Mathematics I

FSV UK, winter semester 2018-19

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1. Sets, propositions and numerical sets

1.1 Sets

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- We say that a set A is **part of a set** B (or A is a **subset** of B), if all elements of A are also elements of B. We write $A \subset B$ (**inclusion**).
- Two sets are **equal** ($A = B$), if they have the same elements, that is to say $A \subset B$ and $B \subset A$ both hold at the same time.

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- We define $\bigcup_{\alpha \in I} A_\alpha$ as the set of all those elements, which belong to at least one of the sets A_α.

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- If two sets have an empty intersection we call them **disjoint**.
- We define $\bigcap_{\alpha \in I} A_\alpha$ as the set of elements that belong to all of A_α.

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- Let us consider m sets $A_1, \ldots, A_m$. The **Cartesian product** $A_1 \times A_2 \times \cdots \times A_m$ is the set of all ordered m-tuples

$$\{[a_1, a_2, \ldots, a_m];\ a_1 \in A_1, \ldots, a_m \in A_m\}.$$

Theorem 1.1 (de Morgan rules)

Let us consider the sets S, A_α, $\alpha \in I$, where $I \neq \emptyset$. Then

$$S \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (S \setminus A_\alpha) \quad \textit{and}$$

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1.2 Propositional calculus, mathematical proofs

A **statement** is any claim for which it makes sense to say that it either holds (is true), or does not hold (is false).

The **negation** of the statement A is the statement: “It is not true that A holds.”

A	$\neg A$
0	1
1	0

The **conjunction** $A \wedge B$ of the statements A and B is the statement: “Both A and B hold.”

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

The **disjunction** $A \vee B$ of A and B is the statement: "A or B holds."

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

The **implication** is the statement: “If A holds, then B also holds.”

A	B	$A \Rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

The **equivalence** is the statement: “A holds if and only if B holds.”

A	B	$A \Leftrightarrow B$
0	0	1
0	1	0
1	0	0
1	1	1

The end of the first lecture, 3. 10. 2018

A **statement function** is an expression, from which we obtain a statement by substituting an element from a given set into the function as a variable. Generally we can write a statement function as

$$A(x_1, x_2, \ldots, x_m), \quad x_1 \in M_1, x_2 \in M_2, \ldots, x_m \in M_m.$$

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Let $A(x)$, $x \in M$, be a statement function.
"For all $x \in M$ it holds $A(x)$."

$$\forall x \in M\colon\ A(x)$$

The symbol $\forall$ is called the **universal quantifier**.

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The symbol $\exists$ is called the **existential quantifier**.Further we use the notation

$$\exists! x \in M\colon A(x),$$

which we read as "There exists exactly one $x \in M$ such that $A(x)$."

Let us consider the statement function $V(x, y)$, $x \in M_1$, $y \in M_2$. Now we can create new statement functions of a single variable $y \in M_2$ as follows:

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We can create statements from these statement functions using another quantifier as follows:

$$\forall y \in M_2\colon\ \big(\forall x \in M_1\colon\ V(x, y)\big),\quad \forall y \in M_2\colon\ \big(\exists x \in M_1\colon\ V(x, y)\big),$$
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We usually write the statements above in the form

$$\forall y \in M_2\ \forall x \in M_1\colon\ V(x, y),\ \ \forall y \in M_2\ \exists x \in M_1\colon\ V(x, y),$$
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Let A and P be statement functions of one variable. Then

$$\begin{aligned} &\forall x \in M, P(x)\colon A(x) \quad \text{means} \quad \forall x \in M\colon \big(P(x) \Rightarrow A(x)\big), \\ &\exists x \in M, P(x)\colon A(x) \quad \text{means} \quad \exists x \in M\colon \big(P(x) \wedge A(x)\big). \end{aligned}$$

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We read the first statement “For every $x \in M$ satisfying P the statement $A(x)$ holds.” The second statement is read “There exists $x \in M$ satisfying P such that $A(x)$ holds.”

