Part 2 (4 points)

Consider the continuous model and assume that $c_t = {}_tV_x$, ${}_0V_x = 0$ and $\Pi_t = \Pi$ for $t \geq 0$. Show that ${}_tV_x = \Pi \cdot \bar{s}_{\overline{t}|}$, where $\bar{s}_{\overline{t}|} = \frac{e^{\delta \cdot t} - 1}{\delta}$.

$(*)$ $c(s) = {}_sV_x = V(s)$; ${}_0V_x = V(0) = 0$; $\Pi(s) = \Pi$

Thiele: $\Pi(s) + \delta \cdot V(s) = V'(s) + (c(s) - V(s)) \cdot \mu_{x+s}$

$(*) \Rightarrow \quad \Pi + \delta\, {}_sV_x = d/ds\, {}_sV_x + ({}_sV_x - {}_sV_x)\, \mu_{x+s}$

$\Pi + \delta\, {}_sV_x = d/ds\, {}_sV_x$ ... ODE

$$d/ds\, {}_sV_x = \Pi + \delta \cdot {}_sV_x$$

$$\frac{d\, {}_sV_x}{\Pi + \delta \cdot {}_sV_x} = ds$$

$$\int \frac{1}{\Pi + \delta \cdot {}_sV_x}\, d\, {}_sV_x = \underbrace{\int 1\, ds}_{= s}$$

substituce: $\begin{bmatrix} u = \Pi + \delta \cdot {}_sV_x \\ du = \delta \cdot d\, {}_sV_x \end{bmatrix} \Rightarrow d\, {}_sV_x = du/\delta$

$$\int \frac{1}{\delta u}\, du = s$$

$$1/\delta \int 1/u\, du = s$$

$$1/\delta \ln|u| = s + C \overset{\text{sub}}{\Rightarrow} 1/\delta \ln|\Pi + \delta \cdot {}_sV_x| = s + C$$

Počáteční podmínka: $V(0) = {}_0V_x = 0$

$\underline{s=0}$: $1/\delta \ln|\Pi| = C \quad \Rightarrow \quad 1/\delta \ln|\Pi + \delta \cdot {}_sV_x| = s + 1/\delta \ln|\Pi| \quad / \cdot \delta$

$$\ln|\Pi + \delta \cdot {}_sV_x| = \delta s + \ln|\Pi| \quad / \exp \qquad \Pi > 0,\ {}_sV_x > 0,\ \delta > 0$$

$$\Pi + \delta \cdot {}_sV_x = e^{\delta s} \cdot \Pi$$

$$\delta \cdot {}_sV_x = \Pi (e^{\delta s} - 1)$$

$${}_sV_x = \frac{\Pi (e^{\delta s} - 1)}{\delta}$$

$${}_sV_x = \Pi \cdot \left.\frac{e^{\delta s} - 1}{\delta}\right\} \bar{s}_{\overline{s}|}$$

$$\boxed{{}_sV_x = \Pi \cdot \bar{s}_{\overline{s}|}}$$

Part 4 (5 points)

Let $J = 1$ represent death by accidental means and $J = 2$ death by other means. Assume that $\delta = 0.05$, $\mu_{1,x+t} = 0.005$ for $t \geq 0$, where $\mu_{1,x+t}$ is the force of decrement for death by accidental means and $\mu_{2,x+t} = 0.02$ for $t \geq 0$.
A 20-year term insurance policy, payable at the moment of death, is issued to a life age x providing a benefit of 2 if death is by accidental means and providing a benefit of 1 for other deaths.
Find the expectation and variance of the present value of benefits random variable.

$$\overset{c_1(T)}{Z_1} = 2v^T, \; T < 20 \quad \text{... umrtí v důsledku nehody}$$
$$= 0, \; T \geq 20$$

$$\overset{c(T)}{Z} = Z_1 + Z_2 \qquad \pi(s) = 0$$

$$\overset{c_2(T)}{Z_2} = v^T, \; T < 20 \quad \text{... umrtí z jiné příčiny}$$
$$= 0, \; T \geq 20$$

- $$E(Z) = NSP = \int_0^{20} 2 \cdot v^s \, {}_sp_x \cdot \mu_{1,x+s} \, ds + \int_0^{20} 1 \cdot v^s \, {}_sp_x \cdot \mu_{2,x+s} \, ds$$
$$= \int_0^{20} 2 \cdot e^{-\delta s} \cdot {}_sp_x \cdot 0{,}005 \, ds + \int_0^{20} e^{-\delta s} \, {}_sp_x \cdot 0{,}02 \, ds$$

$$\sum_j \mu_{j,x+s} = \mu_{1,x+s} + \mu_{2,x+s} = 0{,}005 + 0{,}02 = 0{,}025 = \mu_{x+s} \text{ ... constant}$$
$$\Rightarrow {}_sp_x = \exp\left(-\int_0^{20} \mu_{x+s} \, ds\right) = \exp\left(-\int_0^{20} \mu \, ds\right) = e^{-\mu s} = e^{-0{,}025 \cdot s}$$

$$\Rightarrow E(Z) = 0{,}01 \int_0^{20} e^{-0{,}05s} \cdot e^{-0{,}025s} \, ds + 0{,}02 \int_0^{20} e^{-0{,}05s} \cdot e^{-0{,}025s} \, ds$$
$$= 0{,}03 \int_0^{20} e^{-0{,}075s} \, ds = 0{,}03 \left[-\frac{1}{0{,}075} e^{-0{,}075s} \right]_0^{20} = 0{,}03 \left[-\frac{1}{0{,}075} e^{-0{,}075 \cdot 20} + \frac{1}{0{,}075} \right] =$$
$$= 0{,}03 \left[-\frac{e^{-1{,}5}}{0{,}075} + \frac{1}{0{,}075} \right] = 0{,}03 \cdot \frac{1}{0{,}075} (1 - e^{-1{,}5}) = \frac{30}{75} (1 - e^{-1{,}5}) = \boxed{0{,}3107}$$

... expectation of the present value of the benefits random variable

- $$Var(Z) = EZ^2 - (EZ)^2$$
$$EZ^2 = \int_0^{20} 4 \cdot e^{-2 \cdot 0{,}05s} \cdot e^{-0{,}025s} \cdot 0{,}005 \, ds + \int_0^{20} e^{-2 \cdot 0{,}05s} \cdot e^{-0{,}025s} \cdot 0{,}02 \, ds =$$
$$= 0{,}02 \int_0^{20} e^{-0{,}125s} \, ds + 0{,}02 \int_0^{20} e^{-0{,}125s} \, ds = 0{,}04 \int_0^{20} e^{-0{,}125s} \, ds$$
$$= 0{,}04 \left[-\frac{1}{0{,}125} e^{-0{,}125s} \right]_0^{20} = 0{,}04 \left(-\frac{e^{-2{,}5}}{0{,}125} + \frac{1}{0{,}125} \right) = \underline{0{,}2937}$$

$$Var(Z) = EZ^2 - (EZ)^2 = 0{,}2937 - (0{,}3107)^2 = \boxed{0{,}197}$$

... variance of present value of the benefits random variable