NMFP432 Mathematics of Life Insurance 2

Collection of solved examples

Version: April 15 2024

This collection of solved examples follows the collection intended for the subject Mathematics of Life Insurance 1 located at
`https://www2.karlin.mff.cuni.cz/~vejmelp/MLI1_Collection.pdf`.

Make sure you understand the examples in the previous collection.

7 Net Premium Reserve - Review

Example 7.1. *Consider standard decreasing term insurance with duration n years $(DA)^1_{x:\overline{n|}}$, where the annual net premium is paid m years ($m < n$). Derive the total loss, the annual net premium and the net premium reserve.*

Solution:

Firstly, the total loss has the following form:

$$\boxed{L = \begin{cases} (n-K)\cdot v^{K+1} - P\cdot \sum_{k=0}^{K} v^k, & K = 0,1,\dots,m-1 \\ (n-K)\cdot v^{K+1} - P\cdot \sum_{k=0}^{m-1} v^k, & K = m,\dots,n-1 \\ 0 - P\cdot \sum_{k=0}^{m-1} v^k, & K = n,n+1,\dots \end{cases}}$$

Secondly, one can write the annual net premium as follows:

$$\mathbb{E}L = 0 = (DA)^1_{x:\overline{n|}} - P\cdot \ddot{a}_{x:\overline{m|}}$$
$$\Downarrow$$
$$P = \frac{(DA)^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

Finally, the net premium reserve can be written as:

$$\boxed{{}_kV_x = \begin{cases} (DA)^{\;1}_{x+k,\overline{n-k|}} - P\cdot \ddot{a}_{x+k:\overline{m-k|}}, & k = 0,\dots,m-1 \\ (DA)^{\;1}_{x+k,\overline{n-k|}}, & k = m,\dots,n-1 \end{cases}}$$

Example 7.2. *Consider m-years deferred standard increasing temporary life annuity in advance with duration n years* ${}_{m|}(I\ddot{a})_{x:\overline{n|}}$*, where the annual net premium is paid* m' *years* ($m' < m$)*. Derive the total loss, the annual net premium and the net premium reserve.*

Solution:

Total loss:

$$\boxed{L = \begin{cases} 0 - P \cdot \sum_{k=0}^{K} v^k, & K = 0, 1, \ldots, m'-1 \\ 0 - P \cdot \sum_{k=0}^{m'-1} v^k, & K = m', \ldots, m-1 \\ \sum_{k=m}^{K} (k-m+1) \cdot v^k - P \cdot \sum_{k=0}^{m'-1} v^k, & K = m, \ldots, m+n-1 \\ \sum_{k=m}^{m+n-1} (k-m+1) \cdot v^k - P \cdot \sum_{k=0}^{m'-1} v^k, & K = m+n, m+n+1, \ldots \end{cases}}$$

Annual net premium:

$$\mathbb{E}L = 0 = {}_{m|}(I\ddot{a})_{x:\overline{n|}} - P \cdot \ddot{a}_{x:\overline{m|}}$$
$$\Downarrow$$
$$P = \frac{{}_{m|}(I\ddot{a})_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

Net premium reserve:

$$\boxed{{}_kV_x = \begin{cases} {}_{m-k|}(I\ddot{a})_{x+k:\overline{n|}} - P \cdot \ddot{a}_{x+k:\overline{m'-k|}}, & k = 0, 1, \ldots, m'-1 \\ {}_{m-k|}(I\ddot{a})_{x+k:\overline{n|}}, & k = m', \ldots, m-1 \\ (I\ddot{a})_{x+k:\overline{n+m-k|}} + (k-m) \cdot \ddot{a}_{x+k:\overline{n+m-k|}}, & k = m, \ldots, m+n-1 \\ 0, & k = m+n, m+n+1, \ldots \end{cases}}$$

Example 7.3. *Use the net premium reserve for conversion of an insurance and reduction of the sum insured. Consider m-years deferred temporary life annuity in arrear with duration n years, sum insured* C_1 *and annual net premium paid yearly over the deferment period. However, premium payment ended after* $m' < m$ *years, but the contract continues with reduced sum insured* C_2*. Derive an explicit formula for* C_2*.*

Solution:

Premium calculated for the sum insured C_1 has the following form

$$P = \frac{C_1 \cdot {}_{m|}a_{x:\overline{n|}}}{\ddot{a}_{x:\overline{m|}}}.$$

At time m', when the premium payments were stopped, the reserve is

$${}_{m'}V_x = C_1 \cdot {}_{m-m'|}a_{x+m':\overline{n|}} - P \cdot \ddot{a}_{x+m':\overline{m-m'|}}.$$

This reserve can be used as a single payment for the net single premium of the same insurance with the sum insured C_2 for the remaining years.

$$ {}_{m'}V_x = NSP = C_2 \cdot {}_{m-m'|}a_{x+m':\overline{n}|} $$

$$ \Downarrow $$

$$ C_2 = \frac{C_1 \cdot {}_{m-m'|}a_{x+m':\overline{n}|} - P \cdot \ddot{a}_{x+m':\overline{m-m'}|}}{{}_{m-m'|}a_{x+m':\overline{n}|}} = C_1 \cdot \left(1 - \frac{{}_{m|}a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}} \cdot \frac{\ddot{a}_{x+m':\overline{m-m'}|}}{{}_{m-m'|}a_{x+m':\overline{n}|}}\right) $$

Example 7.4. *Derive the net premium reserve for the term insurance contract for* $n = 20$ *years of a person at age* $x = 40$ *when*

1. *the sum insured* $C = 10,000$ *CZK is constant over the whole contract life,*
2. *the sum insured is constant over the first 10 years and than increases by* $5,000$ *CZK each year.*

Assume that the premium is payed regularly over the whole contract duration.

Solution:

1)

In the first case, the sum insured is assumed to be a constant. Thus,

$$ P = \frac{10,000 \cdot A^{1}_{40:\overline{20}|}}{\ddot{a}_{40:\overline{20}|}}. $$

The reserve at time k can be written as follows

$$ \boxed{{}_{k}V_{40} = 10,000 \cdot A^{1}_{40+k:\overline{20-k}|} - P \cdot \ddot{a}_{40+k:\overline{20-k}|}, \qquad k = 0, 1, ..., 19} $$

2)

When assuming the adjustment of increasing the sum insured by $5,000$ each year after first 10 years, the premium and the reserve must be changed to

$$ P = \frac{10,000 \cdot A^{1}_{40:\overline{20}|} + 5,000 \cdot {}_{10|}(IA)^{1}_{40:\overline{10}|}}{\ddot{a}_{40:\overline{20}|}} $$

and

$$ \boxed{{}_{k}V_{40} = \begin{cases} 10,000 \cdot A^{1}_{40+k:\overline{20-k}|} + 5,000 \cdot {}_{10-k|}(IA)^{1}_{40+k:\overline{10}|} - P \cdot \ddot{a}_{40+k:\overline{20-k}|}, & k = 0, ..., 9 \\ 10,000 \cdot A^{1}_{40+k:\overline{20-k}|} + 5,000 \cdot \left[(IA)^{1}_{40+k:\overline{20-k}|} + (k-10) \cdot A^{1}_{40+k:\overline{20-k}|}\right] & \\ -P \cdot \ddot{a}_{40+k:\overline{20-k}|}, & k = 10, ..., 19 \end{cases}} $$

Example 7.5. *Consider a person at age $x = 35$ who has signed an insurance contract for life annuity in advance deferred to age 65 with monthly payments of CZK 5000. Moreover, assume that the premium refund agreement is active during the deferment period. Derive the formula for the net premium reserve.*

Solution:

First of all, it is worth reminding (when assuming a life annuity, but the concept is similar also for the other usual types of insurance contracts) that $\ddot{a}_x^{(m)}$ is a sign for a whole life annuity whose payments are of size $\frac{1}{m}$ and are made m-times a year.

Since payments are made monthly, to obtain the sum insured, one has to multiply $5,000$ by 12. We can write down the total loss and then derive the formula for the premium.

$$\boxed{L = \begin{cases} 12 \cdot (K + S^{(12)}) \cdot P \cdot v^{K+S^{(12)}} - P \cdot \sum_{k=0}^{12\cdot(K+S^{(12)})-1} v^{\frac{k}{12}}, & K = 0, 1, \ldots, 29 \\ 5,000 \cdot \sum_{k=360}^{12\cdot(K+S^{(12)})-1} v^{\frac{k}{12}} - P \cdot \sum_{k=0}^{359} v^{\frac{k}{12}}, & K = 30, 31, \ldots \end{cases}}$$

Therefore,

$$P = \frac{5,000 \cdot 12 \cdot {}_{30|}\ddot{a}_{35}^{(12)}}{12 \cdot \ddot{a}_{35:\overline{30}|}^{(12)} - 12 \cdot (I^{(12)}A^{(12)})^{1}_{35:\overline{30}|}} = \frac{5,000 \cdot {}_{30|}\ddot{a}_{35}^{(12)}}{\ddot{a}_{35:\overline{30}|}^{(12)} - (I^{(12)}A^{(12)})^{1}_{35:\overline{30}|}},$$

is the value of monthly premium.

