Mathematics II

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- Functions of several variables
- Matrix calculus
- Infinite series
- Integral

V.1. $\mathbf{R}^n$ as a metric and linear space

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Definition

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Definition

Euclidean metric on $\mathbf{R}^n$ is the function $\rho\colon \mathbf{R}^n \times \mathbf{R}^n \to [0, +\infty)$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

The number $\rho(\boldsymbol{x}, \boldsymbol{y})$ is called distance of the point $\boldsymbol{x}$ from the point $\boldsymbol{y}$.

Theorem 5.1 (properties of Euclidean metric)

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Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*

(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbf{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*

(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n, \forall \lambda \in \mathbf{R} \colon \rho(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) = |\lambda| \, \rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*

(v) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbf{R}^n \colon \rho(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + \boldsymbol{z}) = \rho(\boldsymbol{x}, \boldsymbol{y})$. *(translation invariance)*

Definition

Let $\boldsymbol{x} \in \mathbf{R}^n,\ r \in \mathbf{R}, r > 0$. The set $B(\boldsymbol{x}, r)$ defined by

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbf{R}^n;\ \rho(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

is called open ball with radius r centered at $\boldsymbol{x}$.

Definition

Let $M \subset \mathbf{R}^n$. We say that $\boldsymbol{x} \in \mathbf{R}^n$ is an interior point of M, if there exists $r > 0$ such that $B(\boldsymbol{x}, r) \subset M$.

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Theorem 5.2 (properties of open sets)

(i) *The empty set and* $\mathbf{R}^n$ *are open in* $\mathbf{R}^n$.

(ii) *Let sets* $G_\alpha \subset \mathbf{R}^n$, $\alpha \in A \neq \emptyset$, *be open in* $\mathbf{R}^n$. *Then* $\bigcup_{\alpha \in A} G_\alpha$ *is open in* $\mathbf{R}^n$.

(iii) *Let sets* G_i, $i = 1, \dots, m$, *be open in* $\mathbf{R}^n$. *Then* $\bigcap_{i=1}^{m} G_i$ *is open in* $\mathbf{R}^n$.

Theorem 5.3 (properties of closed sets)

(i) *The empty set and* $\mathbf{R}^n$ *are closed in* $\mathbf{R}^n$.

(ii) *Let sets* $F_\alpha \subset \mathbf{R}^n$, $\alpha \in A \neq \emptyset$, *be closed in* $\mathbf{R}^n$. *Then* $\bigcap_{\alpha \in A} F_\alpha$ *is closed in* $\mathbf{R}^n$.

(iii) *Let sets* F_i, $i = 1, \dots, m$, *are closed in* $\mathbf{R}^n$. *Then* $\bigcup_{i=1}^m F_i$ *is closed in* $\mathbf{R}^n$.

Definition

Let $M \subset \mathbf{R}^n$ and $\boldsymbol{x} \in \mathbf{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M, if for each $r > 0$ we have $B(\boldsymbol{x}, r) \cap M \neq \emptyset$ and $B(\boldsymbol{x}, r) \cap (\mathbf{R}^n \setminus M) \neq \emptyset$.

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Boundary of M is the set of all boundary points of M (notation $\operatorname{bd} M$).

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Interior of M is the set of all interior points of M (notation $\operatorname{int} M$).

Definition

Let $\boldsymbol{x}^j \in \mathbf{R}^n$ for each $j \in \mathbf{N}$ and $\boldsymbol{x} \in \mathbf{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$, if $\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0$. The vector $\boldsymbol{x}$ is called limit of the sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$.

Theorem 5.4

Let $\boldsymbol{x}^j \in \mathbf{R}^n$ for each $j \in \mathbf{N}$ and $\boldsymbol{x} \in \mathbf{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \ldots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number x_i.

Definition

Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbf{R}$. We say that f is continuous at $\boldsymbol{x}$ with respect to M, if we have

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0\ \exists \delta \in \mathbf{R}, \delta > 0\ \forall \boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M\colon f(\boldsymbol{y}) \in B(f(\boldsymbol{x}), \varepsilon).$$

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We say that f is continuous at the point $\boldsymbol{x}$, it it is continuous at $\boldsymbol{x}$ with respect to a neighborhood of $\boldsymbol{x}$, i.e.,

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0\ \exists \delta \in \mathbf{R}, \delta > 0\ \forall \boldsymbol{y} \in B(\boldsymbol{x}, \delta) \colon f(\boldsymbol{y}) \in B(f(\boldsymbol{x}), \varepsilon).$$

Remark

Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, $f\colon M \to \mathbf{R}$, $g\colon M \to \mathbf{R}$, and $c \in \mathbf{R}$. If f and g are continuous at the point $\boldsymbol{x}$ with respect to M, then the functions cf, $f + g$ a fg are continuous at $\boldsymbol{x}$ with respect to M. If the function g is nonzero at each point of M, then also the function f/g is continuous at $\boldsymbol{x}$ with respect to M.

Theorem 5.5 (Heine)

Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbf{R}$. Then the following are equivalent.

(i) *The function f is continuous at $\boldsymbol{x}$ with respect to M.*

(ii) *For each sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ such that $\boldsymbol{x}^j \in M$ pro $j \in \mathbf{N}$ a $\lim\limits_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$, we have $\lim\limits_{j\to\infty} f(\boldsymbol{x}^j) = f(\boldsymbol{x})$.*

Remark

Let $r, s \in \mathbf{N}$, $M \subset \mathbf{R}^s$, $L \subset \mathbf{R}^r$, and $\boldsymbol{y} \in M$. Let $\varphi_1, \dots, \varphi_r$ are functions defined on M, which are continuous at $\boldsymbol{y}$ with respect to M and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in L$ for each $\boldsymbol{x} \in M$. Let $f\colon L \to \mathbf{R}$ be continuous at the point $[\varphi_1(\boldsymbol{y}), \dots, \varphi_r(\boldsymbol{y})]$ with respect to L. Then the composed function $F\colon M \to \mathbf{R}$ defined by

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in M,$$

is continuous at $\boldsymbol{y}$ with respect to M.

Definition

Let $M \subset \mathbf{R}^n$ a $f\colon M \to \mathbf{R}$. We say that f is continuous on M, if it is continuous at each point $\boldsymbol{x} \in M$ with respect to M.

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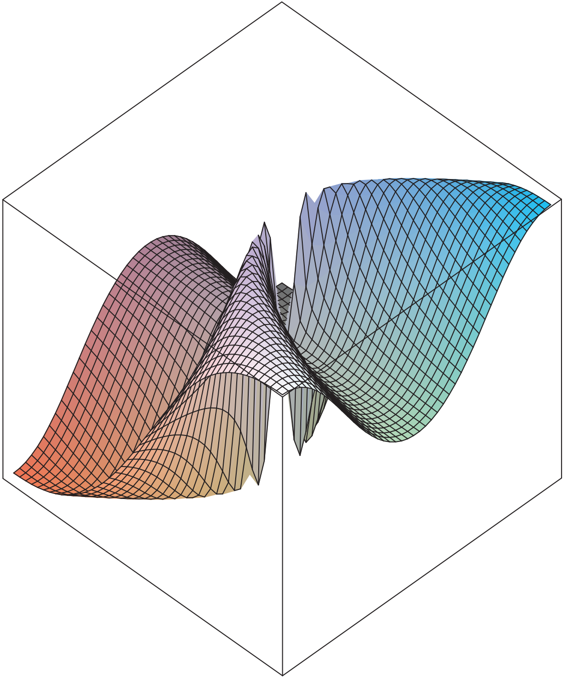
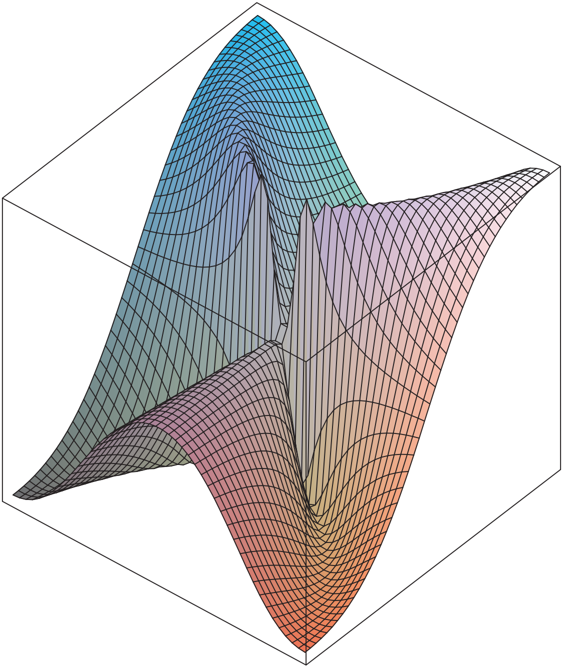
Remark

The projection $\pi_j\colon \mathbf{R}^n \to \mathbf{R}$, $\pi_j(\boldsymbol{x}) = x_j$, $1 \leq j \leq n$, are continuous on $\mathbf{R}^n$.





Let f be a continuous function on $\mathbf{R}^n$ and $c \in \mathbf{R}$. Then we have:

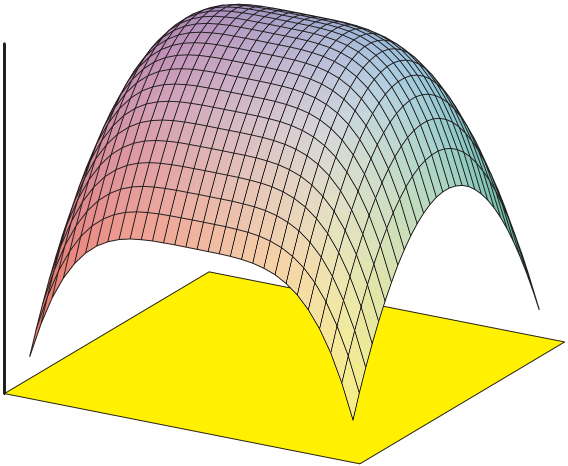
(i) The set $\{\boldsymbol{x} \in \mathbf{R}^n;\ f(\boldsymbol{x}) < c\}$ is open in $\mathbf{R}^n$.

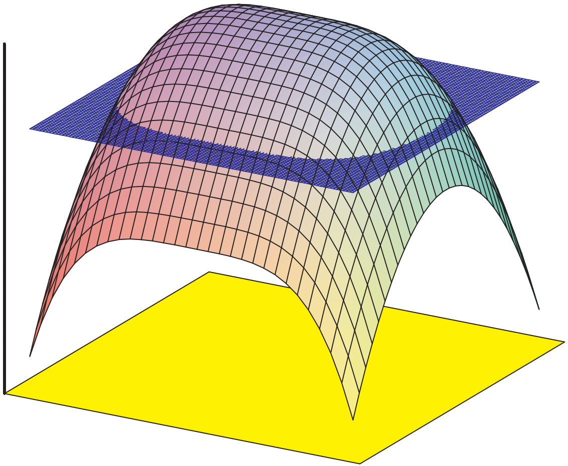
(ii) The set $\{\boldsymbol{x} \in \mathbf{R}^n;\ f(\boldsymbol{x}) > c\}$ is open in $\mathbf{R}^n$.

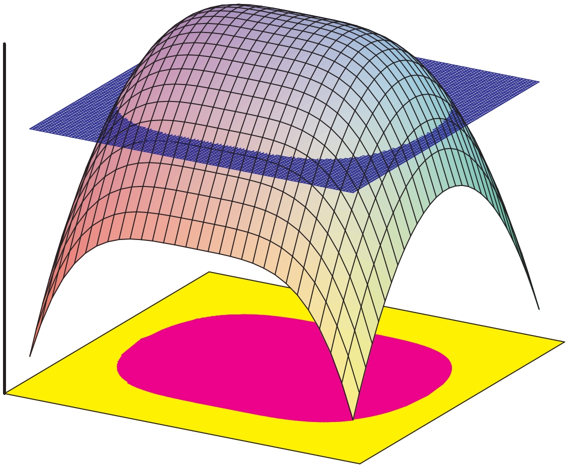
(iii) The set $\{\boldsymbol{x} \in \mathbf{R}^n;\ f(\boldsymbol{x}) \leq c\}$ is closed in $\mathbf{R}^n$.

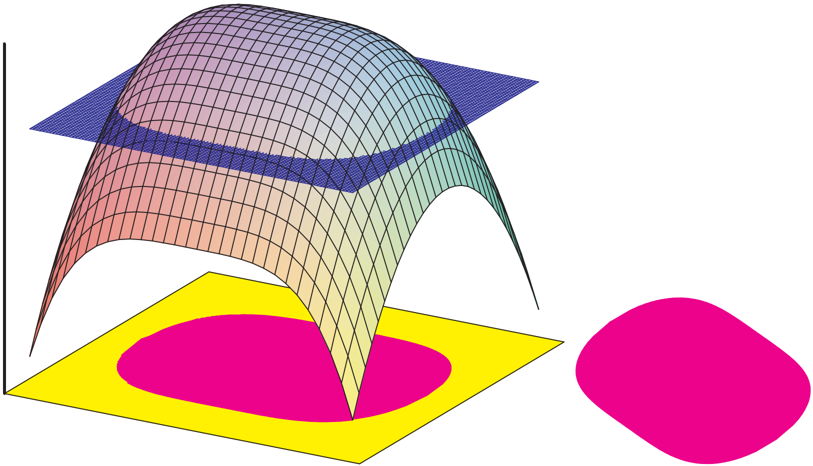
(iv) The set $\{\boldsymbol{x} \in \mathbf{R}^n;\ f(\boldsymbol{x}) \geq c\}$ is closed in $\mathbf{R}^n$.

