresponds to the principle of equivalence). For $k = n$, the reserve would be ${}_nV_x = 1$ (the insurer must pay the sum insured to the beneficiary).

3)

$$P = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n'}|}}$$

$$\boxed{{}_kV_x = \begin{cases} A_{x+k:\overset{1}{\overline{n-k}|}} - P \cdot \ddot{a}_{x+k:\overline{n'-k}|}, & k = 0, 1, \dots, n'-1 \\ A_{x+k:\overset{1}{\overline{n-k}|}}, & k = n', n'+1, \dots, n-1 \end{cases}}$$

Example 6.2. *Derive the net premium reserve for whole life insurance with variable sum insured:*

$$c_1 = 50, \; c_2 = 55, \dots, c_{10} = 95, \; c_{11} = 100, \; c_{12} = 100, \; \dots$$

Consider standard increasing premium paid yearly over the whole insurance duration, i.e. P, $2P$, $3P$, $\dots$.

Solution:

First of all, we should derive the total loss and the formula for the annual net premium.

$$\boxed{L = \begin{cases} 45 \cdot v^{K+1} + 5 \cdot (K+1) \cdot v^{K+1} - P \cdot \sum_{k=0}^{K} \left[(k+1) \cdot v^k\right], & K = 0, 1, \ldots, 10 \\ 100 \cdot v^{K+1} - P \cdot \sum_{k=0}^{K} \left[(k+1) \cdot v^k\right], & K = 11, 12, \ldots \end{cases}}$$

$$\mathbb{E}L = 0 = 45 \cdot A_x + 5 \cdot (IA)^1_{x:\overline{11}|} + 55 \cdot {}_{11|}A_x - P \cdot (I\ddot{a})_x$$

$$\Downarrow$$

$$P = \frac{45 \cdot A_x + 5 \cdot (IA)^1_{x:\overline{11}|} + 55 \cdot {}_{11|}A_x}{(I\ddot{a})_x}$$

Now, we can proceed to the derivation of the formula for the reserve.

$$\boxed{{}_kV_x = \begin{cases} 45 \cdot A_{x+k} + 5 \cdot (IA)^{\ 1}_{x+k:\overline{11-k}|} + 5 \cdot k \cdot A^{\ 1}_{x+k:\overline{11-k}|} \\ +55 \cdot {}_{11-k|}A_{x+k} - P \cdot (I\ddot{a})_{x+k} - P \cdot k \cdot \ddot{a}_{x+k}, & k = 0, 1, \ldots, 10 \\ \\ 100 \cdot A_{x+k} - P \cdot (I\ddot{a})_{x+k} - P \cdot k \cdot \ddot{a}_{x+k}, & k = 11, 12, \ldots \end{cases}}$$

Remark: Be careful with the standard increasing term insurance and life annuity because, when dealing with reserves, a special term must be added. For example, when assuming the term insurance, $(IA)^{\ 1}_{x+k:\overline{11-k}|}$ starts again with payment 1, but at time k we already need to begin with payment $k+1$.

Example 6.3. *Derive the net premium reserve for the insurance with premium refund introduced in Example 5.2.*

Solution:

$$\boxed{\begin{aligned} {}_kV_x =& A_{x+k:\overline{n-k}|}^{\ \ \ \ \ \ \ 1} + P \cdot (IA)^{\ 1}_{x+k:\overline{n-k}|} + P \cdot k \cdot A^{\ 1}_{x+k:\overline{n-k}|} \\ & - P \cdot \ddot{a}_{x+k:\overline{n-k}|}, \qquad\qquad k = 0, 1, \ldots, n-1 \end{aligned}}$$

where the premium P was calculated in **Example 5.2.**

Example 6.4. *Derive the net premium reserve for the insurance with premium refund introduced in Example 5.3.*

Solution:

We can assume again two cases. In the first case, the premium is paid during the whole deferment period.

$$\boxed{{}_kV_x = \begin{cases} {}_{m-k|}\ddot{a}_{x+k} + P_1 \cdot (IA)^{\ 1}_{x+k:\overline{m-k}|} + P_1 \cdot k \cdot A^{\ 1}_{x+k:\overline{m-k}|} \\ -P_1 \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, 1, \ldots, m-1 \\ \\ \ddot{a}_{x+k}, & k = m, m+1, \ldots \end{cases}}$$

where the premium P_1 was calculated in **Example 5.3.**

Similarly, we can derive the reserve for the second case, in which the premium is paid only over m' years.