Values of the net premium reserve will be examined at the ends of years.

$$\boxed{{}_kV_{35} = \begin{cases} 5,000 \cdot 12 \cdot {}_{30-k|}\ddot{a}_{35+k}^{(12)} + 12 \cdot P \cdot (I^{(12)}A^{(12)})^{1}_{35+k:\overline{30-k}|} & \\ +12 \cdot P \cdot k \cdot A^{(12)}_{\overset{1}{35+k}:\overline{30-k}|} - 12 \cdot P \cdot \ddot{a}^{(12)}_{35+k:\overline{30-k}|}, & k = 0, 1, ..., 29 \\ & \\ 5,000 \cdot 12 \cdot \ddot{a}^{(12)}_{35+k}, & k = 30, 31, ... \end{cases}}$$

8 General Net Premium Reserve

Example 8.1. *Apply the general recursive formula for net premium reserve and decompose the premium into savings and risk components. Consider*

1. *endowment with duration n years and net annual premium paid over the whole contract duration,*
2. *m-years deferred temporary life annuity in arrear with duration n year and annual net premium paid over the deferment period,*
3. *m-years deferred whole life annuity in advance with net single premium paid at once at the beginning,*
4. *m-years deferred term insurance with duration n years and net annual premium paid over first $m_1 < m$.*

Solution:

Recursive formula for general net-premium reserve can be written as

$$ {}_kV_x = \sum_{j=0}^{\infty} \left(c_{k+j+1} \cdot v^{j+1} \cdot {}_jp_{x+k} \cdot q_{x+k+j} \right) - \sum_{j=0}^{\infty} \left(\Pi_{k+j} \cdot v^j \cdot {}_jp_{x+k} \right). $$

The premium can be decomposed to the savings premium and the risk premium using the following formulas

$$ \begin{aligned} \Pi_k^s &= v \cdot {}_{k+1}V_x - {}_kV_x, \\ \Pi_k^r &= (c_{k+1} - {}_{k+1}V_x) \cdot v \cdot q_{x+k}. \end{aligned} $$

1)

The endowment can be assumed in terms of the values c_l and Π_l as a general insurance with

$$ c_1 = \cdots = c_n = 1, \qquad c_{n+1} = c_{n+2} = \cdots = 0 $$
$$ \Pi_0 = \cdots = \Pi_{n-1} = P = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}, \qquad \Pi_n = -1, \qquad \Pi_{n+1} = \Pi_{n+2} = \cdots = 0 $$

Therefore, the reserve can be written as

$$ \boxed{ \begin{aligned} {}_kV_x = &\sum_{j=0}^{n-k-1} \left(v^{j+1} \cdot {}_jp_{x+k} \cdot q_{x+k+j} \right) - \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \cdot \sum_{j=0}^{n-k-1} \left(v^j \cdot {}_jp_{x+k} \right) \\ &+ v^{n-k} {}_{n-k}p_{x+k}, \qquad\qquad k = 0, \ldots, n-1 \end{aligned} } $$

and for $k = n$ the reserve would be ${}_nV_x = 1$.

The savings and risk premiums could be calculated now with the use of the formulas stated above.

2)

The temporary life annuity in arrear with duration n years deferred by m years with annual premium paid over the deferment period can be assumed in terms of the values c_l and Π_l as a general insurance with

$$c_1 = c_2 = \cdots = 0$$
$$\Pi_0 = \cdots = \Pi_{m-1} = P = \frac{{}_{m|}a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}, \qquad \Pi_m = 0, \qquad \Pi_{m+1} = \cdots = \Pi_{m+n} = -1$$

Thus, the reserve can be written as

$${}_kV_x = -\frac{{}_{m|}a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}} \cdot \sum_{j=0}^{m-k-1} \left(v^j \cdot {}_jp_{x+k}\right) + \sum_{j=m-k+1}^{m-k+n} \left(v^j \cdot {}_jp_{x+k}\right).$$

It is also better to distinguish two forms of the reserve based on k. We can split this formula into

$$\boxed{{}_kV_x = \begin{cases} -\frac{{}_{m|}a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}} \cdot \sum\limits_{j=0}^{m-k-1} \left(v^j \cdot {}_jp_{x+k}\right) + \sum\limits_{j=m-k+1}^{m-k+n,} \left(v^j \cdot {}_jp_{x+k}\right), & k = 0, \ldots, m-1 \\ \sum\limits_{j=\max\{0,m-k+1\}}^{m-k+n} \left(v^j \cdot {}_jp_{x+k}\right), & k = m, \ldots, m+n \end{cases}}$$

and the savings and risk premiums could be calculated.

3)

The whole life annuity in advance deferred by m years with net single premium paid at the beginning can be assumed in terms of the values c_l and Π_l as a general insurance with

$$c_1 = c_2 = \cdots = 0$$
$$\Pi_0 = NSP = {}_{m|}\ddot{a}_x, \qquad \Pi_1 = \cdots = \Pi_{m-1} = 0, \qquad \Pi_m = \Pi_{m+1} = \cdots = -1$$

Hence, the reserve can be written as

$$\boxed{{}_kV_x = \begin{cases} -{}_{m|}\ddot{a}_x + \sum\limits_{j=m}^{\infty} \left(v^j \cdot {}_jp_x\right), & k = 0 \\ \sum\limits_{j=\max\{0,m-k\}}^{\infty} \left(v^j \cdot {}_jp_{x+k}\right), & k = 1, 2, \ldots \end{cases}}$$

and the savings and risk premiums could be calculated.

4)

Considered m-years deferred term insurance with duration n years and net annual premium paid over first $m_1 < m$ can be assumed in terms of the values c_l and Π_l as a general insurance with

$$c_1 = \ldots = c_m = 0, \qquad c_{m+1} = \ldots = c_{m+n} = 1, \qquad c_{m+n+1} = c_{m+n+2} = \ldots = 0,$$
$$\Pi_0 = \ldots = \Pi_{m_1-1} = P = \frac{{}_{m|}A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m_1}|}}, \qquad \Pi_{m_1} = \Pi_{m_1+1} = \ldots = 0$$

The reserve is as follows

$$\boxed{{}_kV_x = \begin{cases} \displaystyle\sum_{j=m-k}^{m-k+n-1} \left(v^{j+1} \cdot {}_jp_{x+k} \cdot q_{x+k+j}\right) - \frac{{}_{m|}A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m_1}|}} \cdot \sum_{j=0}^{m_1-k-1} \left(v^{j} \cdot {}_jp_{x+k}\right), & k = 0, \ldots, m_1 - 1 \\ \displaystyle\sum_{j=\max\{0,m-k\}}^{m-k+n-1} \left(v^{j+1} \cdot {}_jp_{x+k} \cdot q_{x+k+j}\right), & k = m_1, \ldots, m+n-1 \end{cases}}$$

Example 8.2. *Consider a fully discrete 3-year insurance issued to life aged x for which*

k	c_{k+1}	q_{x+k}
0	2	0.20
1	3	0.25
2	4	0.50

Level annual net premiums of 1 *are paid at the beginning of each year while the person is alive. The effective annual interest rate is $i = \frac{1}{9}$. Calculate the reserves at the end of each year, allocate the total loss to policy years and calculate* $Var(\Lambda_1)$.

Solution:

To calculate reserves we can use the following recursive formula

$${}_kV_x + \Pi_k = v \cdot (c_{k+1} \cdot q_{x+k} + {}_{k+1}V_x \cdot p_{x+k}).$$

Since we know that ${}_0V_x = 0$, we can use this value for computing ${}_1V_x$.

For $k = 0$:

$${}_0V_x + \Pi_0 = v \cdot (c_1 \cdot q_x + {}_1V_x \cdot p_x)$$
$$\Downarrow$$
$${}_1V_x = \frac{\frac{\Pi_0}{v} - c_1 \cdot q_x}{p_x} = \frac{\frac{1}{\frac{1}{1+\frac{1}{9}}} - 2 \cdot 0.2}{0.8} = 0.8889.$$

For $k = 1$:

$$ {}_1V_x + \Pi_1 = v \cdot (c_2 \cdot q_{x+1} + {}_2V_x \cdot p_{x+1}) $$
$$ \Downarrow $$
$$ {}_2V_x = \frac{\frac{{}_1V_x + \Pi_1}{v} - c_2 \cdot q_{x+1}}{p_{x+1}} = \frac{\frac{1}{\frac{0.8889+1}{1+\frac{1}{9}}} - 3 \cdot 0.25}{0.75} = 1.7984. $$

For $k = 2$:

$$ {}_2V_x + \Pi_2 = v \cdot (c_3 \cdot q_{x+2} + {}_3V_x \cdot p_{x+2}) $$
$$ \Downarrow $$
$$ {}_3V_x = \frac{\frac{{}_2V_x + \Pi_2}{v} - c_3 \cdot q_{x+2}}{p_{x+2}} = \frac{\frac{1}{\frac{1.7984+1}{1+\frac{1}{9}}} - 4 \cdot 0.5}{0.5} = 2.2186. $$

Loss incurred by insurer during year $k+1$ (denoted as Λ_k) is defined as

$$ \Lambda_k = \begin{cases} 0, & K < k, \\ c_{k+1} \cdot v - ({}_kV_x + \Pi_k), & K = k, \\ v \cdot {}_{k+1}V_x - ({}_kV_x + \Pi_k), & K > k. \end{cases} $$

For $k = 0$:

$$ \Lambda_0 = \begin{cases} c_1 \cdot v - ({}_0V_x + \Pi_0) = 2 \cdot \frac{1}{1+\frac{1}{9}} - (0+1) = 0.8, & K = 0, \\ v \cdot {}_1V_x - ({}_0V_x + \Pi_0) = \frac{1}{1+\frac{1}{9}} \cdot 0.8889 - (0+1) = -0.2, & K > 0. \end{cases} $$

For $k = 1$:

$$ \Lambda_1 = \begin{cases} 0, & K < 1 \\ c_2 \cdot v - ({}_1V_x + \Pi_1) = 3 \cdot \frac{1}{1+\frac{1}{9}} - (0.8889+1) = 0.8111, & K = 1, \\ v \cdot {}_2V_x - ({}_1V_x + \Pi_1) = \frac{1}{1+\frac{1}{9}} \cdot 1.7984 - (0.8889+1) = -0.2703, & K > 1. \end{cases} $$

For $k = 2$:

$$ \Lambda_2 = \begin{cases} 0, & K < 2 \\ c_3 \cdot v - ({}_2V_x + \Pi_2) = 4 \cdot \frac{1}{1+\frac{1}{9}} - (1.7984+1) = 0.8016, & K = 2, \\ v \cdot {}_3V_x - ({}_2V_x + \Pi_2) = \frac{1}{1+\frac{1}{9}} \cdot 2.2186 - (1.7984+1) = -0.8016, & K > 2. \end{cases} $$

Calculation of $\mathrm{Var}(\Lambda_1)$ can be done in two ways.