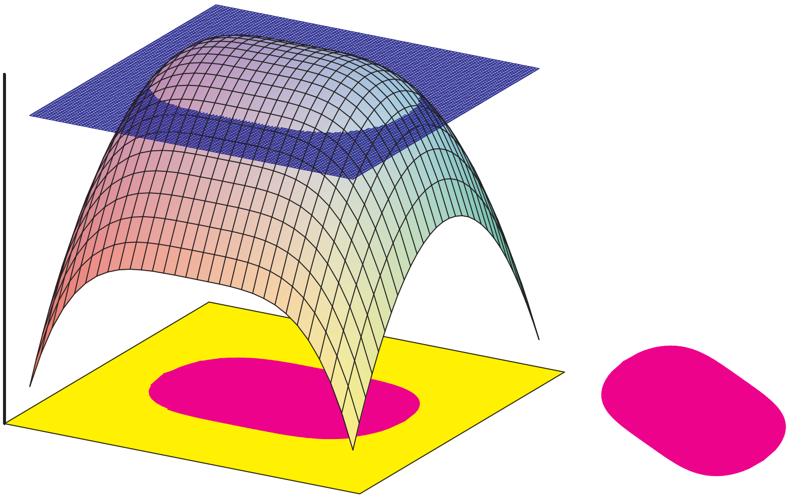
(v) The set $\{\boldsymbol{x} \in \mathbf{R}^n;\ f(\boldsymbol{x}) = c\}$ is closed in $\mathbf{R}^n$.

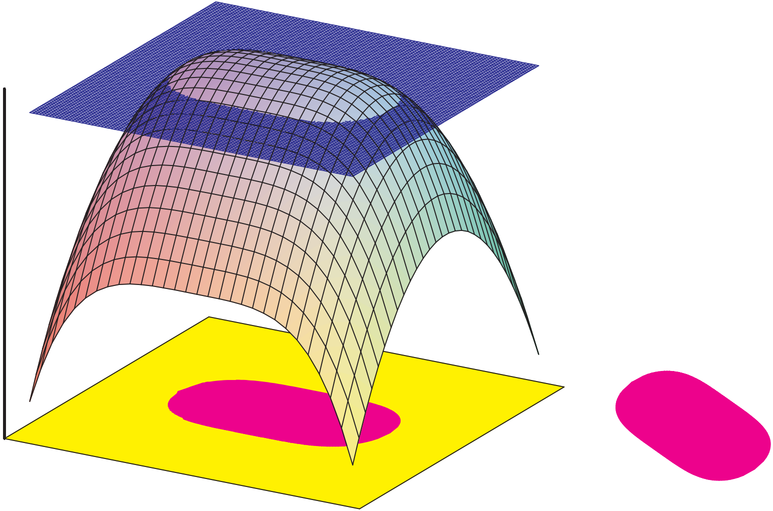












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We say that a set $M \subset \mathbf{R}^n$ is compact, if for each sequence of elements of M there exists a convergent subsequence with limit in M.

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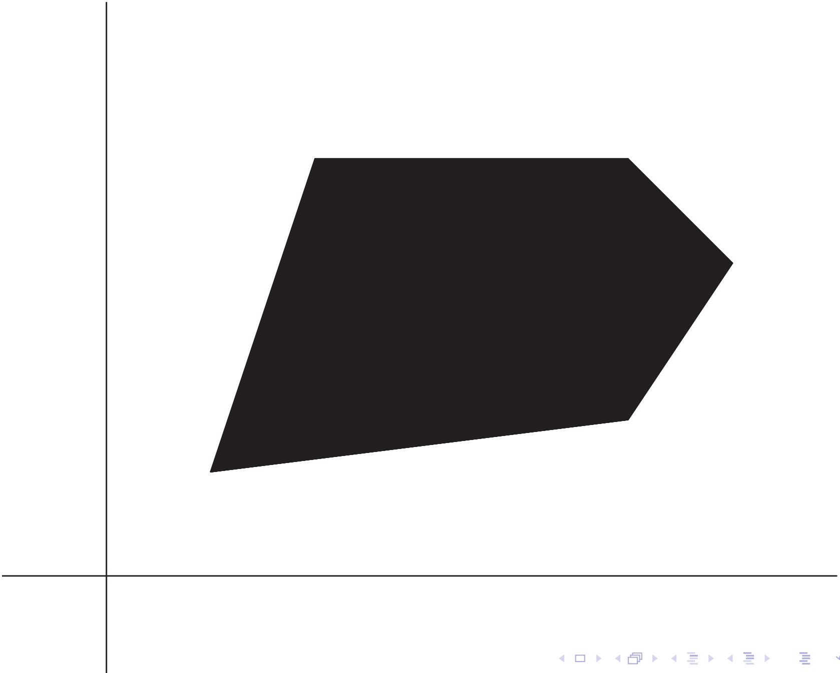
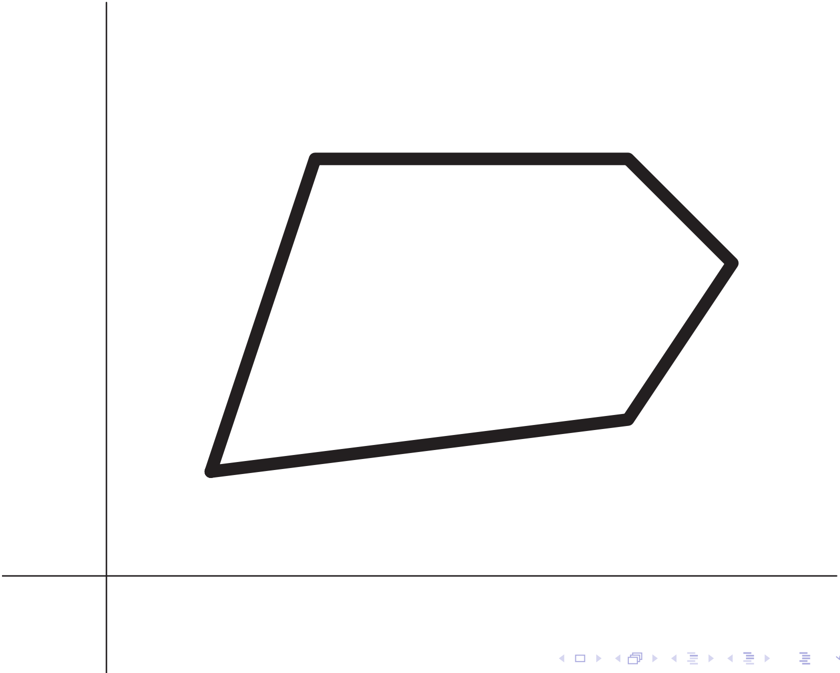
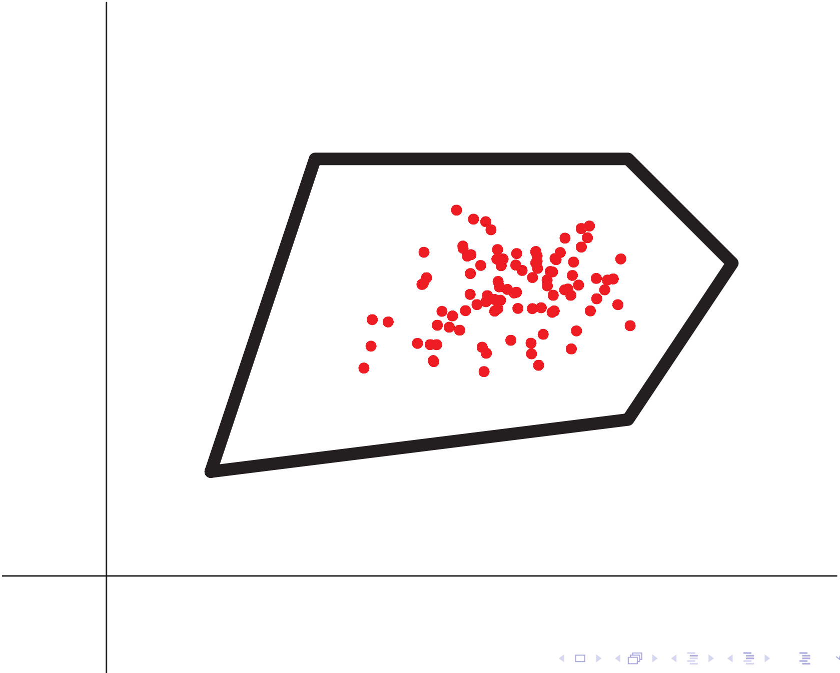
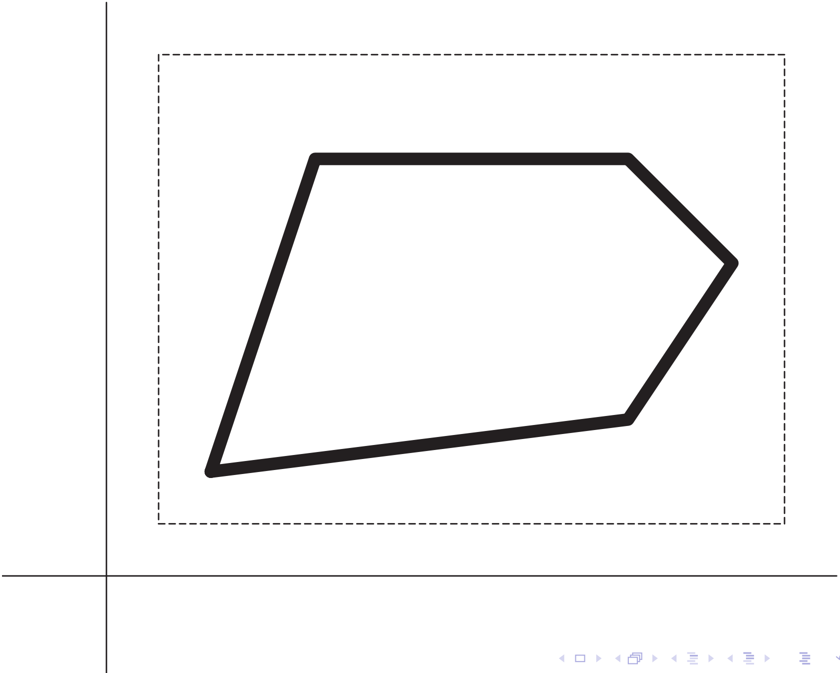
Theorem 5.6 (characterization of compact subsets of $\mathbf{R}^n$)

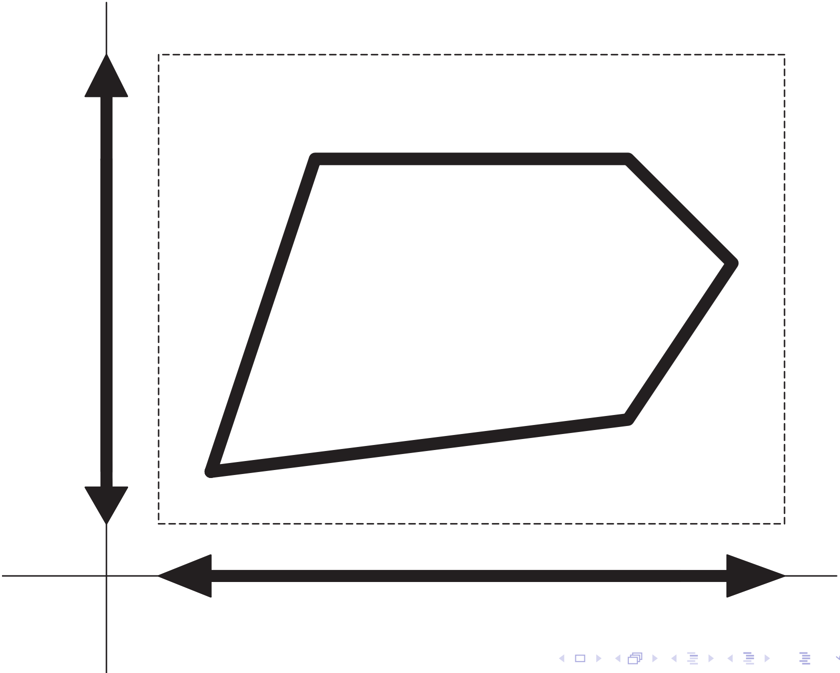
The set $M \subset \mathbf{R}^n$ is compact if and only if M is bounded and closed.

