Mathematics of Life Insurance 1 - HW1

Solution

Part 1 (4 points)

Consider a mortgage for $54,000$\$ that should be paid by monthly payments (paid at the end of each month) of the same amount in 30 years with a nominal interest rate of 4%.

1. Calculate the monthly payment.
2. For each payment compute what part of the current debt is paid by the payment and plot these values.

Solution

1)

$$54000 = P \cdot a_{\overline{30}|}^{(12)} \qquad \rightarrow \qquad P = \frac{54000 \cdot i^{(12)}}{12 \cdot (1 - v^{20})} = 257.8\$$$

2)

See Excel sheet

Part 2 (8 points)

Let

$$\mathrm{P}(T_0 > t) = \exp\left\{ -\left(Kt + \frac{1}{2}Lt^2 + \frac{M}{\log N} N^t - \frac{M}{\log N} \right) \right\},$$

where K, L, M and N are all positive.

1. Derive a formula for $\mathrm{P}(T_x > t)$.
2. Derive a formula for μ_x.
3. Consider

$$K = 0.0001, L = 0.000007, M = 0.00035, N = 1.08.$$

   (a) Calculate ${}_tp_{20}$ for $t = 1, 10, 50, 90$.
   (b) Calculate ${}_tq_{30}$ for $t = 1, 10, 50$.
   (c) Calculate ${}_{t|15}q_{35}$ for $t = 1, 10, 50$.
   (d) Calculate $\mathring{e}_x$ for $x = 65, 66, 67, 68, 69, 70$.

Solution

1)

$$\mathrm{P}(T_x > t) = \frac{\mathrm{P}(T_0 > x+t)}{\mathrm{P}(T_0 > x)} = \frac{\exp\left\{-K(x+t) - \frac{1}{2}L(x^2+2tx+t^2) - \frac{M}{\log N}N^{x+t} + \frac{M}{\log N}\right\}}{\exp\left\{-Kx - \frac{1}{2}Lx^2 - \frac{M}{\log N}N^x + \frac{M}{\log N}\right\}}$$
$$= \exp\left\{-Kt - \frac{1}{2}L(2tx+t^2) - \frac{M}{\log N}(N^{x+t} + N^x)\right\}$$

2)

$$\mu_x = \frac{d}{dx}\log_x p_0 = -\frac{d}{dx}\left(-Kx - \frac{1}{2}Lx^2 - \frac{M}{\log N}N^x + \frac{M}{\log N}\right) = K + Lx + MN^x$$

3a)

$\mathrm{P}(T_x > t)$ derived in 1) is the same as ${}_tp_x$, therefore, after the values substitution we get ${}_tp_{20} = \mathrm{P}(T_{20} > t)$

t	1	10	50	90
${}_tp_{20}$	0.9981	0.9731.3	0.3702	4.05×10^{-10}

3b)

${}_tq_{30} = 1 - {}_tp_{30}$

t	1	10	50
${}_tq_{30}$	0.004	0.055	0.881

3c)

${}_{t|15}q_{35} = {}_{t+15}q_{35} - {}_tq_{35} = {}_tp_{35} - {}_{t+15}p_{35}$

t	1	10	50
${}_{t\|15}q_{35}$	0.150	0.254	0.044

3d)

$\mathring{e}_x = \int_0^\infty {}_tp_x\, dt$

t	65	66	67	68	69	70
$\mathring{e}_x$	9.948	9.479	9.025	8.585	8.161	7.751

Part 3 (5 points)

Compute commutation functions $(C_x, D_x, M_x, N_x, R_x, S_x)$ for **Men, Women** and **Unisex** using Czech life tables for **2022**. Use TIR $= i = 2\%$. For preparing Unisex tables use $l_{Unisex} = 0.5 \cdot l_{Men} + 0.5 \cdot l_{Women}$.

When the commutation functions are calculated, plot the net single premiums for the following capital life insurances for ages $x = 20, \ldots, 60$ and all three life tables.

1. Pure endowment for $n = 65 - x$ years,
2. Term insurance until 65 years, i.e., for $n = 65 - x$ years.

For each insurance give one graph with three lines with respect to used life tables and add a short comment explaining the behaviour of the net single premiums.

Solution

See Excel sheet

Part 4 (4 points)

Consider Unisex life tables for the Czech Republic for year 2022.

Calculate ${}_{0.2}q_{52.4}$

1. under the linearity assumption,
2. under the assumption of constant force of mortality.