1)

We know that $\mathrm{Var}(\Lambda_1) = \mathrm{E}(\Lambda_1^2) - [\mathrm{E}(\Lambda_1)]^2$.
Now it is necessary to calculate $\mathrm{P}(K < 1), \mathrm{P}(K = 1)$ and $\mathrm{P}(K > 1)$.

$$ \begin{aligned} &\mathrm{P}(K < 1) = \mathrm{P}(K = 0) = q_x = 0.2, \\ &\mathrm{P}(K = 1) = {}_1p_x \cdot q_{x+1} = 0.8 \cdot 0.25 = 0.2, \\ &\mathrm{P}(K > 1) = 1 - \mathrm{P}(K \le 1) = 1 - (0.2 + 0.2) = 0.6. \end{aligned} $$

Then

$$
\begin{aligned}
\mathrm{E}(\Lambda_1) &= 0 \cdot 0.2 + 0.8111 \cdot 0.2 - 0.2703 \cdot 0.6 = 0, \\
\mathrm{E}(\Lambda_1^2) &= 0^2 \cdot 0.2 + 0.8111^2 \cdot 0.2 + (-0.2703)^2 \cdot 0.6 = 0.1754, \\
\mathrm{Var}(\Lambda_1) &= \mathrm{E}(\Lambda_1^2) - [\mathrm{E}(\Lambda_1)]^2 = 0.1754.
\end{aligned}
$$

2)

We can also use knowledge resulting from the theorem proved during lectures. It says that $\mathrm{E}(\Lambda_k) = 0$ which corresponds to the result obtained in 1). Moreover, the variance can be calculated directly as

$$\mathrm{Var}(\Lambda_k) = (c_{k+1} - {}_{k+1}V_x)^2 \cdot v^2 \cdot {}_{k+1}p_x \cdot q_{x+k}.$$

Hence,

$$\mathrm{Var}(\Lambda_1) = (c_2 - {}_2V_x)^2 \cdot v^2 \cdot {}_2p_x \cdot q_{x+1},$$

and rewriting ${}_2p_x$ as $p_x \cdot p_{x+1}$, which is $0.8 \cdot 0.75$, we get

$$\mathrm{Var}(\Lambda_1) = (3 - 1.7984)^2 \cdot \left(\frac{1}{1+\frac{1}{9}}\right)^2 \cdot 0.8 \cdot 0.75 \cdot 0.25 = 0.1754.$$

9 Net Premium Reserve II - Continuous Model

Example 9.1. *During lectures, there was constructed a retrospective relation for the net premium reserve*

$$ {}_kV_x = \frac{1}{v^k \cdot {}_kp_x} \cdot \left(\sum_{j=0}^{k-1} \Pi_j \cdot v^j \cdot {}_jp_x - \sum_{j=0}^{k-1} c_{j+1} \cdot v^{j+1} \cdot {}_jp_x \cdot q_{x+j} \right). $$

This construction is related to the discrete case. Consider now the continuous model, assume Thiele's differential equation and derive the retrospective relation for the reserve.

Solution:

Thiele's differential equation:

$$ \Pi(t) + \delta \cdot V(t) = V'(t) + (c(t) - V(t)) \cdot \mu_{x+t}. $$

This can be rewritten as

$$ V'(t) - (\delta + \mu_{x+t}) \cdot V(t) = \Pi(t) - c(t) \cdot \mu_{x+t}. $$

After multiplying both sides of the previous equation by $v^t \cdot {}_tp_x$, one get

$$ v^t \cdot {}_tp_x \cdot [V'(t) - (\delta + \mu_{x+t}) \cdot V(t)] = [\Pi(t) - c(t) \cdot \mu_{x+t}] \cdot v^t \cdot {}_tp_x. \qquad (*) $$

Since for the derivative of $v^t \cdot {}_tp_x$ it holds

$$ \frac{d}{dt} \left(v^t \cdot {}_tp_x \right) = v^t \cdot \ln(v) \cdot {}_tp_x - v^t \cdot {}_tp_x \cdot \mu_{x+t} $$

where $\ln(v) = -\delta$, the left-hand side of the equation $(*)$ corresponds to $\frac{d}{dt}\left(v^t \cdot {}_tp_x \cdot V(t)\right)$. Therefore, we can rewrite $(*)$ as

$$ \frac{d}{dt} \left(v^t \cdot {}_tp_x \cdot V(t) \right) = [\Pi(t) - c(t) \cdot \mu_{x+t}] \cdot v^t \cdot {}_tp_x. $$

For the purpose of the next step we will replace t by r

$$ \frac{d}{dr} \left(v^r \cdot {}_rp_x \cdot V(r) \right) = [\Pi(r) - c(r) \cdot \mu_{x+r}] \cdot v^r \cdot {}_rp_x. $$

The next step is to integrate the previous equation from 0 to t:

$$ \int_0^t \frac{d}{dr} \left(v^r \cdot {}_rp_x \cdot V(r) \right) dr = \int_0^t [\Pi(r) - c(r) \cdot \mu_{x+r}] \cdot v^r \cdot {}_rp_x \, dr. $$

$$ \Downarrow $$

$$ v^t \cdot {}_tp_x \cdot V(t) - v^0 \cdot {}_0p_x \cdot V(0) = \int_0^t [\Pi(r) - c(r) \cdot \mu_{x+r}] \cdot v^r \cdot {}_rp_x \, dr. $$

Since $V(0) = 0$, the second term on the left-hand side of the previous equation is equal to zero and after dividing both sides by $v^t \cdot {}_tp_x$, we obtain

$$ V(t) = \frac{\int_0^t [\Pi(r) - c(r) \cdot \mu_{x+r}] \cdot v^r \cdot {}_rp_x \, dr}{v^t \cdot {}_tp_x}, $$

which is the continuous counterpart of the discrete retrospective relation for the net premium reserve.

Example 9.2. *Consider the continuous (time) model for the net premium reserve calculation. Show that*

$$\mathbb{E}\left[v^T \cdot V(T)\right] = \mathbb{E}\left[\int_0^T \Pi^s(t) \cdot v^t \, dt\right],$$

i.e. that the expected value of discounted net premium reserve at the moment of death is equal to the expected value of discounted savings component of the premium cumulated until the moment of death.

Solution:

We assume the overall loss as

$$L = c(T) \cdot v^T - \int_0^T \Pi(t) \cdot v^t \, dt.$$

Using the principle of equivalence $\mathbb{E}L = 0$, we can prove the given relation as follows:

$$\begin{aligned}
0 = \mathbb{E}L &= \mathbb{E}\left[c(T) \cdot v^T\right] - \mathbb{E}\left[\int_0^T \Pi(t) \cdot v^t \, dt\right] \\
&= \mathbb{E}\left[c(T) \cdot v^T\right] - \mathbb{E}\left[\int_0^\infty \Pi(t) \cdot v^t \cdot \mathbb{I}[T > t] \, dt\right] \\
&= \int_0^\infty c(t) \cdot v^t \cdot {}_tp_x \mu_{x+t} \, dt - \int_0^\infty \Pi(t) \cdot v^t \cdot {}_tp_x \, dt \\
&= \int_0^\infty \left[c(t) \cdot \mu_{x+t} - \Pi(t)\right] \cdot v^t \cdot {}_tp_x \, dt \\
&= \int_0^\infty \left[V(t) \cdot \mu_{x+t} - V'(t) + \delta \cdot V(t)\right] \cdot v^t \cdot {}_tp_x \, dt \\
&= \int_0^\infty V(t) \cdot v^t \cdot {}_tp_x \cdot \mu_{x+t} \, dt - \int_0^\infty \Pi^s(t) \cdot v^t \cdot {}_tp_x \, dt \\
&= \mathbb{E}\left[v^T \cdot V(T)\right] - \mathbb{E}\left[\int_0^T \Pi^s(t) \cdot v^t \, dt\right].
\end{aligned}$$

We used the decomposition of the premium rate $\Pi(t)$ into the saving component and the risk component, which in the continuous model can be written as

$$\Pi(t) = \underbrace{V'(t) - \delta \cdot V(t)}_{\Pi^s(t)} + \underbrace{(c(t) - V(t)) \cdot \mu_{x+t}}_{\Pi^r(t)}.$$

10 Multiple decrements

Notation

- J – cause of decrement (death, disability, cancellation, ...),
- m – number of decrements,
- $T = T_x$ – time of leaving the original status due to one of mutually exclusive decrements,
- ${}_tq_{j,x} = P(T < t,\ J = j)$ – probability of leaving the original status due to decrement j within t years,
- ${}_tp_x = 1 - \sum_{j=1}^{m} {}_tq_{j,x}$ – probability of surviving in the original status at least t years,
- $\mu_{j,x+t} = \lim_{h\to 0^+} \frac{P(T<t+h, J=j|T>t)}{h}$ – force of decrement j,
- $\mu_{x+t} = \sum_{j=1}^{m} \mu_{j,x+t}$,
- $g_j(t) = {}_tp_x \cdot \mu_{j,x+t}$ – probability density function
- $q_{j,x+k} = P(T < k+1,\ J = j \mid T > k)$,
- $q_{x+k} = \sum_{j=1}^{m} q_{j,x+k}$,
- $p_{x+k} = 1 - q_{x+k}$,
- $P(K = k, J = j) = {}_kp_x \cdot q_{j,x+k}$ – joint probability distribution of (K, J)

Example 10.1. *Consider term insurance for n years which provides SI to a beneficiary in the case of death of the insured person and $2 \cdot SI$ in the case of death by accident. Derive the net single premium if the death benefit is paid*

a) at the end of the year of death,

b) immediately on death.

Derive the net annual premium which is paid during the whole contract life/until death.