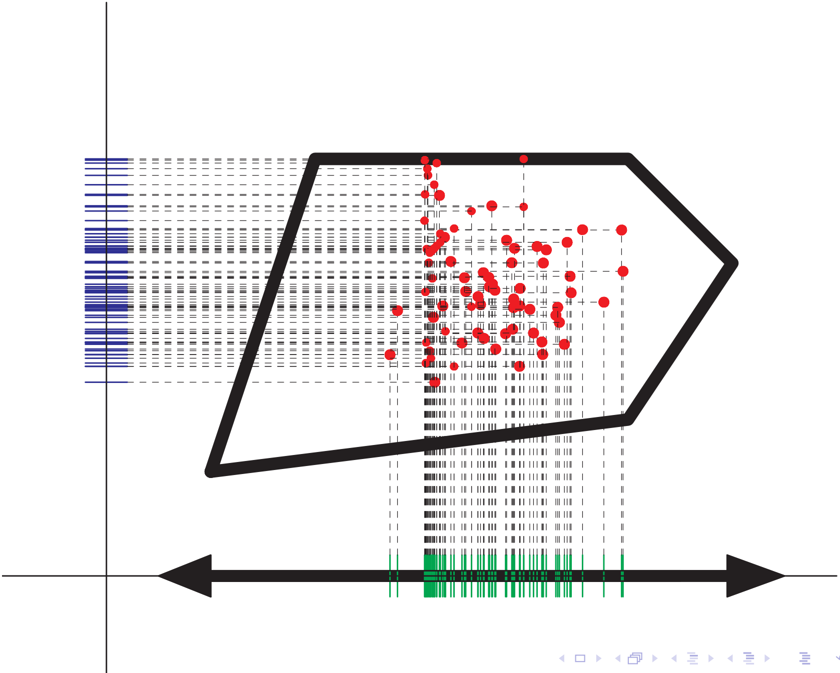


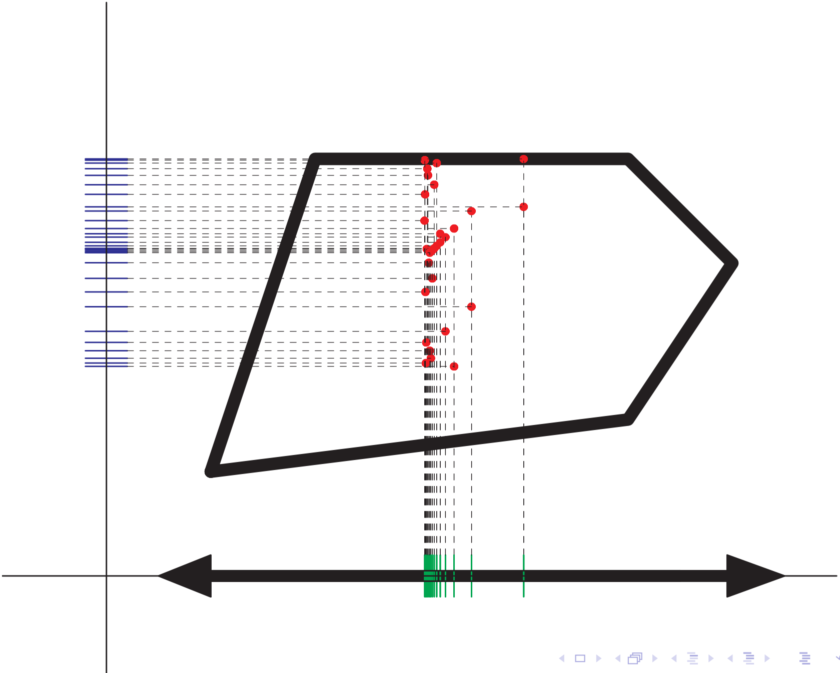


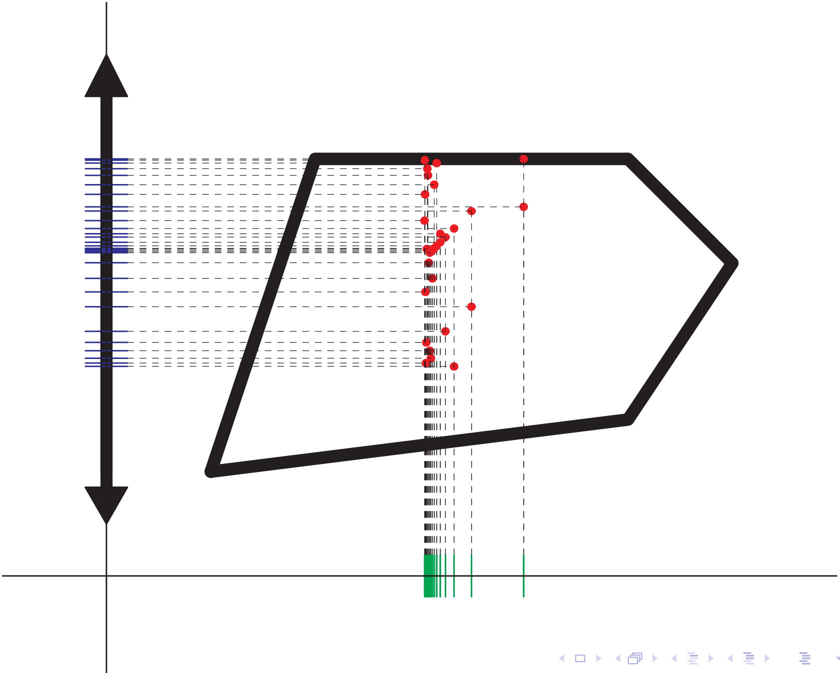


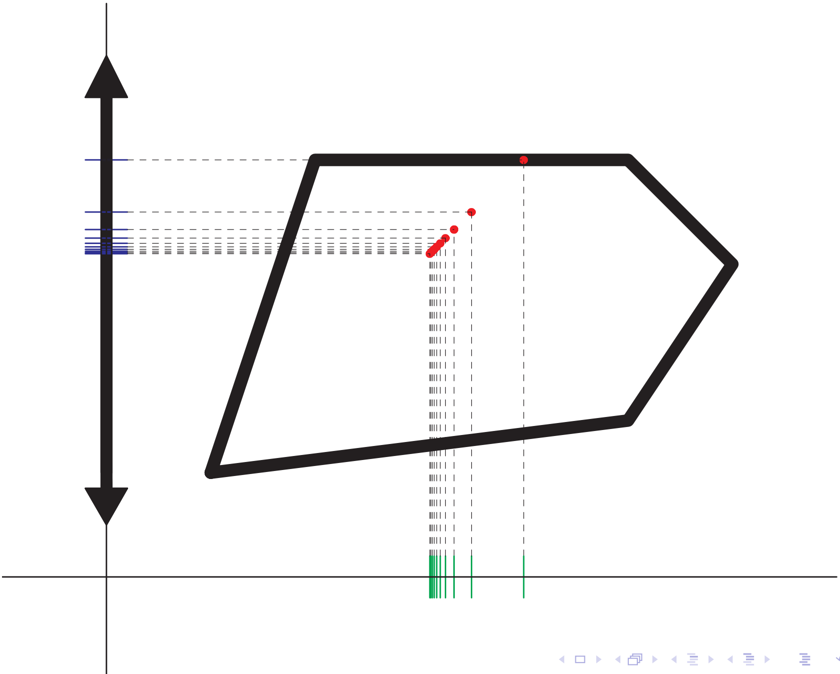












Definition

Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, and f be a function defined at least on M, i.e., $M \subset D_f$. We say that f attains at the point $\boldsymbol{x}$

- maximum on M, if for every $\boldsymbol{y} \in M$ we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$,

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- maximum on M, if for every $\boldsymbol{y} \in M$ we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$,
- local maximum with respect to M, if there exists $\delta > 0$ such that for every $\boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M$ we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$,

Definition

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- local maximum with respect to M, if there exists $\delta > 0$ such that for every $\boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M$ we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$,
- sharp local maximum with respect to M, if there exists $\delta > 0$ such that for every $\boldsymbol{y} \in \big(B(\boldsymbol{x}, \delta) \setminus \{x\}\big) \cap M$ we have $f(\boldsymbol{y}) < f(\boldsymbol{x})$.

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- sharp local maximum with respect to M, if there exists $\delta > 0$ such that for every $\boldsymbol{y} \in \big(B(\boldsymbol{x}, \delta) \setminus \{x\}\big) \cap M$ we have $f(\boldsymbol{y}) < f(\boldsymbol{x})$.

The notions minimum, local minimum, and sharp local minimum with respect to M are defined analogically.

Definition

We say that a function f attains at the point $\boldsymbol{x} \in \mathbf{R}^n$ local maximum, if $\boldsymbol{x}$ is a local maximum with respect to some ball centered at the point $\boldsymbol{x}$. Similarly one can define local minimum, sharp local maximum and sharp local minimum.

Theorem 5.7 (attaining extrema)

Let $M \subset \mathbf{R}^n$ be a nonempty compact set and $f\colon M \to \mathbf{R}$ be continuous on M. Then f attains on M its maximum and minimum.

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Corollary 5.8

Let $M \subset \mathbf{R}^n$ be a nonempty compact set and $f\colon M \to \mathbf{R}$ be continuous on M. Then f is bounded on M.

Definition

We say that function $f\colon \mathbf{R}^n \to \mathbf{R}$ has at a point $\boldsymbol{a} \in \mathbf{R}^n$ limit equal $A \in \mathbb{R}^*$, if we have

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0\ \exists \delta \in \mathbf{R}, \delta > 0\ \forall \boldsymbol{x} \in B(\boldsymbol{a}, \delta) \setminus \{\boldsymbol{a}\} \colon f(\boldsymbol{x}) \in B(A, \varepsilon).$$

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Remark

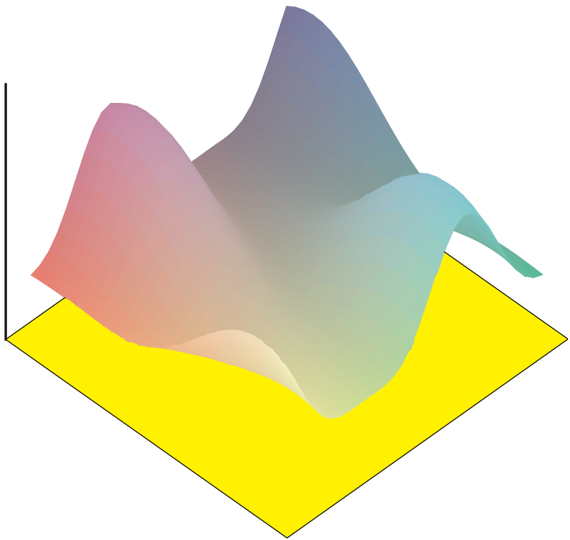
- Each function has at a given point at most one limit. We write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$.
- The function f is continuous at $\boldsymbol{a}$ if and only if $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$.
- For functions of several variables one can prove similar theorems as for functions of one variable (arithmetics, sandwich theorem, ...).

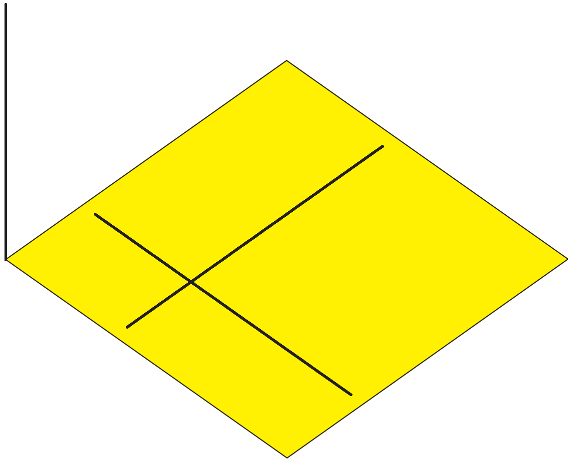
Theorem 5.9

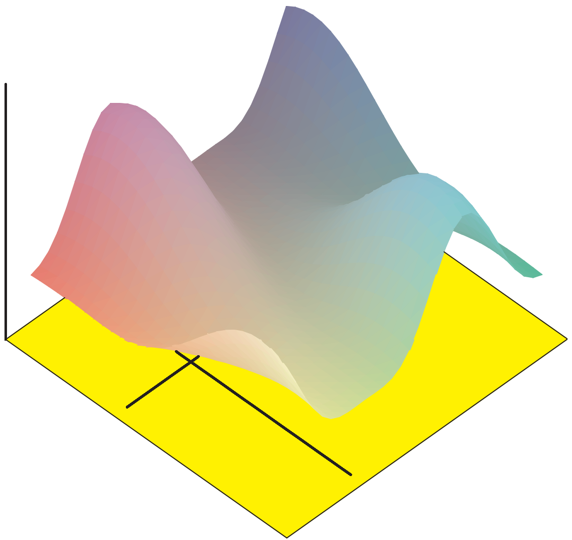
Let $r, s \in \mathbf{N}$, $\boldsymbol{a} \in M \subset \mathbf{R}^s$, $L \subset \mathbf{R}^r$, $\varphi_1, \dots, \varphi_r$ be functions defined on M such that $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \varphi_j(\boldsymbol{x}) = b_j$, $j = 1, \dots, r$, and $\boldsymbol{b} = [b_1, \dots, b_r] \in L$. Let $f \colon L \to \mathbf{R}$ be continuous at the point $\boldsymbol{b}$. We define a function $F \colon M \to \mathbf{R}$ by

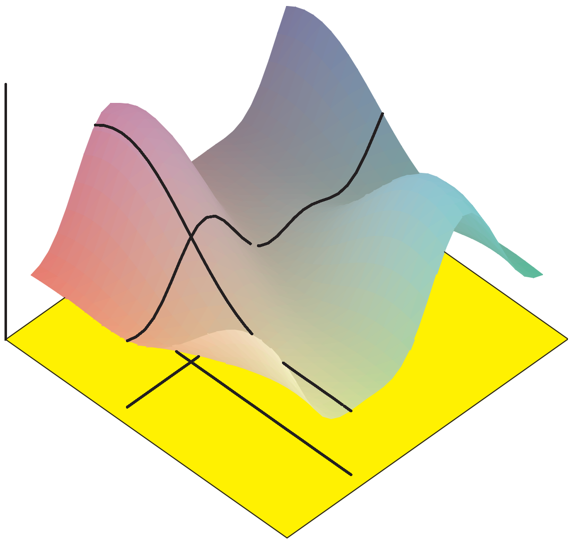
$$F(\boldsymbol{x}) = f(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})), \qquad \boldsymbol{x} \in M.$$

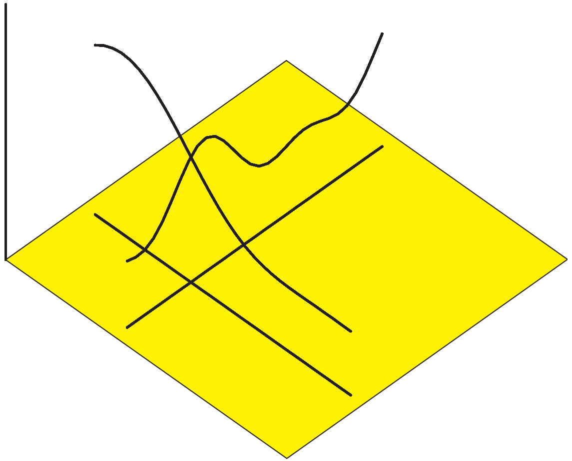
Then $\lim_{\boldsymbol{x}\to\boldsymbol{a}} F(\boldsymbol{x}) = f(\boldsymbol{b})$.

