

6. cvičení - řešení

24. 11. 2022

Příklad 1 a.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{2^n \sqrt[3]{n^3 + n^2} - n \sqrt[3]{8^n + 1}}{\sqrt[n]{2n^2 + 1}} &= \lim_{n \rightarrow \infty} \frac{2^n \left(\sqrt[3]{n^3 + n^2} - n \sqrt[3]{1 + \frac{1}{8^n}} \right)}{2^n \sqrt[n]{1 + \frac{1}{2n^2}}} \\
 &= \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^3 + n^2} - n \sqrt[3]{1 + \frac{1}{8^n}}}{\sqrt[n]{1 + \frac{1}{2n^2}}} \cdot \frac{\sqrt[3]{(n^3 + n^2)^2} + \sqrt[3]{n^3 + n^2} \cdot n \sqrt[3]{1 + \frac{1}{8^n}} + n^2 \sqrt[3]{(1 + \frac{1}{8^n})^2}}{\sqrt[3]{(n^3 + n^2)^2} + \sqrt[3]{n^3 + n^2} \cdot n \sqrt[3]{1 + \frac{1}{8^n}} + n^2 \sqrt[3]{(1 + \frac{1}{8^n})^2}} = \\
 &= \lim_{n \rightarrow \infty} \frac{n^3 + n^2 - n^3 - \frac{n^3}{8^n}}{\sqrt[n]{1 + \frac{1}{2n^2}} \cdot n^2 \cdot \left(\sqrt[3]{(1 + \frac{1}{n})^2} + \sqrt[3]{1 + \frac{1}{n}} \cdot \sqrt[3]{1 + \frac{1}{8^n}} + \sqrt[3]{(1 + \frac{1}{8^n})^2} \right)} = \\
 &= \lim_{n \rightarrow \infty} \frac{1 + \frac{n}{8^n}}{\sqrt[n]{1 + \frac{1}{2n^2}} \cdot \left(\sqrt[3]{(1 + \frac{1}{n})^2} + \sqrt[3]{1 + \frac{1}{n}} \cdot \sqrt[3]{1 + \frac{1}{8^n}} + \sqrt[3]{(1 + \frac{1}{8^n})^2} \right)} \stackrel{\text{VoAL}}{=} \frac{1}{3}
 \end{aligned}$$

Poslední rovnost jsme získali díky opakověnému použití věty o dvou strážnících na výrazy

$$c_n = \sqrt[n]{1 + \frac{1}{2n^2}},$$

$$d_n = \sqrt[3]{\left(1 + \frac{1}{n}\right)^2} + \sqrt[3]{1 + \frac{1}{n}} \cdot \sqrt[3]{1 + \frac{1}{8^n}} + \sqrt[3]{(1 + \frac{1}{8^n})^2}.$$

Pak máme

$$\begin{aligned}
 \sqrt[n]{1} &\leq c_n \leq \sqrt[n]{2} \\
 \lim_{n \rightarrow \infty} \sqrt[n]{1} &= \lim_{n \rightarrow \infty} \sqrt[n]{2} = 1.
 \end{aligned}$$

Dále máme

$$\begin{aligned}
 \sqrt[3]{(1)^2} + \sqrt[3]{1} \cdot \sqrt[3]{1} + \sqrt[3]{(1)^2} &\leq d_n \leq 3 \cdot \sqrt[3]{\left(1 + \frac{1}{n}\right)^2} \\
 \lim_{n \rightarrow \infty} \sqrt[3]{(1)^2} + \sqrt[3]{1} \cdot \sqrt[3]{1} + \sqrt[3]{(1)^2} &= \lim_{n \rightarrow \infty} 3 \cdot \sqrt[3]{\left(1 + \frac{1}{n}\right)^2} \stackrel{\text{VoAL}^+ \text{ spoj.}}{=} 3
 \end{aligned}$$

Příklad 1 b.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left(\sqrt{n^2 + \sqrt{n} + 1} - \sqrt{n^2 + 2\sqrt{n} + 3} \right) \frac{\sqrt[n]{n + n^n}}{[\sqrt{n + 2}]} \\
&= \lim_{n \rightarrow \infty} \frac{n^2 + \sqrt{n} + 1 - (n^2 + 2\sqrt{n} + 3)}{\sqrt{n^2 + \sqrt{n} + 1} + \sqrt{n^2 + 2\sqrt{n} + 3}} \frac{\sqrt[n]{n + n^n}}{[\sqrt{n + 2}]} \\
&= \lim_{n \rightarrow \infty} \frac{-\sqrt{n} - 2}{\sqrt{n^2 + \sqrt{n} + 1} + \sqrt{n^2 + 2\sqrt{n} + 3}} \frac{\sqrt[n]{n + n^n}}{[\sqrt{n + 2}]} \\
&= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n} \frac{-1 - \frac{2}{\sqrt{n}}}{\sqrt{1 + \frac{1}{n\sqrt{n}} + \frac{1}{n^2}} + \sqrt{1 + \frac{2}{n\sqrt{n}} + \frac{3}{n^2}}} \frac{\sqrt[n]{n + n^n}}{[\sqrt{n + 2}]} \\
&\stackrel{\text{AL, spoj.}}{=} -\frac{1}{2} \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n} \frac{\sqrt[n]{n + n^n}}{[\sqrt{n + 2}]} \stackrel{\text{AL}}{=} -\frac{1}{2} \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{[\sqrt{n + 2}]} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^n + n}}{n} = (*)
\end{aligned}$$