Solution:

We define two causes of decrement: $J = 1$ for death by accident, $J = 2$ for death from other causes. Then the net single premium is equal to

a)

$$\text{NSP} = \sum_{k=0}^{n-1} 2 \cdot SI \cdot v^{k+1} \cdot {}_kp_x \cdot q_{1,x+k} + \sum_{k=0}^{n-1} SI \cdot v^{k+1} \cdot {}_kp_x \cdot q_{2,x+k},$$

b)

$$\text{NSP} = \int_0^n 2 \cdot SI \cdot v^t \cdot {}_tp_x \cdot \mu_{1,x+t}\, dt + \int_0^n SI \cdot v^t \cdot {}_tp_x \cdot \mu_{2,x+t}\, dt.$$

The net annual premium can be derived in the standard form (using the equivalence principle)

$$\text{NAP} = \frac{\text{NSP}}{\ddot{a}_{x\overline{n}|}},$$

where $\ddot{a}_{x\overline{n}|}$ is the NSP for the standard life annuity due (=based on life tables without special decrements, i.e. with death only) for n years.

Example 10.2. *Consider special life tables with two causes of decrement for extreme sports and four races:*

year	*prob. of death*	*prob. of disability*	*survival prob.*
0	*0.15*	*0.25*	*0.60*
1	*0.10*	*0.20*	*0.70*
2	*0.05*	*0.15*	*0.80*
3	*0.00*	*0.10*	*0.90*

In the beginning, 1,000 extreme (iid) sportsmen start. Compute/estimate

a) expected number and variance of survivors over four races,

b) expected number and variance of deaths during four races,

c) distribution of the causes of decrement J,

d) conditional distribution of ending the season during third race.

Solution:

We will consider three causes of decrement: $J = 1$ death, $J = 2$ disability, $J = 3$ finishing the season. In this case, we must slightly modify our table[1]

k	$q_{1,k}$	$q_{2,k}$	$q_{3,k}$	p_k
0	0.15	0.25	0	0.60
1	0.10	0.20	0	0.70
2	0.05	0.15	0	0.80
3	0.00	0.10	0.90	0

a)

Let X_1 denote the r.v. of the number of survivors over four races. We can solve it directly or we can realize that we work with binomial distribution with parameters

$$n = 1000, \; p^{(a)} = p_0 \cdot p_1 \cdot p_2 \cdot q_{3,3} = 0.3024.$$

Therefore, $\mathbb{E}X_1 = n \cdot p^{(a)} = 302.4$ and $\text{Var}(\text{X}_1) = n \cdot p^{(a)} \cdot (1 - p^{(a)}) = 210.95$.

[1] You must be always sure that you are working with probability distribution.

b)

Let X_2 denote the r.v. of the number of deaths over four races. Again we can employ the binomial distribution

$$n = 1000,\ p^{(b)} = q_{1,0} + p_0 \cdot q_{1,1} + p_0 \cdot p_1 \cdot q_{1,2} + p_0 \cdot p_1 \cdot p_2 \cdot q_{1,3} = 0.231.$$

Hence, $\mathbb{E}X_2 = n \cdot p^{(b)} = 231$ and $\mathrm{Var(X_2)} = n \cdot p^{(b)} \cdot (1 - p^{(b)}) = 177,64$.

c)

We have already some of the marginal probabilities, namely $P(J = 1) = p^{(b)}$ and $P(J = 3) = p^{(a)}$. Therefore,

$$P(J = 2) = 1 - p^{(a)} - p^{(b)} = q_{2,0} + p_0 \cdot q_{2,1} + p_0 \cdot p_1 \cdot q_{2,2} + p_0 \cdot p_1 \cdot p_2 \cdot q_{2,3} = 0.4666.$$

d)

In general, we can derive the conditional probabilities as

$$P(J = j|K = k) = \frac{P(J = j, K = k)}{P(K = k)} = \frac{{}_kp_x \cdot q_{j,x+k}}{{}_kp_x \cdot q_{x+k}} = \frac{q_{j,x+k}}{q_{x+k}}.$$

Thus, in our case we have

$$\begin{aligned} P(J = 1|K = 2) &= \frac{q_{1,2}}{q_2} = \frac{0.05}{0.20} = 0.25, \\ P(J = 2|K = 2) &= \frac{q_{2,2}}{q_2} = \frac{0.15}{0.20} = 0.75, \\ P(J = 3|K = 2) &= \frac{q_{3,2}}{q_2} = \frac{0}{0.20} = 0. \end{aligned}$$

Example 10.3. *Pension plan for employees: Consider a person at age $x = 30$ and the following insurance:*

- *In the case of death in the original employment, there is a single payment of 5 mil. CZK to a beneficiary at the end of the year of death.*
- *If the employee stays with the same employer up to 70 years, he/she is entitled an annuity with annual payment* 3000*times number of years in employment.*
- *if the employee leaves before reaching the age of 70, he/she is entitled an annuity with annual payment* 3000*times number of finished years in employment with payments starting at the age of 70.*

Define a proper probabilistic model and derive a formula for the net single premium.

Solution:

We can consider two causes of decrement: $J = 1$ death, $J = 2$ leaving the employer, i.e., $m = 2$. We can use slightly generalized formula for the net single premium (inspired by the general formula for net premium reserve at time 0, i.e. by ${}_0V_x$):

$$\text{NSP} = \sum_{j=1}^{m}\sum_{k=0}^{\infty} c_{j,k+1} \cdot v^{k+1} \cdot {}_kp_x \cdot q_{j,x+k} - \sum_{k=0}^{\infty} \pi_k \cdot v^k \cdot {}_kp_x.$$

In our case, we set

- $c_{1,k+1} = \begin{cases} 5 \cdot 10^6, & k = 0, \ldots, 39, \\ 0, & k \geq 40, \end{cases}$
- $c_{2,k+1} = \begin{cases} 3000 \cdot k \cdot {}_{39-k|}\ddot{a}_{31+k}, & k = 0, \ldots, 39, \\ 0, & k \geq 40, \end{cases}$
- $\pi_k = \begin{cases} -3000 \cdot 40 \cdot \ddot{a}_{70}, & k = 40, \\ 0, & \text{otherwise.} \end{cases}$

It is important to realize that the NSP for the annuities are based on the standard life tables (= with the cause of decrement death only).

We can consider also the third cause of decrement $J = 3$ for staying with the original employer up to 70 years, but the situation is then much more difficult. You must be very careful with the definition of the probability distribution, compare with Example 10.2.

Example 10.4. *Consider a person at age $x = 30$ and the following insurance valid until reaching the age of 65:*
In the case of death, there is a single payment of 5 mil. CZK at the end of the year of death. In the case of disability, an annuity of 0.5 mil. CZK is paid until reaching the age 65 from which age it is increased by 50 thousand CZK every year until the death. However, the payment in the case of death is not further valid.
Define a proper probabilistic model and derive a formula for the net single premium.

Solution:

We can consider two causes of decrements: $J = 1$ death, and $J = 2$ disability, i.e. $m = 2$. Using the general formula, we can set

- $c_{1,k+1} = \begin{cases} 5 \cdot 10^6, & k = 0, \ldots, 34, \\ 0, & \text{otherwise,} \end{cases}$
- $c_{2,k+1} = \begin{cases} 0.5 \cdot 10^6 \cdot \ddot{a}_{31+k} + 5 \cdot 10^4 \cdot {}_{34-k|}(I\ddot{a})_{31+k}, & k = 0, \ldots, 34, \\ 0, & \text{otherwise,} \end{cases}$
- $\pi_k = 0$ for all k.

The NSP for the annuities are based on the standard life tables (= with the cause of decrement death only).

11 Construction of life tables with multiple decrements

When we want to prepare a new life insurance product based on multiple decrements, we must construct suitable life tables (with all considered causes of decrement). However, very often the causes of decrements are parametrized separately. Then, our goal is to mix them. You can find the proper way below.

Consider (continuous) compound model with one decrement where we consider only one cause of decrement. Then, the probability of surviving over time t without observing the cause of decrement j is

$$ {}_tp'_{j,x} = \exp\left\{ -\int_0^t \mu_{j,x+s}\, ds \right\}, $$

and we set

$$ {}_tq'_{j,x} = 1 - {}_tp'_{j,x}. $$

Example 11.1. *Show that*

a) ${}_tp_x \leq {}_tp'_{j,x}$,

b) ${}_tq_x \geq {}_tq'_{j,x}$,

c) ${}_tq_{j,x} \leq {}_tq'_{j,x}$.

Solution:

a)

$$ \begin{aligned} {}_tp_x &= \exp\left\{ -\int_0^t \mu_{x+s}\, ds \right\} \\ &= \exp\left\{ -\int_0^t \sum_{j=1}^m \mu_{j,x+s}\, ds \right\} \\ &= \prod_{j=1}^m \exp\left\{ -\int_0^t \mu_{j,x+s}\, ds \right\} \\ &= \prod_{j=1}^m {}_tp'_{j,x}. \end{aligned} $$

Since ${}_tp'_{j,x} \in (0,1)$ for all j, we obtain the inequality.

b)

It is a consequence of a) if we realize that ${}_tq_x = 1 - {}_tp_x$ and ${}_tq'_{j,x} = 1 - {}_tp'_{j,x}$.

c)

It is again a consequence of a) if we realize

$$ {}_tq_{j,x} = \int_0^t {}_sp_x \cdot \mu_{j,x+s}\, ds \le \int_0^t {}_sp'_{j,x} \cdot \mu_{j,x+s}\, ds = {}_tq'_{j,x}. $$

Example 11.2. *Under the assumption of linearity for each cause of decrement, i.e.,*

$$ {}_uq'_{j,x} = u \cdot q'_{j,x},\ \ u \in (0,1), $$

derive an exact relation between $q_{j,x}$ and $q'_{j,x}$.