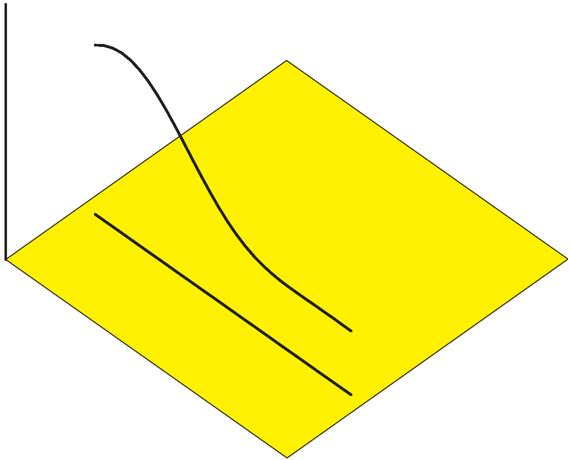


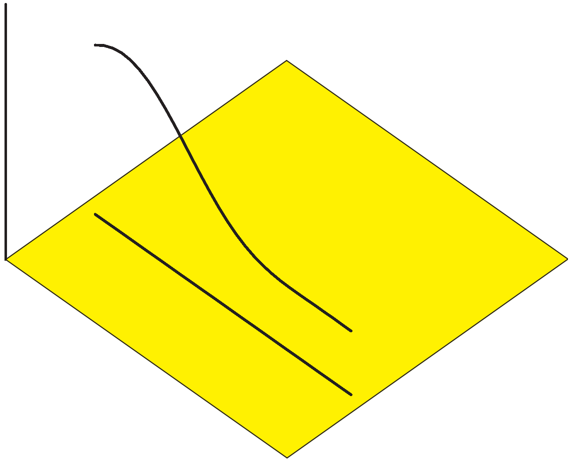


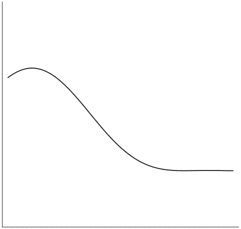


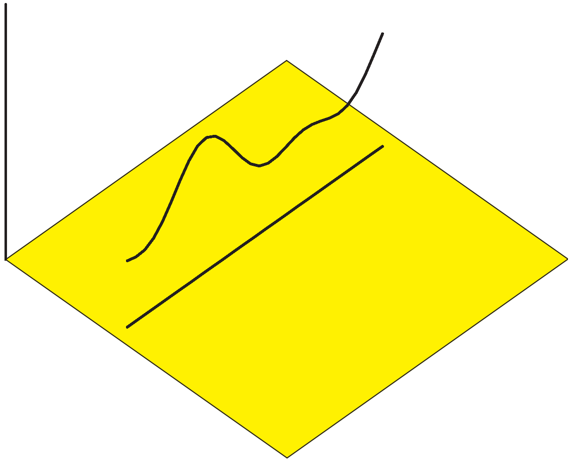


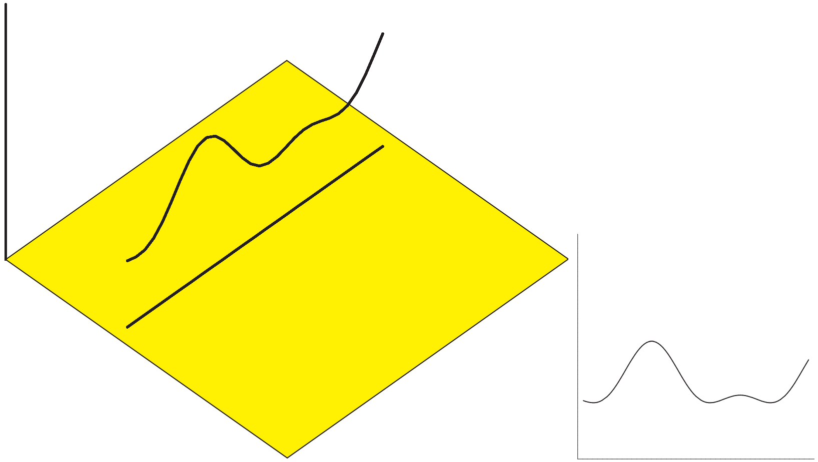












Definition

Let f be a function of n variables, $j \in \{1, \ldots, n\}$, $\boldsymbol{a} \in \mathbf{R}^n$. Then the number

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t}$$

is called partial derivatives (of first order) of function f according to j-th variable at the point $\boldsymbol{a}$ (if it exists).

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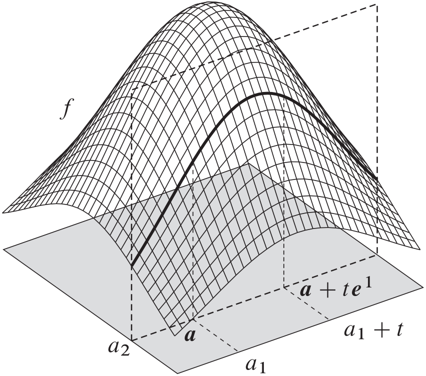
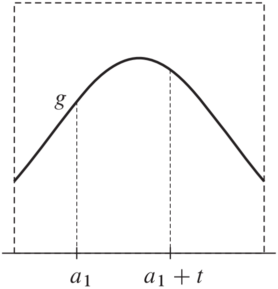
$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t}$$
$$= \lim_{t \to 0} \frac{f(a_1, \ldots, a_{j-1}, a_j + t, a_{j+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{t}$$

is called partial derivatives (of first order) of function f according to j-th variable at the point $\boldsymbol{a}$ (if it exists).



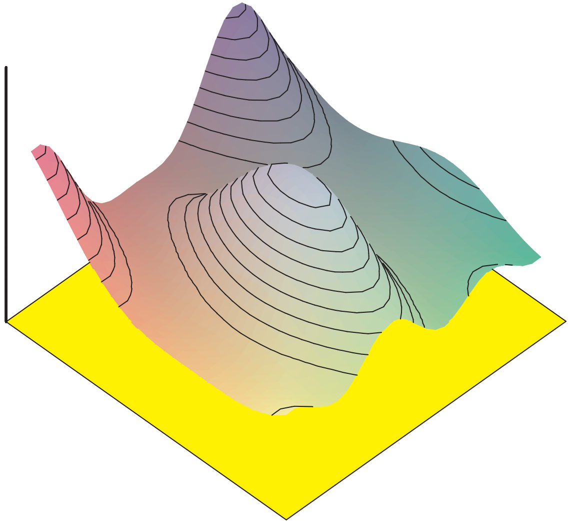
f
a + te^1
a_1 + t
a
a_2
a_1

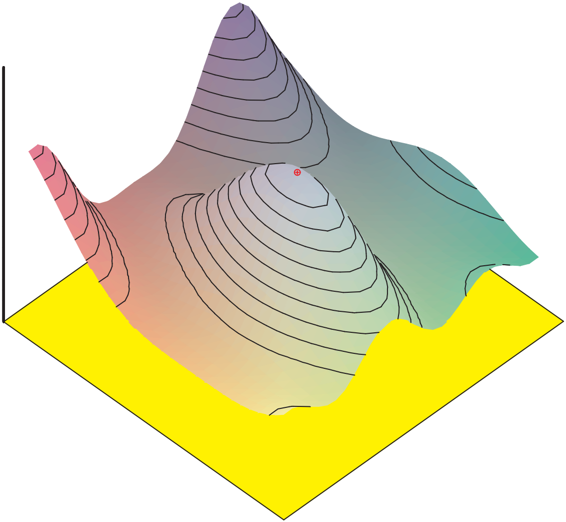


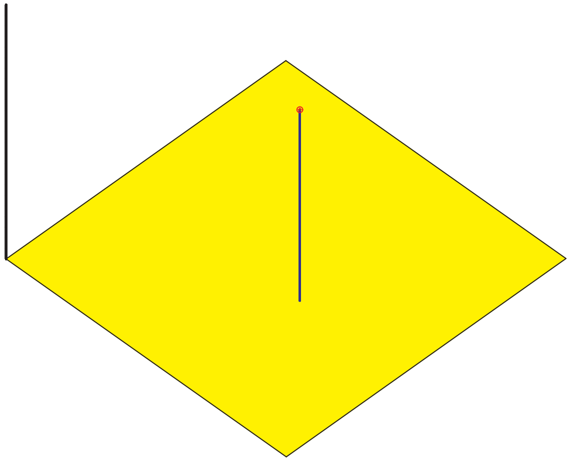
g
a_1
a_1 + t

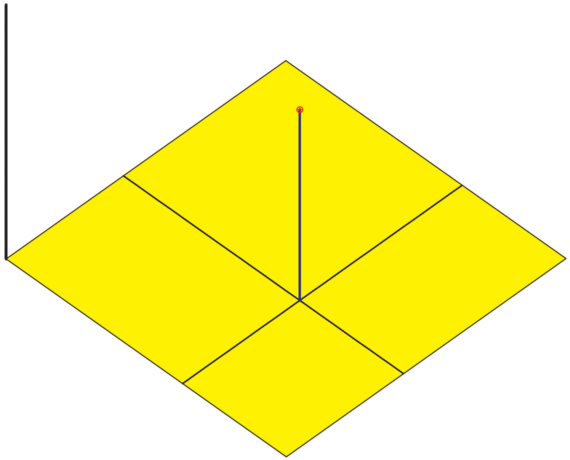
Theorem 5.10 (necessary condition of existence of local extremum)

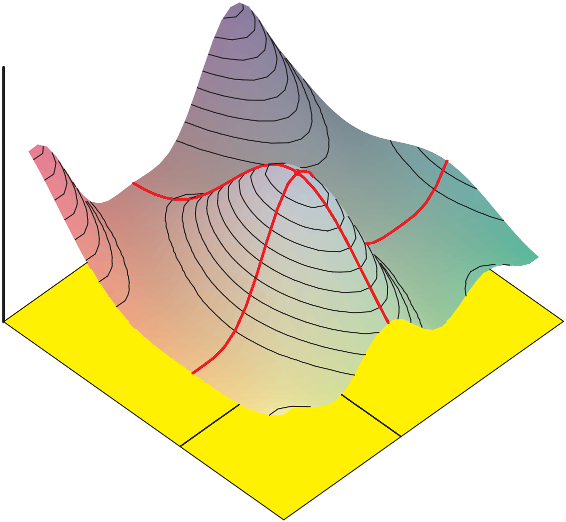
Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, and a function $f\colon G \to \mathbf{R}$ have at the point $\boldsymbol{a}$ local extremum. Then for each $j \in \{1, \dots, n\}$ we have:
The partial derivative $\frac{\partial f}{\partial x_j}(\boldsymbol{a})$ either does not exit or is zero.

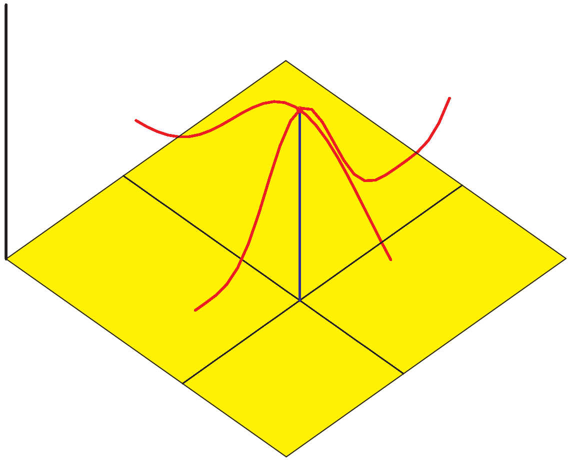


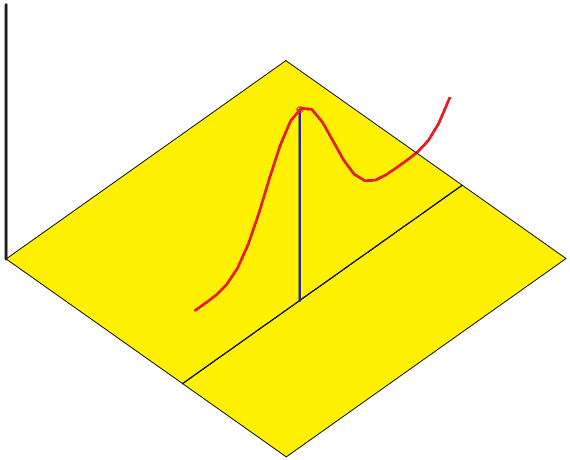


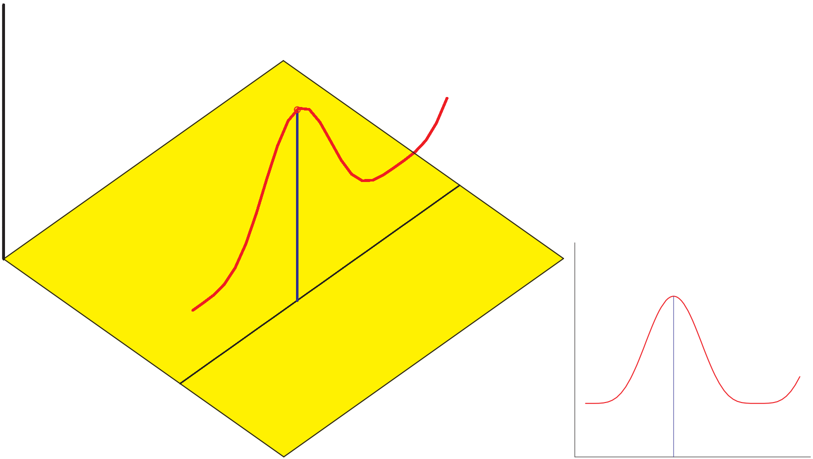












Definition

Let $G \subset \mathbf{R}^n$ be a nonempty open set. Let a function $f\colon G \to \mathbf{R}$ have at each point of the set G all partial derivatives continuous (i.e., function $\boldsymbol{x} \mapsto \frac{\partial f}{\partial x_j}(\boldsymbol{x})$ are continuous on G for each $j \in \{1, \ldots, n\}$). Then we say that f is of the class $\mathcal{C}^1$ on G. The set of all these functions is denoted by $\mathcal{C}^1(G)$.