Platí

$$\sqrt{n} - 1 \leq [\sqrt{n}] \leq [\sqrt{n + 2}] \leq \sqrt{n + 2},$$

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$$\frac{\sqrt{n}}{\sqrt{n + 2}} \leq \frac{\sqrt{n}}{[\sqrt{n + 2}]} \leq \frac{\sqrt{n}}{\sqrt{n} - 1}.$$

Zároveň

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n + 2}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{2}{n}}} \stackrel{\text{AL, spoj.}}{=} 1$$

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$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n} - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{\sqrt{n}}} \stackrel{\text{AL}}{=} 1.$$

Z Věty o dvou policajtech pak platí $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{[\sqrt{n + 2}]} = 1$. Dále platí

$$1 = \frac{\sqrt[n]{n^n}}{n} \leq \frac{\sqrt[n]{n^n + n}}{n} \leq \frac{\sqrt[n]{2n^n}}{n} = \sqrt[n]{2}$$

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$$\lim_{n \rightarrow \infty} 1 = 1 = \lim_{n \rightarrow \infty} \sqrt[n]{2}.$$

Opět použitím Věty o dvou policajtech dostáváme

$$(*) = -\frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$$

Příklad 1 c.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left(\sqrt[3]{2n+a} - \sqrt[3]{2n+b} \right) \cdot \sqrt[3]{(n+1)(3n+2)} = \\
&= \lim_{n \rightarrow \infty} \sqrt[3]{(n+1)(3n+2)} \left(\sqrt[3]{2n+a} - \sqrt[3]{2n+b} \right) \cdot \frac{(\sqrt[3]{2n+a})^2 + \sqrt[3]{2n+a} \cdot \sqrt[3]{2n+b} + (\sqrt[3]{2n+b})^2}{(\sqrt[3]{2n+a})^2 + \sqrt[3]{2n+a} \cdot \sqrt[3]{2n+b} + (\sqrt[3]{2n+b})^2} = \\
&= \lim_{n \rightarrow \infty} \frac{((2n+a) - (2n+b)) \cdot \sqrt[3]{(n+1)(3n+2)}}{(\sqrt[3]{2n+a})^2 + \sqrt[3]{2n+a} \cdot \sqrt[3]{2n+b} + (\sqrt[3]{2n+b})^2} = \\
&= \lim_{n \rightarrow \infty} \frac{(a-b)n^{\frac{2}{3}} \sqrt[3]{(1+\frac{1}{n})(3+\frac{2}{n})}}{n^{\frac{2}{3}} \left((\sqrt[3]{2+\frac{a}{n}})^2 + \sqrt[3]{2+\frac{a}{n}} \cdot \sqrt[3]{2+\frac{b}{n}} + (\sqrt[3]{2+\frac{b}{n}})^2 \right)} \stackrel{\text{VoAL}}{=} \frac{(a-b) \cdot \sqrt[3]{3}}{3 \cdot 2^{\frac{2}{3}}} = (a-b) \cdot 6^{-\frac{2}{3}}
\end{aligned}$$

Použili jsme vzorec

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \dots + A^2B^{n-3} + AB^{n-2} + B^{n-1}).$$

Příklad 1 d. Pro $k \in \mathbb{N}$ označme $P_{\leq k}(n)$ polynom proměnné n stupně nejvýše k .

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{(n^2+1)^{100} - (n+2)^{200} + 400n^{199}}{1+2+\dots+n^{99}} \\
&= \lim_{n \rightarrow \infty} \frac{n^{200} + 100n^{198} + P_{\leq 196}(n) - n^{200} - 400n^{199} - 4 \cdot \binom{200}{2} n^{198} - P_{\leq 197}(n) + 400n^{198}}{\frac{n^{99}(n^{99}+1)}{2}} \\
&= \lim_{n \rightarrow \infty} \frac{n^{198}(100 - 79600 + \frac{P_{\leq 197}(n)}{n^{198}})}{n^{198}(\frac{1}{2} + \frac{1}{n^{99}})} = \lim_{n \rightarrow \infty} \frac{100 - 79600 + \frac{P_{\leq 197}(n)}{n^{198}}}{\frac{1}{2} + \frac{1}{n^{99}}} \stackrel{\text{AL}}{=} -159000.
\end{aligned}$$

Příklad 2

Vzorové řešení doc. Johanise.