Solution:

$$
\begin{aligned}
q'_{j,x} &= 1 - p'_{j,x} \\
&= 1 - \exp\left\{ - \int_0^1 \mu_{j,x+u}\, du \right\} \\
&\qquad \text{with the use of the ass. of linearity : } \mu_{j,x+u} = \frac{q_{j,x}}{1 - u\, q_x} \\
&= 1 - \exp\left\{ -q_{j,x} \cdot \int_0^1 \frac{1}{1 - u\, q_x}\, du \right\} \\
&= 1 - \exp\left\{ -q_{j,x} \cdot \left[\frac{-1}{q_x} \cdot \ln(1 - u\, q_x) \right]_0^1 \right\} \\
&= 1 - \exp\left\{ \frac{q_{j,x}}{q_x} \cdot \ln(1 - \ q_x) \right\} \\
&= 1 - (1 - \ q_x)^{\frac{q_{j,x}}{q_x}}.
\end{aligned}
$$

Then, we can express

$$ q_{j,x} = q_x \cdot \frac{\ln(1 - q'_{j,x})}{\ln(1 - q_x)} = q_x \cdot \frac{\ln\, p'_{j,x}}{\ln\, p_x}. $$

Example 11.3. *Under the assumption of constant force of decrement for each cause, i.e. $\mu_{j,x+u} = \mu_{j,x+\frac{1}{2}}$, $u \in (0,1)$, derive an exact relation between $q_{j,x}$ and $q'_{j,x}$.*

Solution:

The assumption stays valid also for the aggregate force of decrement, i.e., $\mu_{x+u} = \mu_{x+\frac{1}{2}}$, $u \in (0,1)$.

Then

$$
\begin{aligned}
q_{j,x} &= \int_0^1 {}_u p_x \cdot \mu_{j,x+u}\, du \\
&= \int_0^1 {}_u p_x \cdot \mu_{j,x+\frac{1}{2}}\, du \\
&= \frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} \cdot \int_0^1 {}_u p_x \cdot \mu_{x+\frac{1}{2}}\, du \\
&= \frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} \cdot \int_0^1 {}_u p_x \cdot \mu_{x+u}\, du \\
&= \frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} \cdot q_x.
\end{aligned}
$$

Moreover, under our assumption

$$
\begin{aligned}
p'_{j,x} &= \exp\left\{-\int_0^1 \mu_{j,x+u}\, du\right\} = \exp\left\{-\mu_{j,x+\frac{1}{2}}\right\}, \\
p_x &= \exp\left\{-\int_0^1 \mu_{x+u}\, du\right\} = \exp\left\{-\mu_{x+\frac{1}{2}}\right\},
\end{aligned}
$$

i.e., we get the ratio

$$
\frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} = \frac{-\ln p'_{j,x}}{-\ln p_x}.
$$

Thus, we get the same formula as in the previous example

$$
q_{j,x} = q_x \cdot \frac{\ln(1-q'_{j,x})}{\ln(1-q_x)} = q_x \cdot \frac{\ln p'_{j,x}}{\ln p_x}.
$$

We can summarize the construction of multiple decrement life tables from m compound models with one decrement:

1. Compute $q'_{j,x}$ and $p'_{j,x} = 1 - q'_{j,x}$ for all j.
2. Derive
$$
p_x = \prod_{j=1}^{m} p'_{j,x} \text{ and } q_x = 1 - p_x.
$$
3. Apply the formula
$$
q_{j,x} = q_x \cdot \frac{\ln p'_{j,x}}{\ln p_x}.
$$

Example 11.4. *Using the above introduced approach, construct the multiple decrement life table with $m = 3$ given the following compound models with one decrement:*

x	$q'_{1,x}$	$q'_{2,x}$	$q'_{3,x}$
25	*0.020*	*0.030*	*0.200*
26	*0.022*	*0.034*	*0.100*
27	*0.028*	*0.040*	*0.120*

i.e., compute $q_{1,x}$, $q_{2,x}$, $q_{3,x}$.

Solution:

x	$q'_{1,x}$	$q'_{2,x}$	$q'_{3,x}$	p_x	q_x	$q_{1,x}$	$q_{2,x}$	$q_{3,x}$
25	0.020	0.030	0.200	0.760	0.240	0.018	0.027	0.195
26	0.022	0.034	0.100	0.850	0.150	0.021	0.032	0.097
27	0.028	0.040	0.120	0.821	0.179	0.026	0.037	0.116

12 Multiple Life Insurance

We consider m *independent*[2] lives with (random) future lifetimes

$$T_1 := T_{x_1},\ \ldots,\ T_m := T_{x_m}.$$

Notation

- Joint-life status
  - status $u = x_1 : x_2 : \cdots : x_m$ = all m participating lives survive,
  - failure time
$$T(u) = \min\{T_1, \ldots, T_m\},$$
  - survival probability
$${}_tp_{x_1:x_2:\cdots:x_m} = P(T(u) > t) = \prod_{k=1}^{m} P(T_k > t) = \prod_{k=1}^{m} {}_tp_{x_k},$$
    and
$${}_tq_{x_1:x_2:\cdots:x_m} = 1 - {}_tp_{x_1:x_2:\cdots:x_m}.$$
    Under independence, we set
$$l_{x_1:x_2:\cdots:x_m} = \prod_{k=1}^{m} l_{x_k},\ \ d_{x_1:x_2:\cdots:x_m} = l_{x_1:x_2:\cdots:x_m} - l_{x_1+1:x_2+1:\cdots:x_m+1}.$$
    Then
$${}_tp_{x_1:x_2:\cdots:x_m} = \frac{l_{x_1+t:x_2+t:\cdots:x_m+t}}{l_{x_1:x_2:\cdots:x_m}}.$$

- Last-survivor status
  - status $u = \overline{x_1 : x_2 : \cdots : x_m}$ = at least one of the m lives survives,
  - failure time
$$T(u) = \max\{T_1, \ldots, T_m\},$$
  - survival probability
$${}_tp_{\overline{x_1:x_2:\cdots:x_m}} = P(T(u) > t) = S_1^t - S_2^t + \cdots (-1)^{m-1} \cdot S_m^t,$$
    where
$$S_k^t = \sum_{(j_1,\ldots,j_k)\subset\{1,\ldots,m\}} {}_tp_{x_{j_1}:x_{j_2}:\cdots:x_{j_k}},$$
    and
$${}_tq_{\overline{x_1:x_2:\cdots:x_m}} = 1 - {}_tp_{\overline{x_1:x_2:\cdots:x_m}}.$$

[2] The independence is quite questionable assumption, especially when we consider a family insurance. There are several approaches how to elaborate the dependence, e.g., copula functions or conditional forces of mortality.

Example 12.1. *Consider the following insurances for a pair of independent lifes at ages x and y:*

a) joint-life whole life insurance payable on the first death,

b) joint-life life annuity-due.

c) joint-life life annuity-due for n years.

Derive a reasonable generalization of the commutation functions which enable you to simplify the computation of the net single premiums.

Solution:

a)

Under the independence of lifes

$$\begin{aligned} A_{x:y} &= \sum_{k=0}^{\infty} v^{k+1} \cdot {}_k p_{x:y} \cdot q_{x+k:y+k} \\ &= \sum_{k=0}^{\infty} v^{k+1} \cdot \frac{l_{x+k:y+k}}{l_{x:y}} \cdot \frac{d_{x+k:y+k}}{l_{x+k:y+k}} \\ &= \sum_{k=0}^{\infty} \frac{v^{f(x+k+1,y+k+1)}}{v^{f(x,y)}} \cdot \frac{d_{x+k:y+k}}{l_{x:y}}. \end{aligned}$$

There are several possible choices of f:

$$f(x,y) = \frac{x+y}{2}, \;\; f(x,y) = \max\{x,y\}, \;\; f(x,y) = \min\{x,y\}.$$

On the other hand, it is not possible to use simple sum of the ages as f, because we need a transformation which preserves v^k. So, if we define the commutation functions as

$$\begin{aligned} C_{x:y} &= v^{f(x+1,y+1)} \cdot d_{x:y}, & D_{x:y} &= v^{f(x,y)} \cdot l_{x:y}, \\ M_{x:y} &= \sum_{k=0}^{\infty} C_{x+k:y+k}, & N_{x:y} &= \sum_{k=0}^{\infty} D_{x+k:y+k}, \\ R_{x:y} &= \sum_{k=0}^{\infty} M_{x+k:y+k}, & S_{x:y} &= \sum_{k=0}^{\infty} N_{x+k:y+k}, \end{aligned}$$

we can get the standard expression for NSP

$$A_{x:y} = \sum_{k=0}^{\infty} \frac{C_{x+k:y+k}}{D_{x:y}} = \frac{M_{x:y}}{D_{x:y}}.$$

b)

For the life annuity-due, we obtain

$$\begin{aligned}\ddot{a}_{x:y} &= \sum_{k=0}^{\infty} v^k \cdot {}_kp_{x:y} \\ &= \sum_{k=0}^{\infty} v^k \cdot \frac{l_{x+k:y+k}}{l_{x:y}} \\ &= \sum_{k=0}^{\infty} \frac{v^{f(x+k,y+k)}}{v^{f(x,y)}} \cdot \frac{l_{x+k:y+k}}{l_{x:y}} \\ &= \sum_{k=0}^{\infty} \frac{D_{x+k:y+k}}{D_{x:y}} = \frac{N_{x:y}}{D_{x:y}}.\end{aligned}$$

c)

For the life annuity-due for n years, we have

$$\begin{aligned}\ddot{a}_{x:y\overline{n}|} &= \sum_{k=0}^{n-1} v^k \cdot {}_kp_{x:y} \\ &= \sum_{k=0}^{n-1} \frac{D_{x+k:y+k}}{D_{x:y}} = \frac{N_{x:y} - N_{x+n:y+n}}{D_{x:y}}.\end{aligned}$$