Definition

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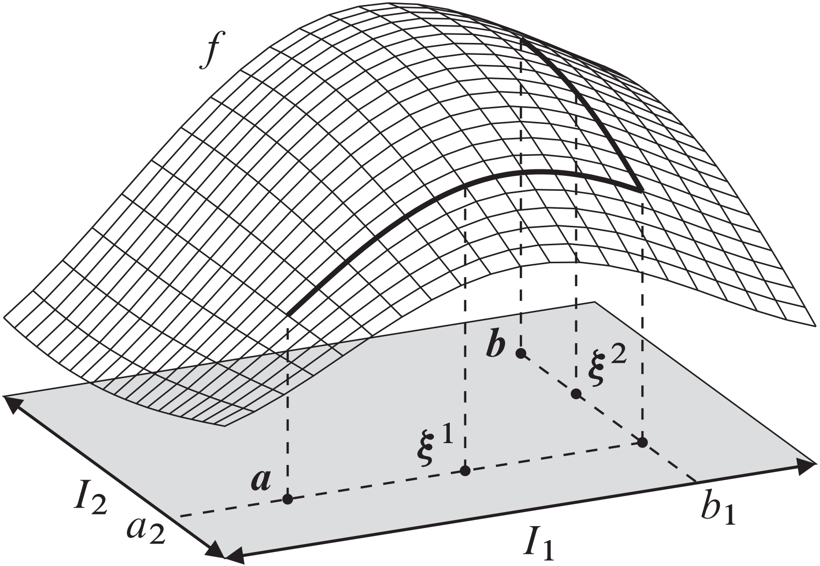
Remark

If $G \subset \mathbf{R}^n$ is a nonempty open set and and $f, g \in \mathcal{C}^1(G)$, then $f + g \in \mathcal{C}^1(G)$, $f - g \in \mathcal{C}^1(G)$, and $fg \in \mathcal{C}^1(G)$. If moreover for each $\boldsymbol{x} \in G$ we have : $g(\boldsymbol{x}) \neq 0$, then $f/g \in \mathcal{C}^1(G)$.

Proposition 5.11 (Lagrange)

Let $n \in \mathbf{N}$, $I_1, \dots, I_n \subset \mathbf{R}$ be open intervals, $I = I_1 \times I_2 \times \cdots \times I_n$, $f \in \mathcal{C}^1(I)$, $\boldsymbol{a}, \boldsymbol{b} \in I$. Then there exist points $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^n \in I$ with $\xi_j^i \in \langle a_j, b_j \rangle$ for each $i, j \in \{1, \dots, n\}$, such that

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{\xi}^i)(b_i - a_i).$$



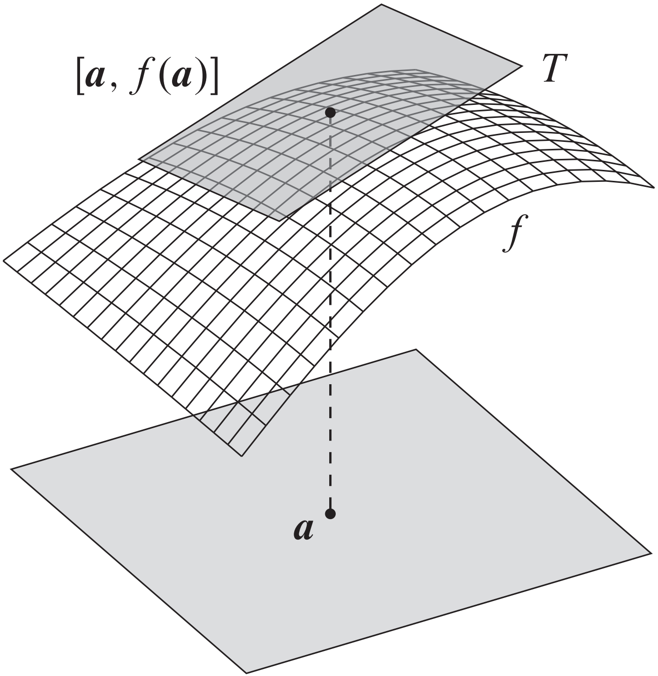
f
b
ξ2
ξ1
a
I2
a2
I1
b1

Definition

Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in \mathcal{C}^1(G)$. Then the graph of the function

$$T\colon \boldsymbol{x} \mapsto f(\boldsymbol{a}) + \frac{\partial f}{\partial x_1}(\boldsymbol{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\boldsymbol{a})(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(\boldsymbol{a})(x_n - a_n), \quad \boldsymbol{x} \in \mathbf{R}^n,$$

is called tangent hyperplane to the graph of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$.



$[\boldsymbol{a}, f(\boldsymbol{a})]$
T
f
$\boldsymbol{a}$

Theorem 5.12

Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in \mathcal{C}^1(G)$, and T be a function, such that its graph is the tangent hyperplane of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$. Then

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 5.12

Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in \mathcal{C}^1(G)$, and T be a function, such that its graph is the tangent hyperplane of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 5.13

Let $G \subset \mathbf{R}^n$ be an open nonempty set and $f \in \mathcal{C}^1(G)$. Then f is continuous on G.

Theorem 5.14

Let $r, s \in \mathbf{N}$, $G \subset \mathbf{R}^s$, $H \subset \mathbf{R}^r$ be open sets. Let $\varphi_1, \dots, \varphi_r \in \mathcal{C}^1(G)$, $f \in \mathcal{C}^1(H)$ and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in H$ for each $\boldsymbol{x} \in G$. Then the composed function $F\colon G \to \mathbf{R}$ defined by

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in G,$$

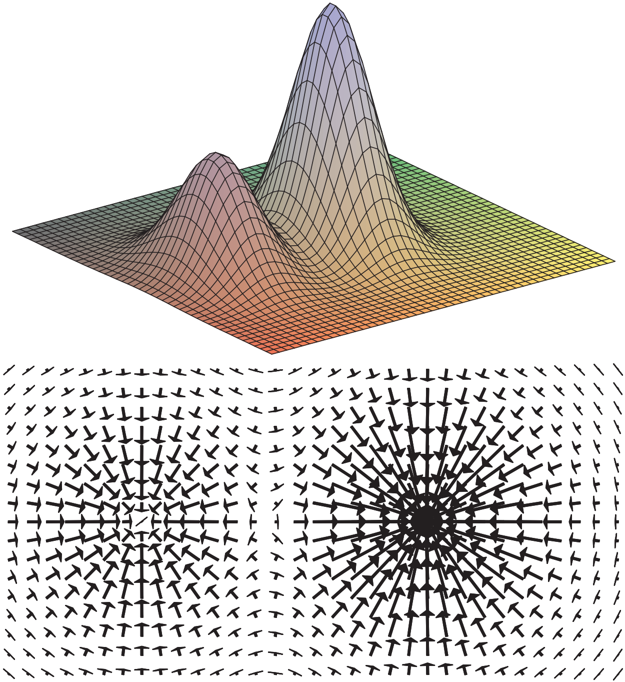
is of the class $\mathcal{C}^1$ on G. Let $\boldsymbol{a} \in G$ and $\boldsymbol{b} = [\varphi_1(\boldsymbol{a}), \dots, \varphi_r(\boldsymbol{a})]$. Then for each $j \in \{1, \dots, s\}$ we have

$$\frac{\partial F}{\partial x_j}(\boldsymbol{a}) = \sum_{i=1}^{r} \frac{\partial f}{\partial y_i}(\boldsymbol{b}) \frac{\partial \varphi_i}{\partial x_j}(\boldsymbol{a}).$$

Definition

Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in \mathcal{C}^1(G)$.
Gradient of f at the point $\boldsymbol{a}$ is defined as the vector

$$\nabla f(\boldsymbol{a}) = \left[\frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \ldots, \frac{\partial f}{\partial x_n}(\boldsymbol{a}) \right].$$



Definition

Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in \mathcal{C}^1(G)$, and $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$. Then the point $\boldsymbol{a}$ is called stationary (or also critical) point of the function f.

Definition

Let $G \subset \mathbf{R}^n$ be an open set, $f\colon G \to \mathbf{R}$, $i, j \in \{1, \dots, n\}$, and $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ exists for each $\boldsymbol{x} \in G$. Then partial derivative of the second order of the function f according to i-th and j-th variable at the point $\boldsymbol{a} \in G$ is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\boldsymbol{a}).$$

If $i = j$ then we use the notation

$$\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a}).$$

Definition

Let $G \subset \mathbf{R}^n$ be an open set, $f\colon G \to \mathbf{R}$, $i, j \in \{1, \ldots, n\}$, and $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ exists for each $\boldsymbol{x} \in G$. Then partial derivative of the second order of the function f according to i-th and j-th variable at the point $\boldsymbol{a} \in G$ is defined by

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If $i = j$ then we use the notation

$$\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a}).$$

Theorem 5.15

Let $i, j \in \{1, \ldots, n\}$ and let both partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ be continuous at a point $\boldsymbol{a} \in \mathbf{R}^n$. Then we have

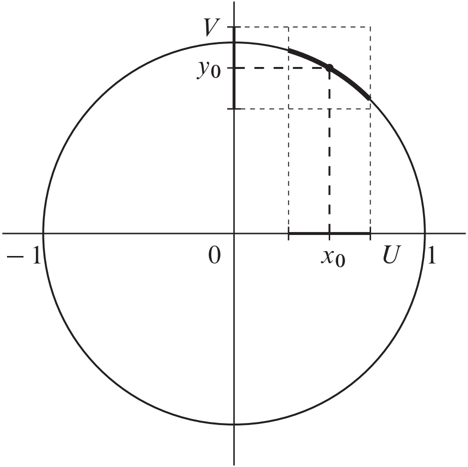
$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a}).$$

Definition

Let $G \subset \mathbf{R}^n$ be an open set and $k \in \mathbf{N}$. We say that a function f is of the class $\mathcal{C}^k$ on G, if all partial derivatives of f till k-th order are continuous on G. The set of all these functions is denoted by $\mathcal{C}^k(G)$.

Definition

Let $G \subset \mathbf{R}^n$ be an open set and $k \in \mathbf{N}$. We say that a function f is of the class $\mathcal{C}^k$ on G, if all partial derivatives of f till k-th order are continuous on G. The set of all these functions is denoted by $\mathcal{C}^k(G)$. We say that a function f is of the class $\mathcal{C}^\infty$ on G, if all partial derivatives of all orders of f are continuous on G. The set of all functions of the class $\mathcal{C}^\infty$ on G is denoted by $\mathcal{C}^\infty(G)$.



V
y_0
-1
0
x_0
U
1

Theorem 5.16 (implicit function theorem)

Let $G \subset \mathbf{R}^{n+1}$ be an open set, $F\colon G \to \mathbf{R}$, $\tilde{\boldsymbol{x}} \in \mathbf{R}^n$, $\tilde{y} \in \mathbf{R}$, $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

Theorem 5.16 (implicit function theorem)

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1. $F \in \mathcal{C}^1(G)$,
2. $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,
3. $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

Then there exist a neighborhood $U \subset \mathbf{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighborhood $V \subset \mathbf{R}$ of the point $\tilde{y}$ such that for each $\boldsymbol{x} \in U$ there exists unique $y \in V$ with the property $F(\boldsymbol{x}, y) = 0$.

Theorem 5.16 (implicit function theorem)

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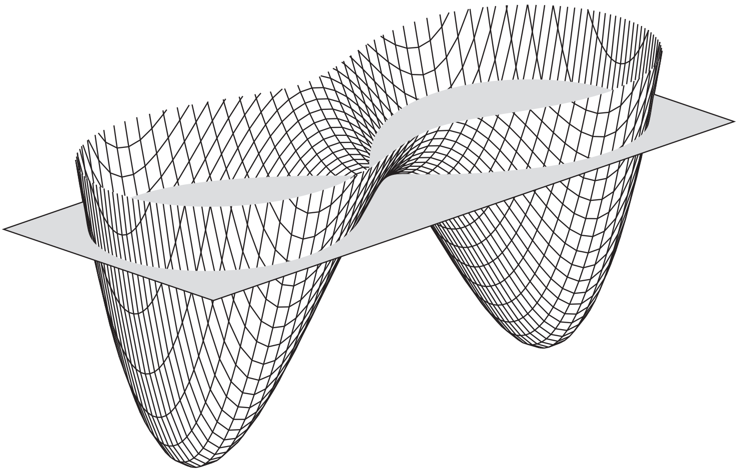
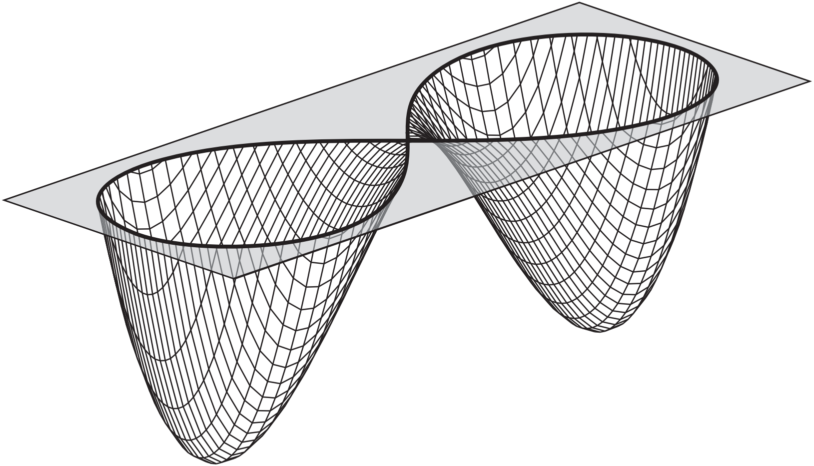
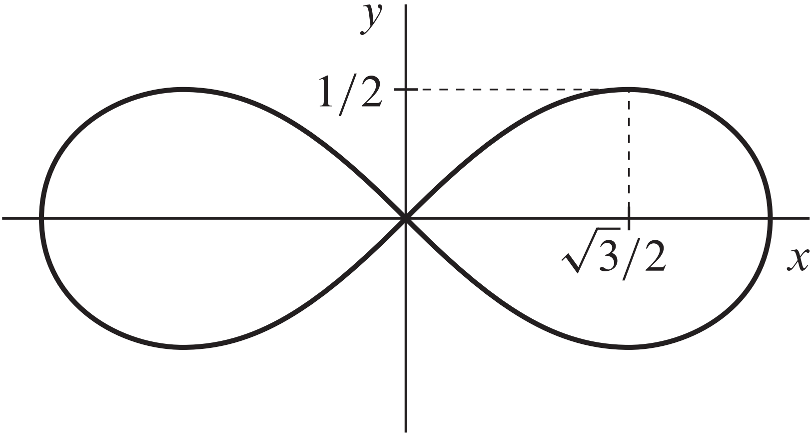
1. $F \in \mathcal{C}^1(G)$,
2. $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,
3. $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