$$
\boxed{{}_kV_x = \begin{cases} {}_{m-k|}\ddot{a}_{x+k} + P_2 \cdot (IA)^{\;1}_{x+k:\overline{m'-k}|} + P_2 \cdot k \cdot A^{\;1}_{x+k:\overline{m'-k}|} & \\ +P_2 \cdot m' \cdot {}_{m'-k|}A_{x+k:\overline{m-m'}|} - P_2 \cdot \ddot{a}_{x+k:\overline{m'-k}|}, & k = 0, 1, \ldots, m'-1 \\ & \\ {}_{m-k|}\ddot{a}_{x+k} + P_2 \cdot m' \cdot A^{\;1}_{x+k:\overline{m-k}|}, & k = m', m'+1, \ldots, m-1 \\ \ddot{a}_{x+k}, & k = m, m+1, \ldots \end{cases}}
$$

where also the premium P_2 was calculated in **Example 5.3.**

Example 6.5. *Use the net premium reserve for conversion of an insurance and reduction of the sum insured*[1]*. Consider*

1. *endowment with duration n years with sum insured C_1 and annual net premium paid yearly over the whole contract duration. However, premium payment ended after $n' < n$ years, but the contract continues with reduced sum insured C_2. Derive a formula for C_2.*
2. *m-years deferred standard increasing whole life annuity in advance with sum insured C_1 and annual net premium paid yearly over the deferment period. However, premium payment ended after $m' < m$ years, but the contract continues with reduced sum insured C_2. Derive a formula for C_2.*

Solution:

1)

Premium calculated for sum insured C_1 has the following form

$$P = \frac{C_1 \cdot A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

At time n', when the premium payments were stopped, the value of the reserve is

$${}_{n'}V_x = C_1 \cdot A_{x+n':\overline{n-n'}|} - P \cdot \ddot{a}_{x+n':\overline{n-n'}|}.$$

This reserve turns to the net single premium for an endowment with sum insured C_2 for the remaining $n - n'$ years.

$${}_{n'}V_x = NSP = C_2 \cdot A_{x+n':\overline{n-n'}|}$$

$$\Downarrow$$

$$C_2 = \frac{C_1 \cdot A_{x+n':\overline{n-n'}|} - P \cdot \ddot{a}_{x+n':\overline{n-n'}|}}{A_{x+n':\overline{n-n'}|}} = C_1 \cdot \left(1 - \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \cdot \frac{\ddot{a}_{x+n':\overline{n-n'}|}}{A_{x+n':\overline{n-n'}|}}\right)$$

[1] The current value of the net premium reserve belongs to the insured person and can be used to modify the insurance policy.

2)

Premium calculated at the beginning of the contract is

$$P = \frac{C_1 \cdot {}_{m|}(I\ddot{a})_x}{\ddot{a}_{x:\overline{m|}}}.$$

At time m', when the premium payments were stopped, the value of the reserve is

$${}_{m'}V_x = C_1 \cdot {}_{m-m'|}(I\ddot{a})_{x+m'} - P \cdot \ddot{a}_{x+m':\overline{m-m'|}}.$$

We can use this reserve as the net single premium for this annuity with sum insured C_2.

$${}_{m'}V_x = NSP = C_2 \cdot {}_{m-m'|}(I\ddot{a})_{x+m'}$$

$$\Downarrow$$

$$C_2 = \frac{C_1 \cdot {}_{m-m'|}(I\ddot{a})_{x+m'} - P \cdot \ddot{a}_{x+m':\overline{m-m'|}}}{{}_{m-m'|}(I\ddot{a})_{x+m'}} = C_1 \cdot \left(1 - \frac{{}_{m|}(I\ddot{a})_x}{\ddot{a}_{x:\overline{m|}}} \cdot \frac{\ddot{a}_{x+m':\overline{m-m'|}}}{{}_{m-m'|}(I\ddot{a})_{x+m'}}\right)$$