Remark
The generalization of the commutation functions to m lifes is straightforward, e.g.,

$$C_{x_1:x_2:\cdots:x_m} = v^{f(x_1+1,\ldots,x_m+1)} \cdot d_{x_1:x_2:\cdots:x_m} \quad ; \quad D_{x_1:x_2:\cdots:x_m} = v^{f(x_1,\ldots,x_m)} \cdot l_{x_1:x_2:\cdots:x_m},$$

where

$$f(x_1,\ldots,x_m) = \frac{\sum_{k=1}^m x_k}{m},$$

or

$$f(x_1,\ldots,x_m) = \max\{x_1,\ldots,x_m\},$$

or

$$f(x_1,\ldots,x_m) = \min\{x_1,\ldots,x_m\}.$$

Note that also the relations between CF which we know from the univariate case are valid, e.g.,

$$M_{x:y} = D_{x:y} - d \cdot N_{x:y}.$$

Example 12.2. *Consider the following insurances for a pair of independent lifes at ages x and y:*

a) last-survival life annuity-due.

b) last-survival whole life insurance payable on the last death, *Using the above introduced commutation functions derive the net single premiums.*

Solution:

a)

$$
\begin{aligned}
\ddot{a}_{\overline{x:y}} &= \sum_{k=0}^{\infty} v^k \cdot {}_kp_{\overline{x:y}} \\
&= \sum_{k=0}^{\infty} v^k \cdot ({}_kp_x + {}_kp_y - {}_kp_{x:y}) \\
&= \sum_{k=0}^{\infty} v^k \cdot {}_kp_x + \sum_{k=0}^{\infty} v^k \cdot {}_kp_y - \sum_{k=0}^{\infty} v^k \cdot {}_kp_{x:y} \\
&= \frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{x:y}}{D_{x:y}}.
\end{aligned}
$$

b)

We can use the previous example to get

$$
\begin{aligned}
\ddot{A}_{\overline{x:y}} &= 1 - d\,\ddot{a}_{\overline{x:y}} \\
&= 1 + 1 - 1 - d \cdot \left(\frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{x:y}}{D_{x:y}} \right) \\
&= \frac{D_x - d\,N_x}{D_x} + \frac{D_y - d\,N_y}{D_y} - \frac{D_{x:y} - d\,N_{x:y}}{D_{x:y}} \\
&= \frac{M_x}{D_x} + \frac{M_y}{D_y} - \frac{M_{x:y}}{D_{x:y}}.
\end{aligned}
$$

Example 12.3. *Consider*

a) *widow's annuity-due (asymmetric) – payment stream of rate 1 starts at the death of husband x and terminates at the death of wife y.*

b) *widow's and widower's annuity-due (symmetric) – payment stream starts at the death of husband x or wife y and terminates at the death of wife y or husband x.*

c) *orphan's annuity-due – payment stream starts at the death of parents x, y and terminates at the death of child z or by reaching the age of 18.*

Solution:

a)

Denote by u the status when wife is living and husband died

$$
{}_kp_u^{(a)} = {}_kp_y \cdot (1 - {}_kp_x).
$$

Then

$$\begin{aligned}\ddot{a}_u^{(a)} &= \sum_{k=0}^{\infty} v^k \cdot {}_kp_u^{(a)} \\ &= \sum_{k=0}^{\infty} v^k \cdot {}_kp_y \cdot (1 - {}_kp_x) \\ &= \ddot{a}_y - \ddot{a}_{x:y}.\end{aligned}$$

b)

Denote by u the status when the wife is living and the husband died or vice versa

$${}_kp_u^{(b)} = {}_kp_y \cdot (1 - {}_kp_x) + {}_kp_x \cdot (1 - {}_kp_y).$$

Then

$$\begin{aligned}\ddot{a}_u^{(b)} &= \sum_{k=0}^{\infty} v^k \cdot {}_kp_u^{(b)} \\ &= \sum_{k=0}^{\infty} v^k \cdot [{}_kp_y \cdot (1 - {}_kp_x) + {}_kp_x \cdot (1 - {}_kp_y)] \\ &= \ddot{a}_x + \ddot{a}_y - 2 \cdot \ddot{a}_{x:y}.\end{aligned}$$

c)

Denote by u the status when the child is living and the parents died and set $n = 18-z$. Then

$${}_kp_u^{(c)} = {}_kp_z \cdot (1 - {}_kp_x) \cdot (1 - {}_kp_y),$$

and

$$\begin{aligned}\ddot{a}_u^{(c)} &= \sum_{k=0}^{n-1} v^k \cdot {}_kp_u^{(c)} \\ &= \sum_{k=0}^{n-1} v^k \cdot {}_kp_z \cdot (1 - {}_kp_x) \cdot (1 - {}_kp_y) \\ &= \ddot{a}_{z\overline{n}|} - \ddot{a}_{x:z\overline{n}|} - \ddot{a}_{y:z\overline{n}|} + \ddot{a}_{x:y:z\overline{n}|}.\end{aligned}$$

Example 12.4. *Consider orphan's annuity-due where payment stream starts at the death of parents x, y and terminates when both children z, w reach the age of 18 or at the death of last child.*

Solution:

Denote by u the status when at least one child is living and both parents died and set $n = 18 - z$ and $t = 18 - w$. Moreover, let us assume that child z is the younger

one.
Then

$$ {}_kp_u = {}_kq_{\overline{x:y}} \cdot {}_kp_{\overline{z:w}}. $$

If the age restriction were not considered then we could write

$$ {}_kq_{\overline{x:y}} = 1 - {}_kp_{\overline{x:y}} = 1 - ({}_kp_x + {}_kp_y - {}_kp_{x:y}) = 1 - {}_kp_x - {}_kp_y + {}_kp_x \cdot {}_kp_y $$

and

$$ {}_kp_{\overline{z:w}} = {}_kp_z + {}_kp_w - {}_kp_{z:w} = {}_kp_z + {}_kp_w - {}_kp_z \cdot {}_kp_w, $$

and therefore

$$ \begin{aligned} {}_kp_u =& {}_kp_z + {}_kp_w - {}_kp_z \cdot {}_kp_w - {}_kp_x \cdot {}_kp_z - {}_kp_x \cdot {}_kp_w + {}_kp_x \cdot {}_kp_z \cdot {}_kp_w \\ & - {}_kp_y \cdot {}_kp_z - {}_kp_y \cdot {}_kp_w + {}_kp_y \cdot {}_kp_z \cdot {}_kp_w + {}_kp_x \cdot {}_kp_y \cdot {}_kp_z \\ & + {}_kp_x \cdot {}_kp_y \cdot {}_kp_w - {}_kp_x \cdot {}_kp_y \cdot {}_kp_z \cdot {}_kp_w. \end{aligned} $$

Since there is the age limit, we can distinguish three situations, now without considering parents because those will be included afterwards.

1. only z is living (u is valid with a maximum of n years): ${}_kp^1 = {}_kp_z \cdot (1 - {}_kp_w)$,
2. only w is living (u is valid with a maximum of t years): ${}_kp^2 = {}_kp_w \cdot (1 - {}_kp_z)$,
3. both z and w are living (u is valid with a maximum of n years): ${}_kp^3 = {}_kp_w \cdot {}_kp_z$.

Therefore, the considered annuity can be rewritten as follows:

$$ \begin{aligned} \ddot{a}_u =& \sum_{k=0}^{n-1} v^k \cdot {}_kq_{\overline{x:y}} \cdot {}_kp^1 + \sum_{k=0}^{t-1} v^k \cdot {}_kq_{\overline{x:y}} \cdot {}_kp^2 + \sum_{k=0}^{n-1} v^k \cdot {}_kq_{\overline{x:y}} \cdot {}_kp^3 \\ =& \sum_{k=0}^{n-1} v^k \cdot {}_kq_{\overline{x:y}} \cdot {}_kp_z + \sum_{k=0}^{t-1} v^k \cdot {}_kq_{\overline{x:y}} \cdot ({}_kp_w - {}_kp_{z:w}) \\ =& \ddot{a}_{z\overline{n}|} + \ddot{a}_{w\overline{t}|} - \ddot{a}_{z:w\overline{t}|} - \ddot{a}_{x:z\overline{n}|} - \ddot{a}_{x:w\overline{t}|} + \ddot{a}_{x:z:w\overline{t}|} \\ & - \ddot{a}_{y:z\overline{n}|} - \ddot{a}_{y:w\overline{t}|} + \ddot{a}_{y:z:w\overline{t}|} + \ddot{a}_{x:y:z\overline{n}|} + \ddot{a}_{x:y:w\overline{t}|} - \ddot{a}_{x:y:z:w\overline{t}|}. \end{aligned} $$

13 Expense-Loaded Premium and Reserve

Expenses can be classified into following groups[3]:

Expenses	Abbreviation	Charged	Proportional to
acquisition	α	at the beginning	SI
collection	β	when premium is collected	premium
administration	γ	during the entire contract period[4]	SI
annuity	δ	when annuity is paid	annuity repayment

Expense-loaded annual premium is estimated using the (generalized) equivalence principle: the expected present value of the premium payments must be equal to the expected present value of the benefits and the incurred costs (expenses).

Expense-loaded premium reserve is defined as the difference between the expected present value of future benefits plus expenses minus expense-loaded premium related to the end of year k.

Example 13.1. *Consider the whole life insurance with the annual premium paid during the whole contract period. Derive the expense-loaded premium and reserve.*

i) Decompose the reserve.

ii) Do not decompose the reserve.

Solution:

The expense-loaded annual premium P^B must satisfy the following (generalized) equivalence principle

$$P^B\,\ddot{a}_x = A_x + \alpha + \beta \cdot P^B\,\ddot{a}_x + \gamma \cdot \ddot{a}_x.$$

If we divide the equation by $\ddot{a}_x$, we obtain the decomposition of the premium in the form

$$P^B = P + P^\alpha + P^\beta + P^\gamma,$$

where in our case it holds

$$P = \frac{A_x}{\ddot{a}_x},\ P^\alpha = \frac{\alpha}{\ddot{a}_x},\ P^\beta = \frac{\beta \cdot P^B \cdot \ddot{a}_x}{\ddot{a}_x} = \beta \cdot P^B,\ P^\gamma = \frac{\gamma \cdot \ddot{a}_x}{\ddot{a}_x} = \gamma.$$

where P is the net annual premium and the remaining components correspond to the expense groups. We can derive the expense-loaded annual premium in an explicit form

$$P^B = \frac{A_x + \alpha + \gamma \cdot \ddot{a}_x}{(1-\beta) \cdot \ddot{a}_x}.$$

[3] We follow the notation by Gerber. Prof. Cipra is using different notation for collection expenses γ (instead of β) and administration expenses β (instead of γ).