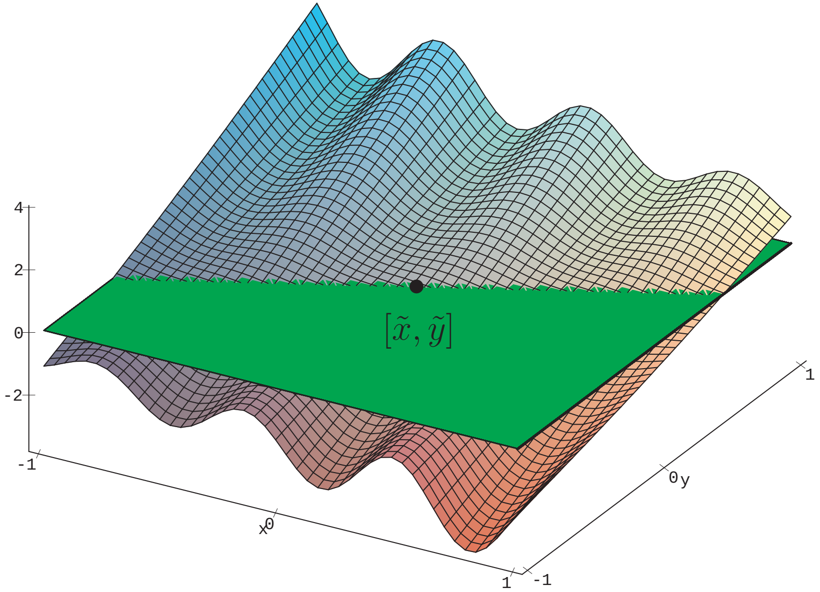
Then there exist a neighborhood $U \subset \mathbf{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighborhood $V \subset \mathbf{R}$ of the point $\tilde{y}$ such that for each $\boldsymbol{x} \in U$ there exists unique $y \in V$ with the property $F(\boldsymbol{x}, y) = 0$. If we denote this y by $\varphi(\boldsymbol{x})$, then the resulting function φ is in $\mathcal{C}^1(U)$ and

$$\frac{\partial \varphi}{\partial x_j}(\boldsymbol{x}) = -\frac{\frac{\partial F}{\partial x_j}(\boldsymbol{x}, \varphi(\boldsymbol{x}))}{\frac{\partial F}{\partial y}(\boldsymbol{x}, \varphi(\boldsymbol{x}))} \quad \text{for } \boldsymbol{x} \in U,\ j \in \{1, \dots, n\}.$$

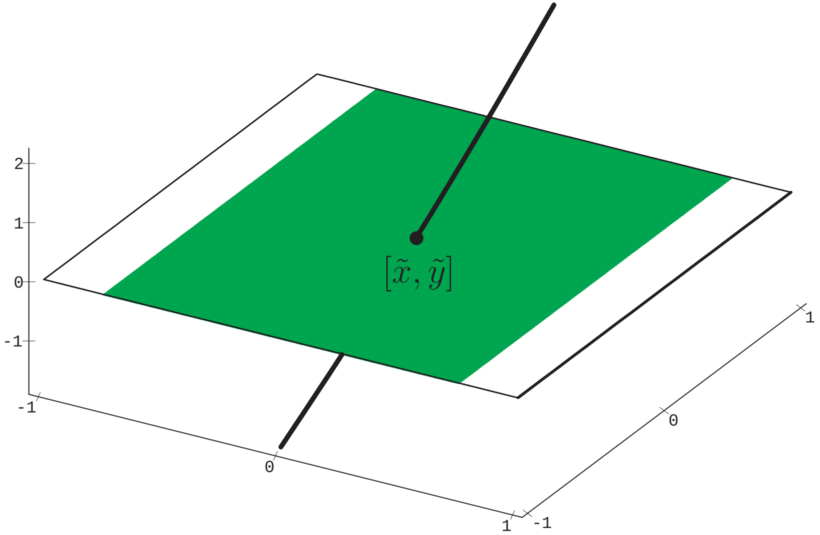


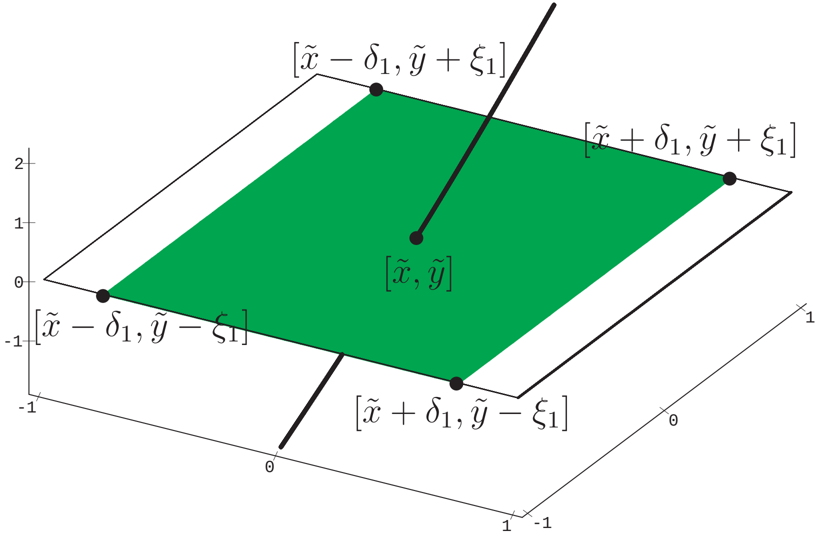



y
1/2
$\sqrt{3}/2$
x

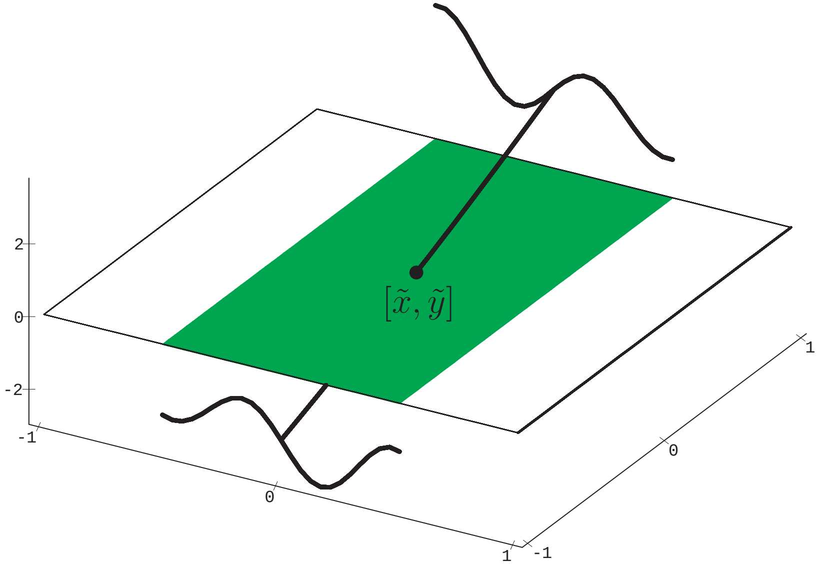


4
2
0
-2
-1
x 0
1
-1
0 y
1
$[\tilde{x}, \tilde{y}]$


2
1
0
-1
-1
0
1
-1
0
1
$[\tilde{x}, \tilde{y}]$



$[\tilde{x} - \delta_1, \tilde{y} + \xi_1]$
$[\tilde{x} + \delta_1, \tilde{y} + \xi_1]$
$[\tilde{x}, \tilde{y}]$
$[\tilde{x} - \delta_1, \tilde{y} - \xi_1]$
$[\tilde{x} + \delta_1, \tilde{y} - \xi_1]$
2
1
0
-1
-1
0
1
-1
0
1


2
0
-2
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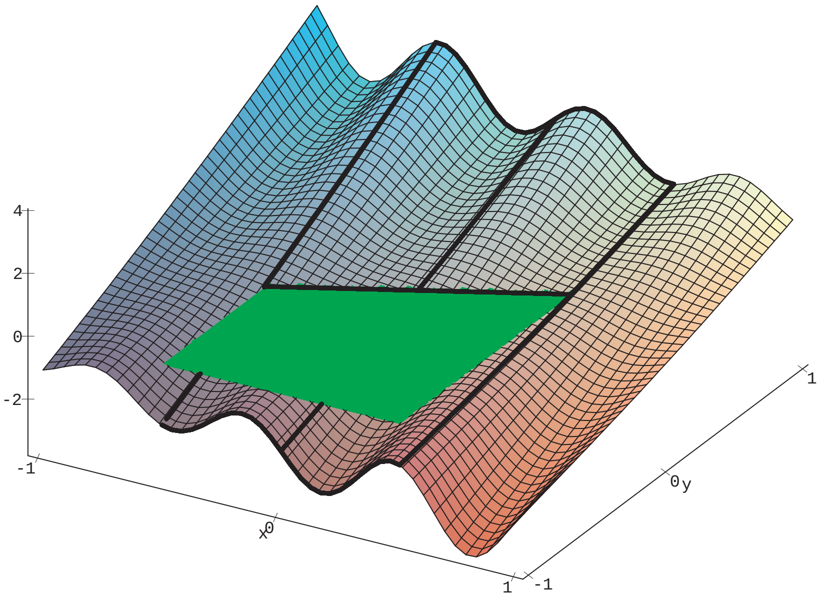
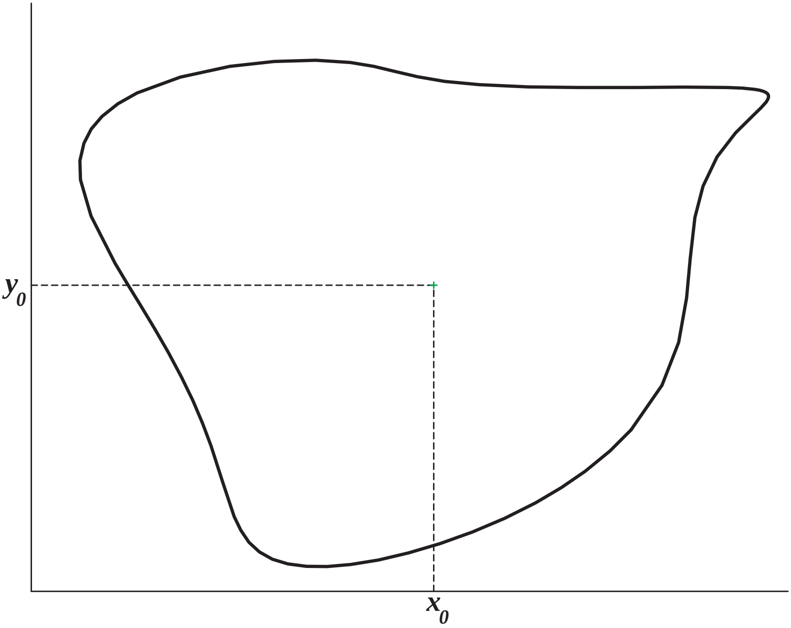
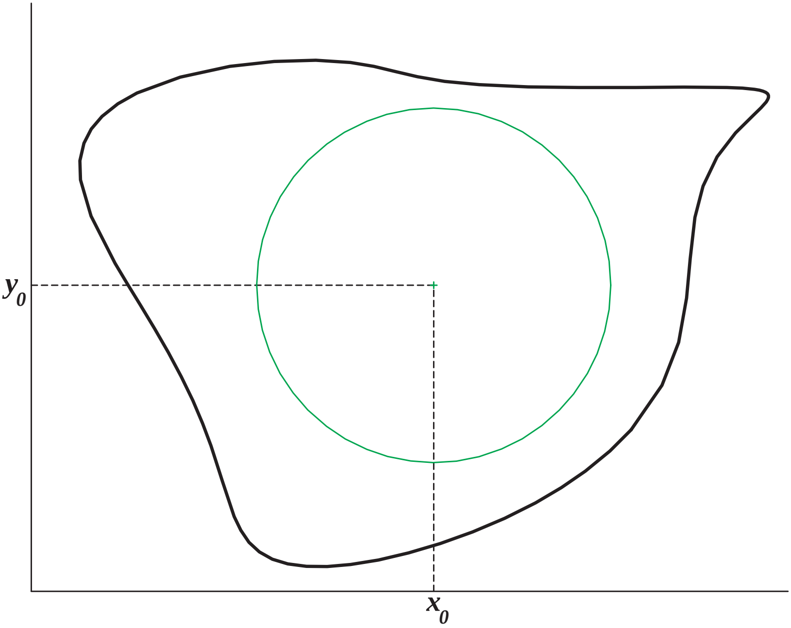
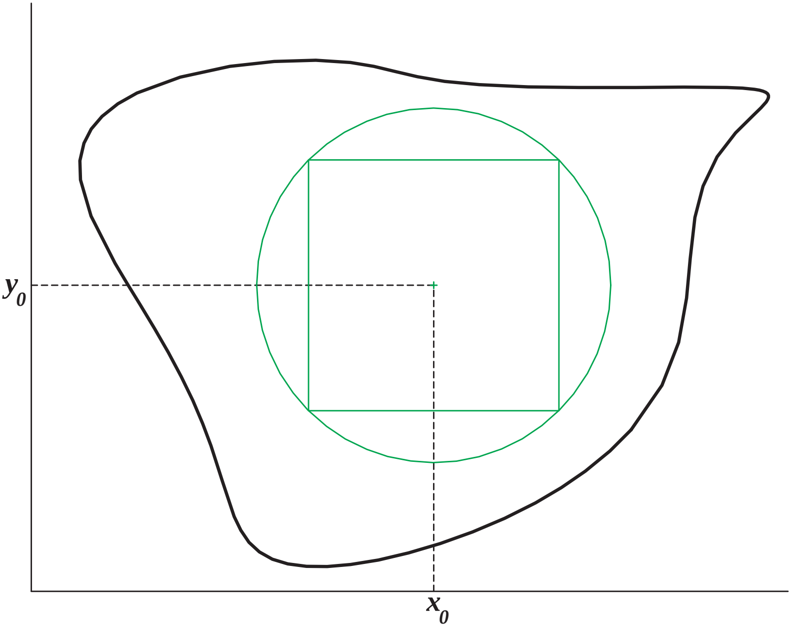
1
-1
0
1
$[\tilde{x}, \tilde{y}]$