Solution

$$
\begin{aligned}
{}_{0.2}q_{52.4} &= \mathrm{P}(T_{52.4} < 0.2) = \mathrm{P}(0.4 < T_{52} < 0.6 | T_{52} > 0.4) = \frac{\mathrm{P}(0.4 < T_{52} < 0.6)}{\mathrm{P}(T_{52} > 0.4)} \\
&= \frac{{}_{0.6}q_{52} - {}_{0.4}q_{52}}{{}_{0.4}p_{52}}
\end{aligned}
$$

1)

$$\frac{{}_{0.6}q_{52} - {}_{0.4}q_{52}}{{}_{0.4}p_{52}} = \frac{0.6 \cdot q_{52} - 0.4 \cdot q_{52}}{1 - 0.4 \cdot q_{52}}$$

$$l_{52} = 95746.1, \qquad l_{53} = 95386.4 \qquad \rightarrow \qquad q_{52} = \frac{l_{52} - l_{53}}{l_{52}} = 0.003757$$

$${}_{0.2}q_{52.4} = 0.000752531$$

2)

$$\frac{{}_{0.6}q_{52} - {}_{0.4}q_{52}}{{}_{0.4}p_{52}} = \frac{1 - (p_{52})^{0.6} - (1 - (p_{52})^{0.4})}{(p_{52})^{0.4}} = \frac{(p_{52})^{0.4} - (p_{52})^{0.6}}{(p_{52})^{0.4}}$$

$$p_{52} = 1 - q_{52} = 0.996243 \qquad \rightarrow \qquad {}_{0.2}q_{52.4} = 0.00752532$$

Part 5 (5 points)

Consider term insurance for 20 years issued to a life aged x. The benefit is paid at the end of the year of death and is of amount 1 if death occurs during the first year, $1+g$ during the second year, $(1+g)^2$ during the third year, and so on.

Plot the net single premium considering Unisex life tables and $g = 0.5\%$ for ages $x = 20, \dots, 80$.

Solution

$$
\begin{aligned}
NSP &= v \cdot q_x + (1+g) \cdot v^2 \cdot {}_{1|}q_x + (1+g)^2 \cdot v^3 \cdot {}_{2|}q_x + \cdots + (1+g)^{n-1} \cdot v^n \cdot {}_{n-1|}q_x \\
&= \sum_{k=0}^{n-1} v^{k+1} \cdot (1+g)^k \cdot {}_{k|}q_x = \frac{1}{1+g} \cdot \sum_{k=0}^{n-1} v^{k+1} \cdot (1+g)^{k+1} \cdot {}_{k|}q_x \\
&= \frac{1}{1+g} \cdot A^1_{x:\overline{n}|\,i^*},
\end{aligned}
$$

where $i^* = \frac{i-g}{1+g}$ and $A^1_{x:\overline{n}|\,i^*}$ corresponds to the NSP of term insurance calculated with interest rate i^*.

See Excel sheet

Part 6 (4 points)

Suppose that a life aged 30 arranged an insurance with the following parameters. If death occurs in the first 20 years, 10,000 is paid. Otherwise, 20,000 is paid. Moreover, it was arranged with the insurance company that a premium refund will be paid in the case of death during the first 5 years. This refund will be 25% of the **net single premium**. Assume that it holds: $l_x = 100 - x, \quad 0 \leq x \leq 100$ and $i = 5\%$.

Calculate the net single premium of this insurance.

Solution

$$\boxed{Z = \begin{cases} (10000 + 0.25 \cdot NSP) \cdot v^{K+1}, & K = 0, 1, \ldots, 4 \\ 10000 \cdot v^{K+1}, & K = 5, 6, \ldots, 19 \\ 20000 \cdot v^{20}, & K = 20, 21, \ldots \end{cases}}$$

We can split the calculation of the NSP to three parts: term insurance for 5 years, term insurance for 20 years and pure endowment for 20 years.

Term insurance for 5 years:

$$NSP_1 = 0.25 \cdot NSP \cdot \sum_{k=0}^{4} v^{k+1} \cdot {}_kp_x \cdot q_{x+k},$$

where

$$v = \frac{1}{1+i} = \frac{1}{1.05} = \frac{20}{21},$$

$${}_kp_x \cdot q_{x+k} = \frac{l_{x+k} - l_{x+k+1}}{l_x} = \frac{1}{100 - x} \qquad \to \qquad {}_kp_{30} \cdot q_{30+k} = \frac{1}{70}.$$

Therefore,

$$NSP_1 = 0.25 \cdot NSP \cdot \sum_{k=0}^{4} \left(\frac{20}{21}\right)^{k+1} \cdot \frac{1}{70} =,$$

Term insurance for 20 years:

$$NSP_2 = 10000 \cdot \sum_{k=0}^{19} v^{k+1} \cdot {}_kp_x \cdot q_{x+k},$$

Hence,

$$NSP_2 = 10000 \cdot \sum_{k=0}^{19} \left(\frac{20}{21}\right)^{k+1} \cdot \frac{1}{70} =,$$

Pure endowment for 20 years:

$$NSP_3 = 20000 \cdot v^{20} \cdot {}_{20}p_x,$$

where

$${}_{20}p_x = \frac{l_{x+20}}{l_x} = \frac{80 - x}{100 - x} \qquad \to \qquad {}_{20}p_{30} = \frac{5}{7}.$$

Thus,

$$NSP_3 = 20000 \cdot \left(\frac{20}{21}\right)^{20} \cdot \frac{5}{7},$$

Altogether, we get

$$NSP = NSP1 + NSP2 + NSP3 = 0.0154624 \cdot NSP + 1780.32 + 5384.14$$

Solving this equation we obtain

$$NSP = 7276.97.$$