[4] In case of the life annuity, γ is considered for the entire contract period as well (different from the lecture).

i)

Now, we will focus on the expense-loaded premium reserve, which can be also decomposed as

$$\boxed{{}_kV_x^B = {}_kV_x + {}_kV_x^\alpha + {}_kV_x^\beta + {}_kV_x^\gamma}$$

where the net premium reserve is equal to

$$\boxed{{}_kV_x = A_{x+k} - P \cdot \ddot{a}_{x+k}, \qquad k = 0, 1, \dots}$$

the reserve for the acquisition expenses is equal to

$$\boxed{{}_kV_x^\alpha = I(k=0) \cdot \alpha - P^\alpha \cdot \ddot{a}_{x+k}, \qquad k = 0, 1, \dots}$$

the reserve for the collection expenses is

$$\boxed{{}_kV_x^\beta = \beta \cdot P^B \, \ddot{a}_{x+k} - P^\beta \cdot \ddot{a}_{x+k} = 0, \qquad k = 0, 1, \dots}$$

which is (nearly) always equal to zero, and the reserve for the administration expenses is

$$\boxed{{}_kV_x^\gamma = \gamma \cdot \ddot{a}_{x+k} - P^\gamma \cdot \ddot{a}_{x+k} = 0, \qquad k = 0, 1, \dots}$$

which is equal to zero only if the premium collection period is the same as the whole contract period (for cases without δ). Realize that the last two components are equal to zero and at the same time the α component is negative (nonpositive). Therefore, in this case, the expense-loaded premium reserve is lower or equal to the net premium reserve.

ii)

When we are not interested into the decomposition and particular components, we can derive the expense-loaded premium reserve directly

$$\boxed{{}_kV_x^B = A_{x+k} + I(k=0) \cdot \alpha + \beta \cdot P^B \cdot \ddot{a}_{x+k} + \gamma \cdot \ddot{a}_{x+k} - P^B \cdot \ddot{a}_{x+k}, \qquad k = 0, 1 \dots}$$

Example 13.2. *Consider the m year deferred life annuity due for n years with premium paid during the deferment period. Derive the expense-loaded premium and reserve*

1. *without premium refund,*
2. *with premium refund (= paid premium is returned to a beneficiary at the end of the year of death of the insured person during the deferment period).*

Solution:

i)

The (generalized) equivalence principle is

$$P^B \cdot \ddot{a}_{x:\overline{m}|} = {}_{m|}\ddot{a}_{x:\overline{n}|} + \alpha + \beta \cdot P^B \cdot \ddot{a}_{x:\overline{m}|} + \gamma \cdot \ddot{a}_{x:\overline{m}|} + \delta \cdot {}_{m|}\ddot{a}_{x:\overline{n}|},$$

i.e., we get the components

$$P = \frac{{}_{m|}\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}, P^\alpha = \frac{\alpha}{\ddot{a}_{x:\overline{m}|}}, P^\beta = \frac{\beta \cdot P^B \cdot \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} = \beta \cdot P^B,$$

$$P^\gamma = \frac{\gamma \cdot \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} = \gamma, P^\delta = \frac{\delta \cdot {}_{m|}\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},$$

and the expense-loaded annual premium

$$P^B = P + P^\alpha + P^\beta + P^\gamma + P^\delta,$$

or

$$P^B = \frac{(1+\delta) \cdot {}_{m|}\ddot{a}_{x\overline{n}|} + \alpha + \gamma \cdot \ddot{a}_{x:\overline{m}|}}{(1-\beta) \cdot \ddot{a}_{x:\overline{m}|}}.$$

The components of the expense-loaded premium reserve are:
the net premium reserve

$$\boxed{{}_kV_x = \begin{cases} {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} - P \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ \ddot{a}_{x+k:\overline{n+m-k}|}, & k = m, \ldots, m+n-1 \end{cases}}$$

the reserve for the acquisition expenses

$$\boxed{{}_kV_x^\alpha = \begin{cases} I(k=0) \cdot \alpha - P^\alpha \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ 0, & k = m, \ldots, m+n-1 \end{cases}}$$

the reserve for the collection expenses

$$\boxed{{}_kV_x^\beta = \begin{cases} \beta \cdot P^B \cdot \ddot{a}_{x+k:\overline{m-k}|} - P^\beta \cdot \ddot{a}_{x+k:\overline{m-k}|} = 0, & k = 0, \ldots, m-1, \\ 0, & k = m, \ldots, m+n-1 \end{cases}}$$

the reserve for the administration expenses

$$\boxed{{}_kV_x^\gamma = \begin{cases} \gamma \cdot \ddot{a}_{x+k:\overline{m-k}|} - P^\gamma \cdot \ddot{a}_{x+k:\overline{m-k}|} = 0, & k = 0, \ldots, m-1, \\ 0, & k = m, \ldots, m+n-1 \end{cases}}$$

and the reserve for the annuity expenses

$$\boxed{{}_kV_x^\delta = \begin{cases} \delta \cdot {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} - P^\delta \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ \delta \cdot \ddot{a}_{x+k:\overline{n+m-k}|}, & k = m, \ldots, m+n-1 \end{cases}}$$

Then we have

$$\boxed{{}_kV_x^B = {}_kV_x + {}_kV_x^\alpha + {}_kV_x^\beta + {}_kV_x^\gamma + {}_kV_x^\delta, \qquad k = 0, \ldots, m+n-1}$$

We provide also the non-decomposed formula for the expense-loaded premium reserve:

$$\boxed{{}_kV_x^B = \begin{cases} (1+\delta) \cdot {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} + I(k=0) \cdot \alpha + \beta \cdot P^B \cdot \ddot{a}_{x+k:\overline{m-k}|} & \\ +\gamma \cdot \ddot{a}_{x+k:\overline{m-k}|} - P^B \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ & \\ (1+\delta) \cdot \ddot{a}_{x+k:\overline{n+m-k}|}, & k = m, \ldots, m+n-1 \end{cases}}$$

ii)

We will modify only the net premium component where the premium refund is incorporated. The (generalized) equivalence principle is

$$\hat{P}^B \cdot \ddot{a}_{x:\overline{m}|} = {}_{m|}\ddot{a}_{x:\overline{n}|} + \hat{P}^B \cdot (IA)^1_{x:\overline{m}|} + \alpha + \beta \cdot \hat{P}^B \cdot \ddot{a}_{x:\overline{m}|} + \gamma \cdot \ddot{a}_{x:\overline{m}|} + \delta \cdot {}_{m|}\ddot{a}_{x:\overline{n}|},$$

i.e., the premium refund is modeled using the standard increasing term insurance component, so the net annual component is

$$\hat{P} = \frac{{}_{m|}\ddot{a}_{x:\overline{n}|} + \hat{P}^B\,(IA)^1_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}},$$

and the expense-loaded annual premium equals to

$$\hat{P}^B = \frac{(1+\delta) \cdot {}_{m|}\ddot{a}_{x:\overline{n}|} + \alpha + \gamma \cdot \ddot{a}_{x:\overline{m}|}}{(1-\beta) \cdot \ddot{a}_{x:\overline{m}|} - (IA)^1_{x:\overline{m}|}}.$$

The premium refund influences the net premium reserve component as follows

$$\boxed{{}_k\hat{V}_x = \begin{cases} {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} + \hat{P}^B \cdot (IA)^{\;1}_{x+k:\overline{m-k}|} + k \cdot \hat{P}^B \cdot A^{\;1}_{x+k:\overline{m-k}|} & \\ -\hat{P} \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ & \\ \ddot{a}_{x+k:\overline{n+m-k}|}, & k = m, \ldots, m+n-1 \end{cases}}$$

where we must use the correction term $k \cdot \hat{P}^B \cdot A^{\;1}_{x+k:\overline{m-k}|}$ to get the right premium refund level after k years. In principle, the β component is also influenced (it contains new $\hat{P}^B$), but it is again equal to zero. Thus, we have obtained

$$\boxed{{}_k\hat{V}_x^B = {}_k\hat{V}_x + {}_kV_x^\alpha (+{}_kV_x^\beta) + {}_kV_x^\gamma + {}_kV_x^\delta}$$

Example 13.3. *Consider the m year deferred standard increasing term insurance for n years with premium paid during the deferment period. Derive the expense-loaded premium and reserve*

1. using the standard definition of expenses,

2. using the modified definition of expenses as follows[5] *(assume $m \geq 3$):*

- *acquisition expenses are divided into three consecutive payments,*
- *collection expenses are standard decreasing,*
- *administration expenses differ between deferment period and subsequent period.*

Solution:

i)

The (generalized) equivalence principle is

$$P^B \cdot \ddot{a}_{x:\overline{m}|} = {}_{m|}(IA)^1_{x:\overline{n}|} + \alpha + \beta \cdot P^B \cdot \ddot{a}_{x:\overline{m}|} + \gamma \cdot \ddot{a}_{x:\overline{m+n}|},$$

i.e., we get components of the expense-loaded annual premium

$$P = \frac{{}_{m|}(IA)^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},\ P^\alpha = \frac{\alpha}{\ddot{a}_{x:\overline{m}|}},\ P^\beta = \frac{\beta \cdot P^B \cdot \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} = \beta \cdot P^B,\ P^\gamma = \frac{\gamma \cdot \ddot{a}_{x:\overline{m+n}|}}{\ddot{a}_{x:\overline{m}|}},$$

and the expense-loaded annual premium

$$P^B = P + P^\alpha + P^\beta + P^\gamma,$$

or

$$P^B = \frac{{}_{m|}(IA)^1_{x:\overline{n}|} + \alpha + \gamma \cdot \ddot{a}_{x:\overline{m+n}|}}{(1-\beta) \cdot \ddot{a}_{x:\overline{m}|}}.$$