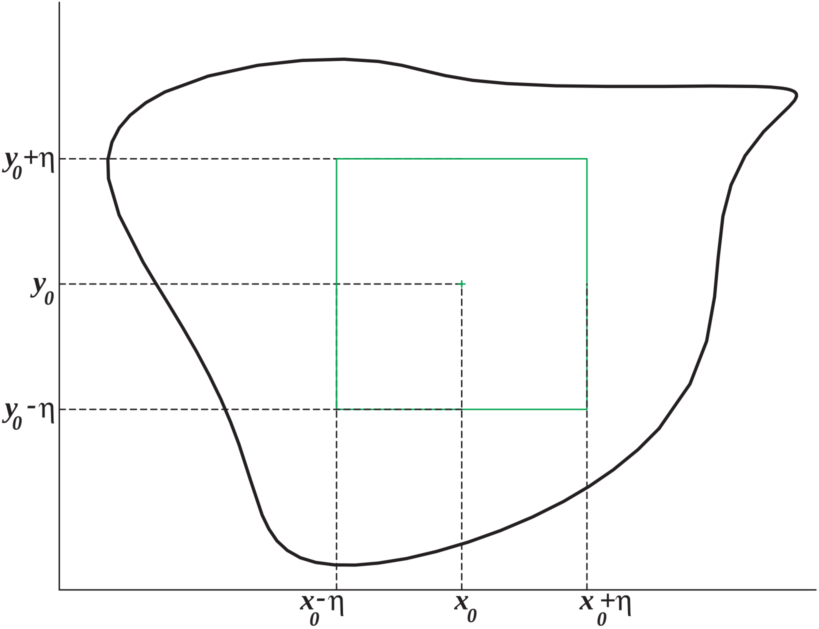
4
2
0
-2
-1
x0
1
-1
0 y
1

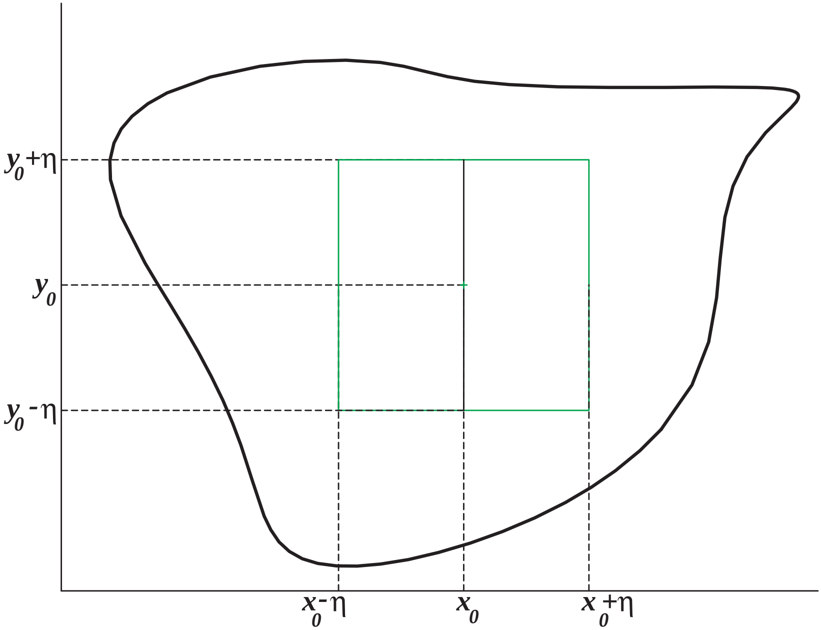

y_0
x_0


y_0
x_0

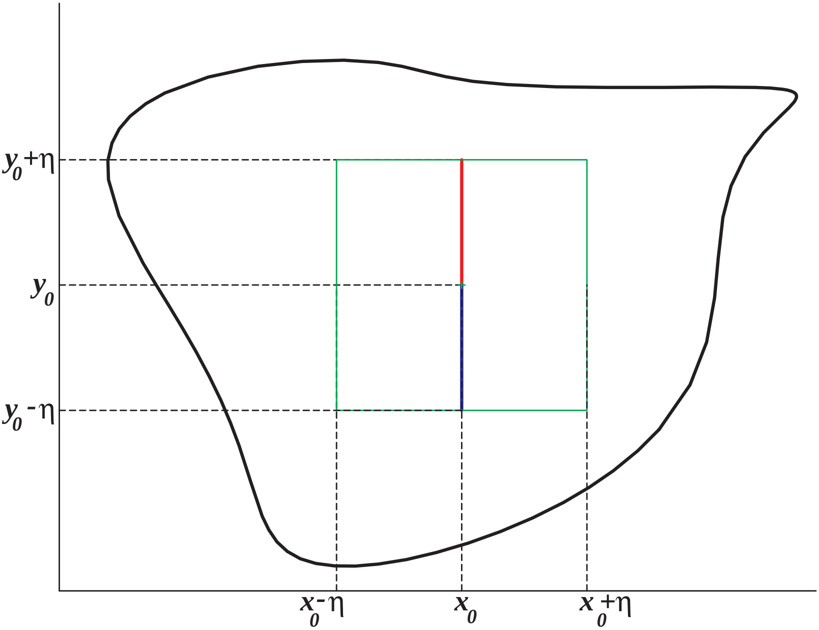


y_0
x_0

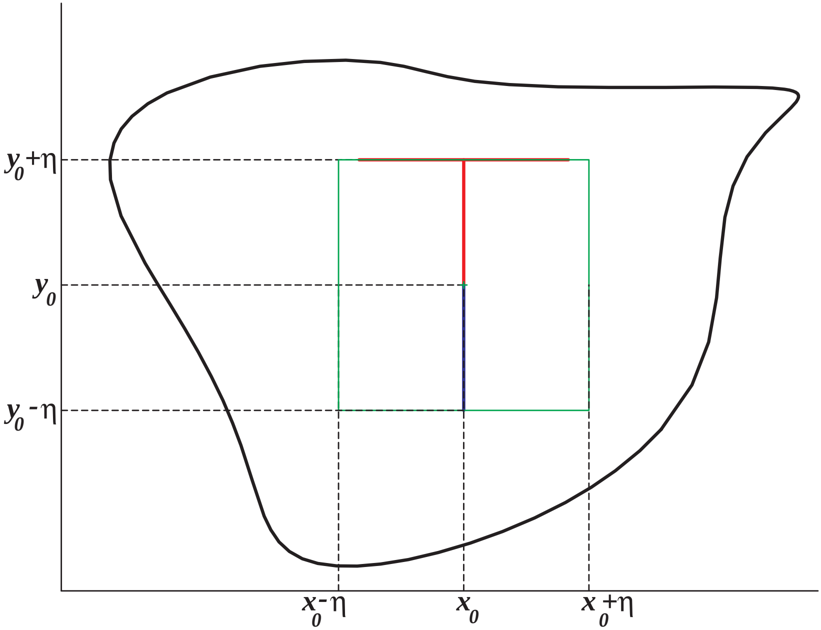

$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\eta$
x_0
$x_0+\eta$

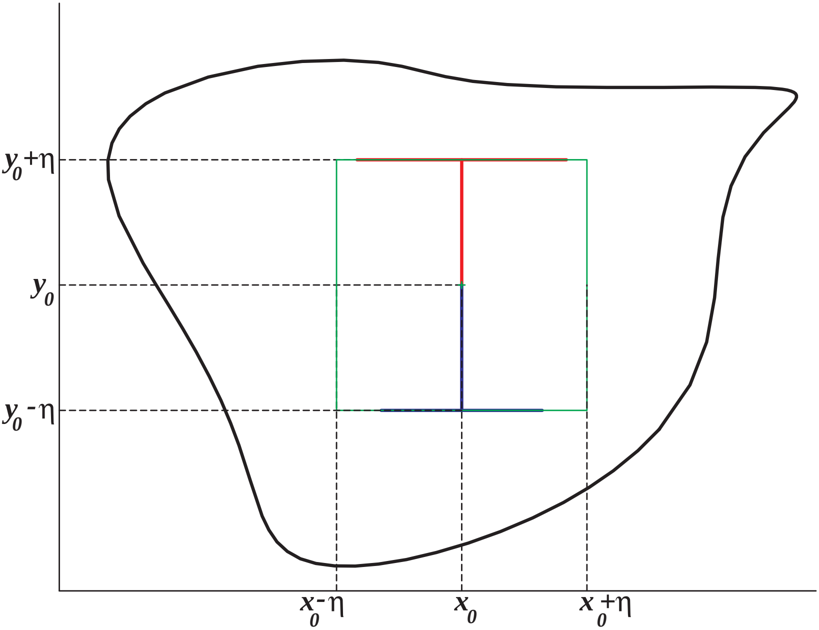


$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\eta$
x_0
$x_0+\eta$

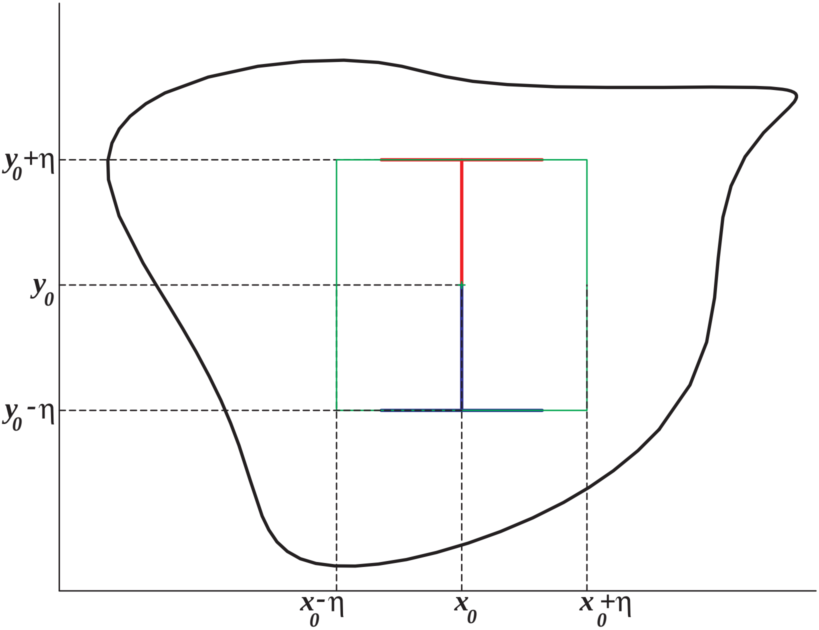


$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\eta$
x_0
$x_0+\eta$

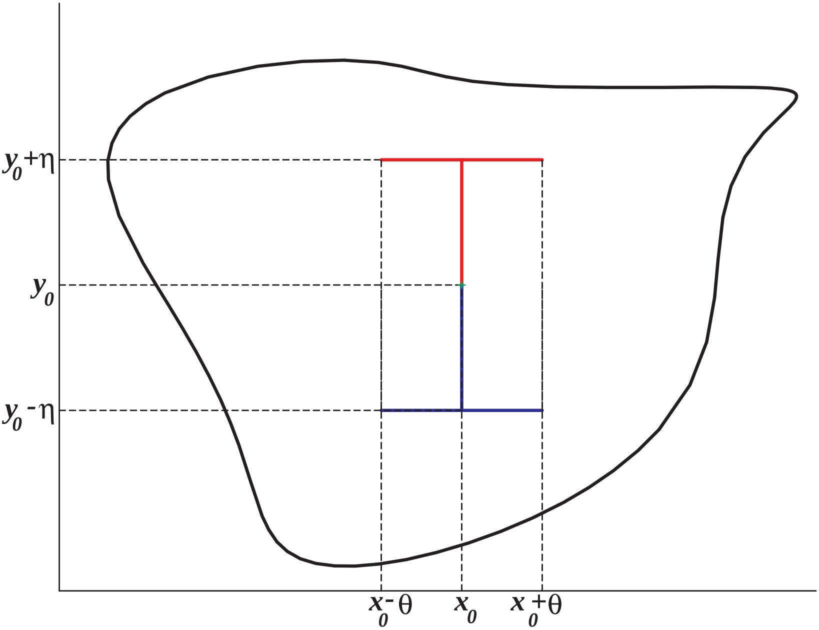

$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\eta$
x_0
$x_0+\eta$

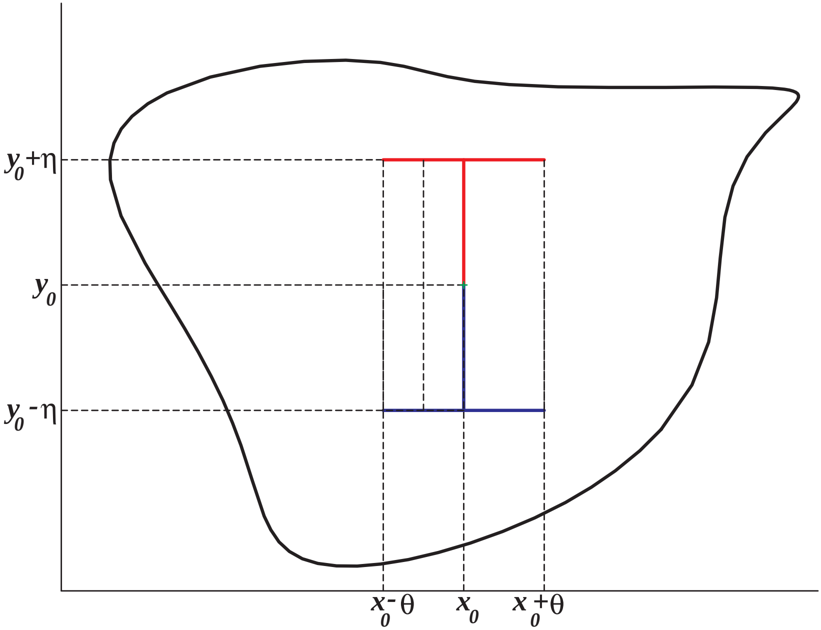


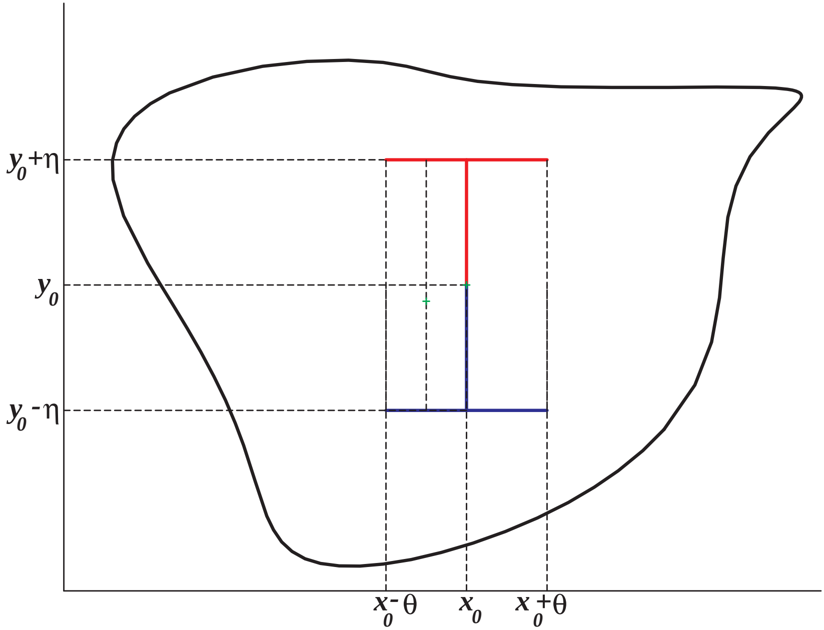
$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\eta$
x_0
$x_0+\eta$