Let V be a statement function of the variable $x \in M$, then

$$\neg\big(\forall x \in M\colon\ V(x)\big) \quad \text{means the same as} \quad \exists x \in M\colon \neg V(x),$$
$$\neg\big(\exists x \in M\colon\ V(x)\big) \quad \text{means the same as} \quad \forall x \in M\colon \neg V(x).$$

Direct proof

By using the validity of the statement A we show the validity of the statement C_1, using C_1 we show the validity of C_2, from which we show C_3, and so on until, using the validity of C_n we show the statement B. We then have discovered the following chain of implications

$$A \Rightarrow C_1, \ C_1 \Rightarrow C_2, \ C_2 \Rightarrow C_3, \dots, C_{n-1} \Rightarrow C_n, \ C_n \Rightarrow B.$$

The end of the second lecture, 4. 10. 2018

Indirect proof

This type of proof is based on the equivalence of the statements $A \Rightarrow B$ and $\neg B \Rightarrow \neg A$. If the second is true then so is the first. Therefore it suffices to find any proof of the second statement.

Proof by contradiction

This method is based on the equivalence of the statements $A \Rightarrow B$ and $\neg(A \wedge \neg B)$. In this method of proof we assume the validity of $A \wedge \neg B$. If we are able to deduce a statement C, which we know to be false, then $A \wedge \neg B$ must also be false (one cannot deduce a false statement from a true statement). It therefore holds $\neg(A \wedge \neg B)$, or $A \Rightarrow B$.

Mathematical induction.

One can use this type of proof to show statements of the following sort

$$\forall n \in \mathbf{N}\colon\ V(n), \tag{1.1}$$

where $V(n)$, $n \in \mathbf{N}$ is a statement function.

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$$\forall n \in \mathbf{N}\colon\ V(n), \tag{1.1}$$

where $V(n)$, $n \in \mathbf{N}$ is a statement function. In the first step of mathematical induction we show the validity of the statement $V(1)$. In the second step we prove the statement

$$\forall n \in \mathbf{N}\colon\ V(n) \Rightarrow V(n+1),$$

that is we assume the validity of $V(n)$ (the so called **induction hypothesis**) and deduce the validity of $V(n+1)$. From these two steps we get the validity of the statement (1.1).

1.3 The set of real numbers

The set of real numbers **R** will be considered as the set with the operations **addition** and **multiplication**, which will be denoted as usual, and a relation **ordering** ($\leq$), such that the following three groups are satisfied.

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II. The relationships of the ordering and the operations addition and multiplication.

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I. The properties of addition and multiplication and their relationships.
II. The relationships of the ordering and the operations addition and multiplication.
III. Infimum axiom.

The end of the third lecture, 11. 10. 2018

Definition of boundedness

Definice

We say, that the set $M \subset \mathbf{R}$ is **bounded from below**, if there exists a number $a \in \mathbf{R}$ such that, for each $x \in M$ we have $x \geq a$.

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Let M be a nonempty bounded from below set. Then there exists a unique number $g \in \mathbf{R}$ such that

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The number g is denoted by $\inf M$ and is called **infimum** of the set M.

Supremum

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Let $M \subset \mathbf{R}$. The number $G \in \mathbf{R}$ satisfying

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Theorem 1.2

Let $M \subset \mathbf{R}$ be a nonempty set which is bounded from above. Then there exists $\sup M$.

Maximum and minimum

Definice

Let $M \subset \mathbf{R}$. We say that a is a **maximum** of the set M (notation $\max M$), if $a \in M$ and a is an upper bound of M. We define analogously **minimum** of M. Maximum and minimum of M is denoted by $\max M$ and $\min M$ respectively.

Theorem 1.3

For every $r \in \mathbf{R}$ there exists an ***integer part*** *of r, i.e., there exists $k \in \mathbf{Z}$ such that $k \leq r < k + 1$. (Integer part of r is denoted by $[r]$).*

Basic properties of real numbers

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Theorem 1.4

For each $x \in \mathbf{R}$ there exists $n \in \mathbf{N}$ such that $x < n$.

Theorem 1.5

For each $x \in \langle 0, +\infty)$ and for each $n \in \mathbf{N}$ there exists a unique $y \in \mathbf{R}$, $y \geq 0$, with $y^n = x$.