The components of the expense-loaded premium reserve are

$$\boxed{{}_kV_x = \begin{cases} {}_{m-k|}(IA)^{\,1}_{x+k:\overline{n}|} - P \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ (IA)^{\,1}_{x+k:\overline{n+m-k}|} + (k-m) \cdot A^{\,1}_{x+k:\overline{n+m-k}|}, & k = m, \ldots, m+n-1 \end{cases}}$$

$$\boxed{{}_kV^\alpha_x = \begin{cases} I(k=0) \cdot \alpha - P^\alpha \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ 0, & k = m, \ldots, m+n-1 \end{cases}}$$

$$\boxed{{}_kV^\beta_x = \begin{cases} \beta \cdot P^B \cdot \ddot{a}_{x+k:\overline{m-k}|} - P^\beta \cdot \ddot{a}_{x+k:\overline{m-k}|} = 0, & k = 0, \ldots, m-1, \\ 0, & k = m, \ldots, m+n-1 \end{cases}}$$

$$\boxed{{}_kV^\gamma_x = \begin{cases} \gamma \cdot \ddot{a}_{x+k:\overline{m+n-k}|} - P^\gamma \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \ldots, m-1, \\ \gamma \cdot \ddot{a}_{x+k:\overline{m+n-k}|}, & k = m, \ldots, m+n-1 \end{cases}}$$

Then we have

$$\boxed{{}_kV^B_x = {}_kV_x + {}_kV^\alpha_x + {}_kV^\beta_x + {}_kV^\gamma_x, \qquad k = 0, \ldots, m+n-1}$$

[5] This can really happen in practice, so always check the definition of expenses.

ii)

The (generalized) equivalence principle with the modified expenses

$$P^B \cdot \ddot{a}_{x:\overline{m}|} = {}_{m|}(IA)^1_{x:\overline{n}|} + \alpha \cdot \ddot{a}_{x:\overline{3}|} + \beta \cdot P^B \cdot (D\ddot{a})_{x:\overline{m}|} + \gamma_1 \cdot \ddot{a}_{x:\overline{m}|} + \gamma_2 \cdot {}_{m|}\ddot{a}_{x:\overline{n}|}.$$

We can derive the components of the expense-loaded annual premium

$$P = \frac{{}_{m|}(IA)^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},\ P^\alpha = \frac{\alpha \cdot \ddot{a}_{x:\overline{3}|}}{\ddot{a}_{x:\overline{m}|}},\ P^\beta = \frac{\beta \cdot P^B \cdot (D\ddot{a})_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}},\ P^\gamma = \frac{\gamma_1 \cdot \ddot{a}_{x:\overline{m}|} + \gamma_2 \cdot {}_{m|}\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},$$

The components of the expense-loaded premium reserve:

$$\boxed{{}_kV_x = \begin{cases} {}_{m-k|}(IA)^{\ 1}_{x+k:\overline{n}|} - P \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \dots, m-1, \\ (IA)^{\ 1}_{x+k:\overline{n+m-k}|} + (k-m) \cdot A^{\ 1}_{x+k:\overline{n+m-k}|}, & k = m, \dots, m+n-1 \end{cases}}$$

$$\boxed{{}_kV_x^\alpha = \begin{cases} \alpha \cdot \ddot{a}_{x+k:\overline{3-k}|} - P^\alpha \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \dots, 2, \\ -P^\alpha \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 3, \dots, m-1, \\ 0, & k = m, \dots, m+n-1 \end{cases}}$$

$$\boxed{{}_kV_x^\beta = \begin{cases} \beta \cdot P^B \cdot (D\ddot{a})_{x+k:\overline{m-k}|} - P^\beta \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \dots, m-1, \\ 0, & k = m, \dots, m+n-1 \end{cases}}$$

which is not equal to zero in this case, and

$$\boxed{{}_kV_x^\gamma = \begin{cases} \gamma_1 \cdot \ddot{a}_{x+k:\overline{m-k}|} + \gamma_2 \cdot {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} - P^\gamma \cdot \ddot{a}_{x+k:\overline{m-k}|}, & k = 0, \dots, m-1, \\ \gamma_2 \cdot \ddot{a}_{x+k:\overline{m+n-k}|}, & k = m, \dots, m+n-1 \end{cases}}$$

Then, we have the expense-loaded premium reserve

$$\boxed{{}_kV_x^B = {}_kV_x + {}_kV_x^\alpha + {}_kV_x^\beta + {}_kV_x^\gamma, \qquad k = 0, \dots, m+n-1}$$

14 Lee-Carter model

You can read more about the Lee-Carter model here (this text was also used for the preparation of this chapter). In this collection we provide only a brief introduction.

With the improvement of medical technologies, life expectancies world over have seen a remarkable improvement across various age groups. This particular exposure to risk is called longevity risk. With life insurance and pensions often lasting for decades it leads to increased payouts and survival benefits to retirees and annuitants. Also, improvements in computational processing has also aided in carrying out intensive statistical algorithms but virtue of which more sophisticated modes of analysis can be carried with greater degrees of accuracy and efficiency.

The widely cited model for projecting mortality rates is the Lee-Carter (1992) model. This model was developed by the namesakes of the model to forecast mortality projections for the US population. The explicit form of the model is given by:

$$\ln(m_{x,t}) = a_x + b_x \cdot k_t + \varepsilon_{x,t},$$

where

- $x \ldots$ age (group),
- $t \ldots$ time,
- $m_{x,t} \ldots$ mortality rate at age x during year t,
- $a_x, b_x \ldots$ age specific constants,
- $k_t \ldots$ unobservable time specific index,
- $\varepsilon_{x,t} \ldots$ white noise $\sim \mathrm{WN}(0, \sigma_\varepsilon^2)$ (often problem with heteroskedasticity).

Certain constraints need to be imposed on the model for accurate results. The constraints put forward were as follows:

$$\sum_{x=x_1}^{x_m} b_x = 1,$$
$$\sum_{t=t_1}^{t_n} k_t = 0,$$

where $x \in \{x_1, \ldots, x_m\}$ and $t \in \{t_1, \ldots, t_n\}$.

Mortality rate can be calculated in the following way:

$$m_x = \frac{\int_0^1 S_0(x+u) \cdot \mu_{x+u}\, du}{\int_0^1 S_0(x+u)\, du},$$

where $S_x(t) = \mathrm{P}(T_x > t) = {}_tp_x$ is the survival function

The integral in the numerator can be rewritten as

$$\int_0^1 S_0(x+u) \cdot \mu_{x+u}\, du = \int_0^1 {}_{x+u}p_0 \cdot \mu_{x+u}\, du = \mathrm{P}(T_0 < x+1 | T_0 > x) = \mathrm{P}(T_x < 1)$$
$$= S_0(x) - S_0(x+1) = {}_xp_0 - {}_{x+1}p_0 = {}_xp_0 \cdot (1 - p_x) = {}_xp_0 \cdot q_x,$$

and the integral in the denominator can be approximated by

$$\int_0^1 S_0(x+u)\, du \approx \frac{S_0(x) + S_0(x+1)}{2} = \frac{{}_xp_0 \cdot (1 + p_x)}{2} = \frac{{}_xp_0 \cdot (2 - q_x)}{2}.$$

Therefore, the approximation of the mortality rate is

$$m_x \approx \frac{2 \cdot q_x}{2 - q_x}.$$

Parameter estimation

1) Ordinary least squares (OLS)

$$O_{LS}(a, b, k) = \sum_x \sum_t (\ln\, m_{x,t} - a_x - b_x \cdot k_t)^2$$

and is minimized such that the normalization conditions hold.
Parameters α_x can be estimated as

$$\hat{a}_x = \frac{1}{t_n - t_1 + 1} \sum_{t=t_1}^{t_n} \ln\, m_{x,t}.$$

These estimates are then used for calculation of $Z_{x,t}$, where

$$Z_{x,t} = \ln\, m_{x,t} - \hat{a}_x$$

for all x and t.

These values are used for

$$\tilde{O}_{LS}(b, k) = \sum_x \sum_t (Z_{x,t} - b_x \cdot k_t)^2.$$

The estimates $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{k}}$ are obtained with the use of singular value decomposition (SVD). The values $Z_{x,t}$ are formed into a matrix $\boldsymbol{A} = \{Z_{x,t}\}_{x,t}$ of size $m \times n$.

Singular value decomposition: $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, where

- $\boldsymbol{U}$ is a matrix of eigenvectors of $\boldsymbol{AA}^T$,
- $\boldsymbol{V}$ is a matrix of eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$,

- $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{S} & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$, where $\boldsymbol{S} = \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r})$ with $r = \min\{m, n\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ are the eigenvalues.

One can construct the best rank 1 approximation of $\boldsymbol{A}$ via estimates $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{k}}$ computed as

$$\hat{\boldsymbol{b}} = \frac{\boldsymbol{u}_1}{\sum_{j=1}^{x_m - x_1 + 1} u_{1,j}},$$
$$\hat{\boldsymbol{k}} = \sqrt{\lambda_1} \cdot \boldsymbol{v}_1 \cdot \left(\sum_{j=1}^{x_m - x_1 + 1} u_{1,j} \right),$$

where software performs some additional normalization.

2) Maximum-likelihood method and generalized GLM

Let us assume

- $D_{x,t} \ldots$ number of deaths at age x,
- $E_{x,t} \ldots$ number of living at age x,

and further
$$D_{x,t} \sim \text{Po}(E_{x,t} \cdot \exp\{a_x + b_x \cdot k_t\}),$$
where $E_{x,t}$ is an offset (=exposure).

Newton-Raphson algorithm and normalization is then used for estimation of the parameters.

Remark: In Renshaw and Haberman (2006) an unobservable cohort effect $c_x \cdot i_{t-x}$ (which corresponds to the age of birth) is added to $a_x + b_x \cdot k_t$.

In terms of prediction, we need a prediction of k_t for $t > t_n$. It is possible to work with k_t as with a time series and predict with the help of ARIMA model.