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For each $x \in \langle 0, +\infty)$ *and for each* $n \in \mathbf{N}$ *there exists a unique* $y \in \mathbf{R}$, $y \geq 0$, *with* $y^n = x$.

Theorem 1.6

Let $a, b \in \mathbf{R}$, $a < b$. *Then there exists* $r \in \mathbf{Q}$ *such that* $a < r < b$.

Kurt Gödel (1906–1978)

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Suppose that to each natural number $n \in \mathbf{N}$ is assigned a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a **sequence** of real numbers. The number a_n is called ***n*-th member** of the sequence. A sequence $\{a_n\}_{n=1}^{\infty}$ equals a sequence $\{b_n\}_{n=1}^{\infty}$, if $a_n = b_n$ holds for every $n \in \mathbf{N}$.

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A sequence $\{a_n\}$ is **monotone**, if it satisfies one of the conditions above. A sequence $\{a_n\}$ is **strictly monotone**, if it is increasing or decreasing.

II.2 Convergence

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We say that a sequence $\{a_n\}$ has a **limit** which equals to a real number A, if

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0 \; \exists n_0 \in \mathbf{N} \; \forall n \in \mathbf{N}, n \geq n_0 : \; |a_n - A| < \varepsilon.$$

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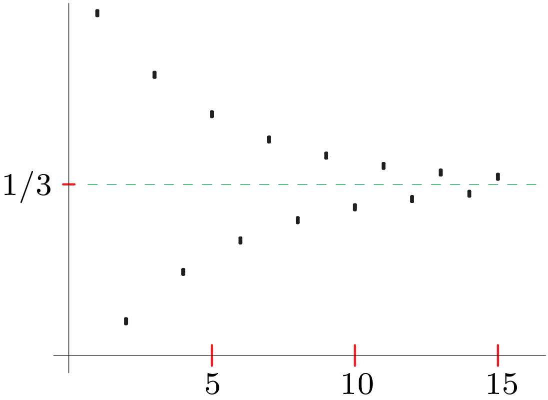
We denote $\lim\limits_{n\to\infty} a_n = A$ or only $\lim a_n = A$. We say that a sequence $\{a_n\}$ is **convergent**, if there exists $A \in \mathbf{R}$ with $\lim a_n = A$.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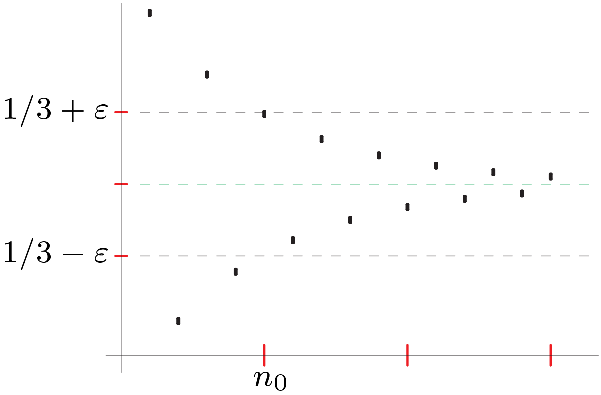
$1/3+\varepsilon$
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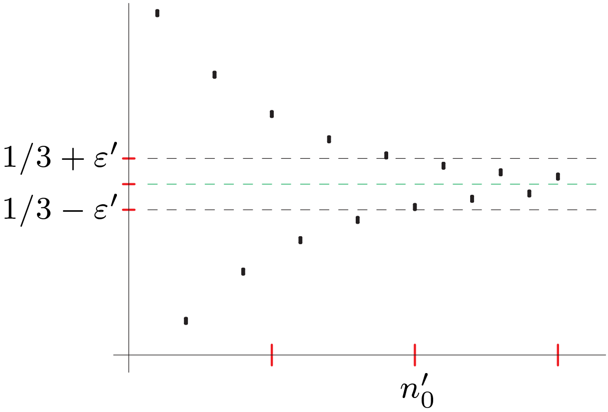
n_0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