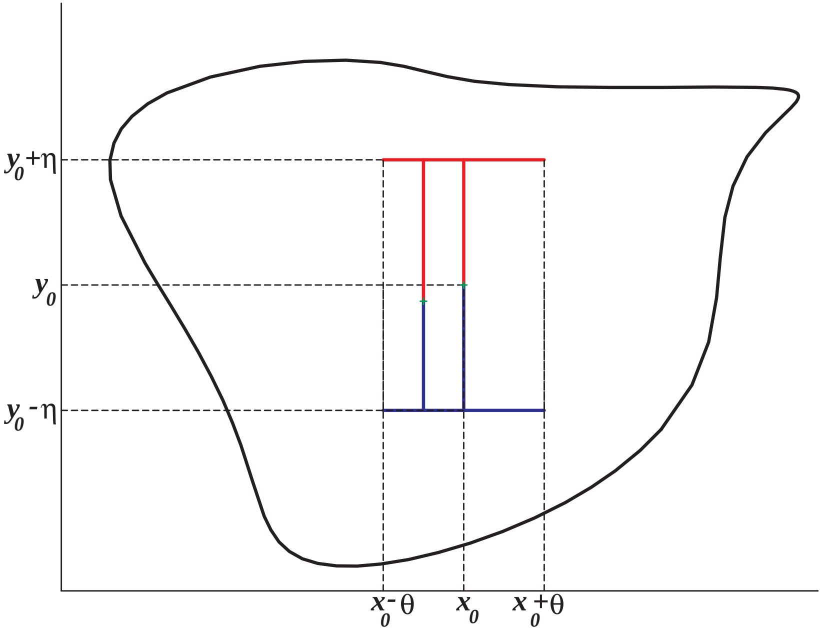
$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\eta$
x_0
$x_0+\eta$

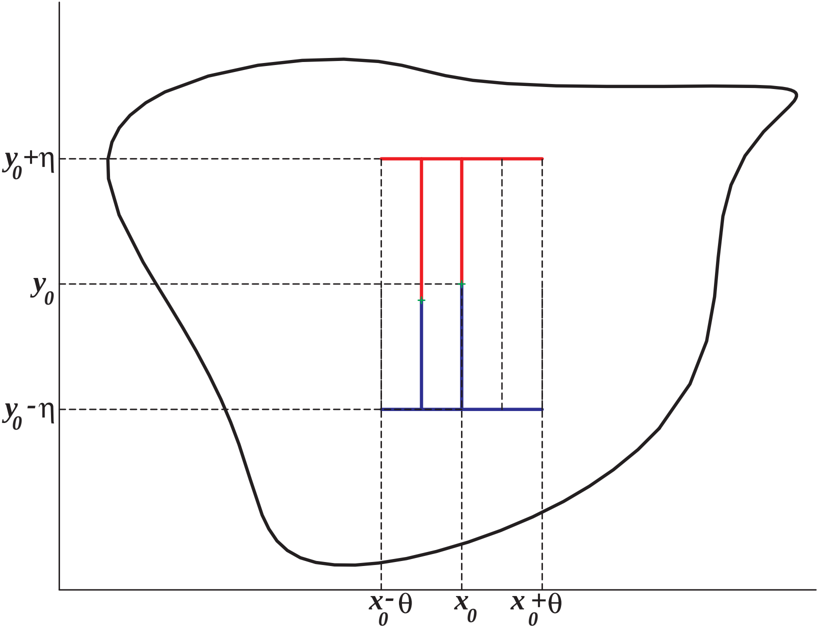

$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$


$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$



$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$


$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$



$y_0+\eta$
y_0
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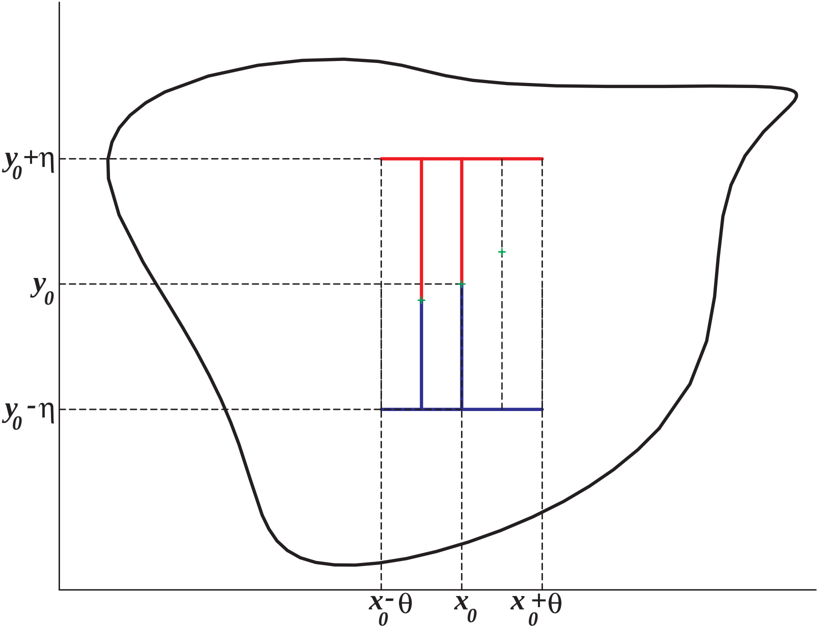
x_0
$x_0+\theta$


$y_0+\eta$
y_0
$y_0-\eta$
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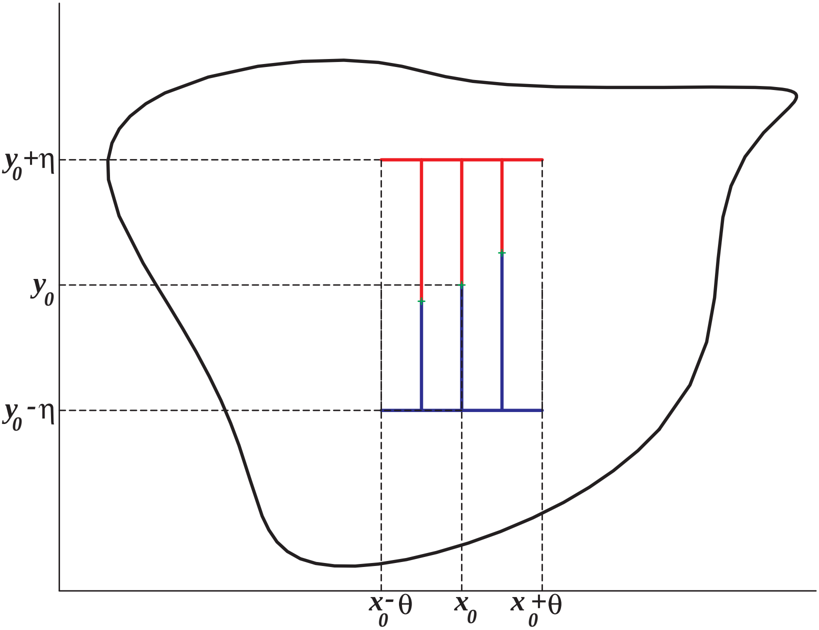
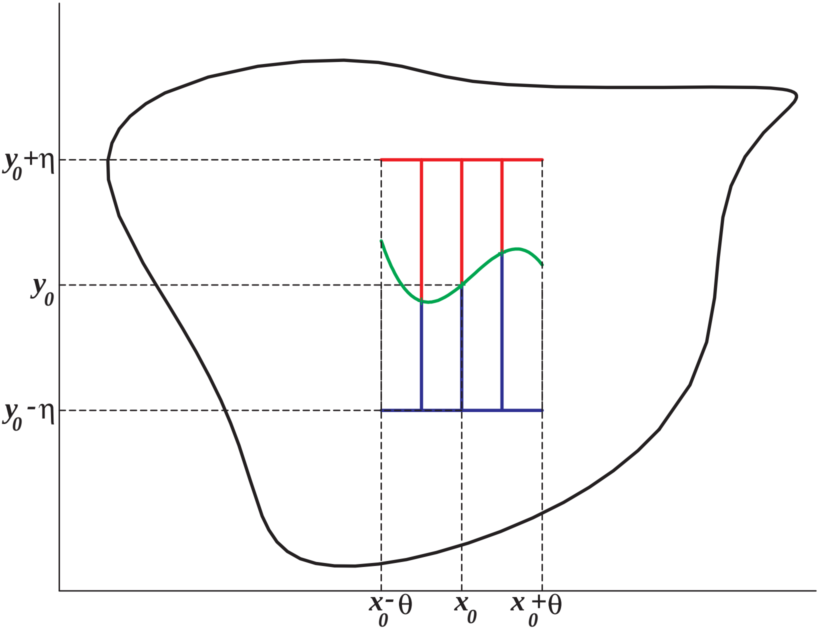
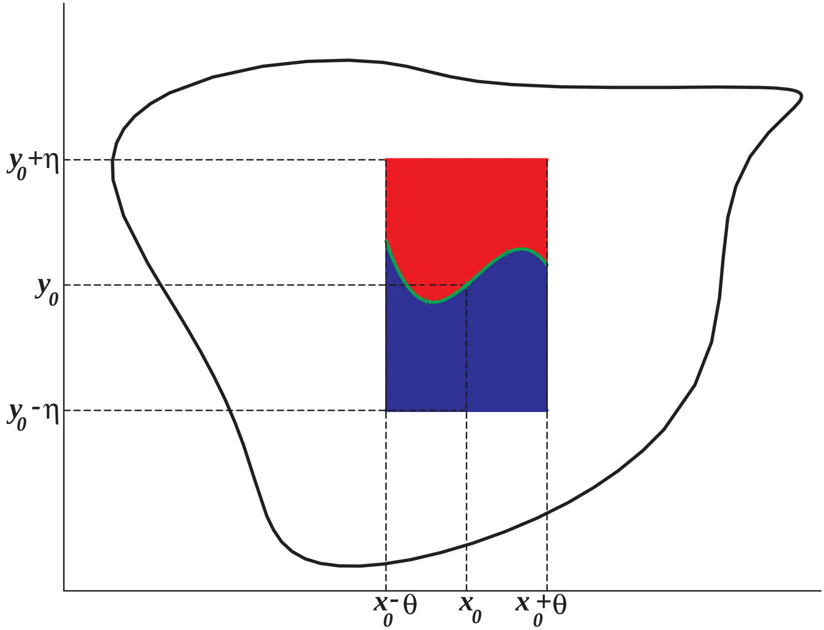
x_0
$x_0+\theta$



$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$



$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$



$y_0+\eta$
y_0
$y_0-\eta$
$x_0-\theta$
x_0
$x_0+\theta$

Theorem 5.17 (implicit function theorem)

Let $m, n \in \mathbf{N}$, $k \in \mathbf{N} \cup \{\infty\}$, $G \subset \mathbf{R}^{n+m}$ *be an open set,* $F_j\colon G \to \mathbf{R}$ *for* $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbf{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbf{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. *Suppose that*

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1. $F_j \in \mathcal{C}^k(G)$ *for each* $j \in \{1, \dots, m\}$,
2. $F_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = 0$ *for each* $j \in \{1, \dots, m\}$,
3. $\begin{vmatrix} \frac{\partial F_1}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_1}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_m}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \end{vmatrix} \neq 0.$

Then there exist a neighborhood $U \subset \mathbf{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighborhood $V \subset \mathbf{R}^m$ of the point $\tilde{\boldsymbol{y}}$ such that for each $\boldsymbol{x} \in U$ there exists unique $\boldsymbol{y} \in V$ with the property $F_j(\boldsymbol{x}, \boldsymbol{y}) = 0$ for each $j \in \{1, \dots, m\}$.

Theorem 5.17 (implicit function theorem)

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Then there exist a neighborhood $U \subset \mathbf{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighborhood $V \subset \mathbf{R}^m$ of the point $\tilde{\boldsymbol{y}}$ such that for each $\boldsymbol{x} \in U$ there exists unique $\boldsymbol{y} \in V$ with the property $F_j(\boldsymbol{x}, \boldsymbol{y}) = 0$ for each $j \in \{1, \dots, m\}$. If we denote coordinates of this $\boldsymbol{y}$ by $\varphi_j(\boldsymbol{x})$, $j = 1, \dots, m$, then the resulting functions φ_j are in $\mathcal{C}^k(U)$.

Remark

The symbol in the condition (3) of Theorem 5.17 is called determinant. The definition will presented later on.

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For $m = 1$ we have $\begin{vmatrix} a \end{vmatrix} = a$, $a \in \mathbf{R}$.

For $m = 2$ we have $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, $a, b, c, d \in \mathbf{R}$.

Theorem 5.18 (Lagrange multiplier theorem)

Let $G \subset \mathbf{R}^2$ be an open set, $f, g \in \mathcal{C}^1(G)$, $M = \{[x, y] \in G;\ g(x, y) = 0\}$, and $[\tilde{x}, \tilde{y}] \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

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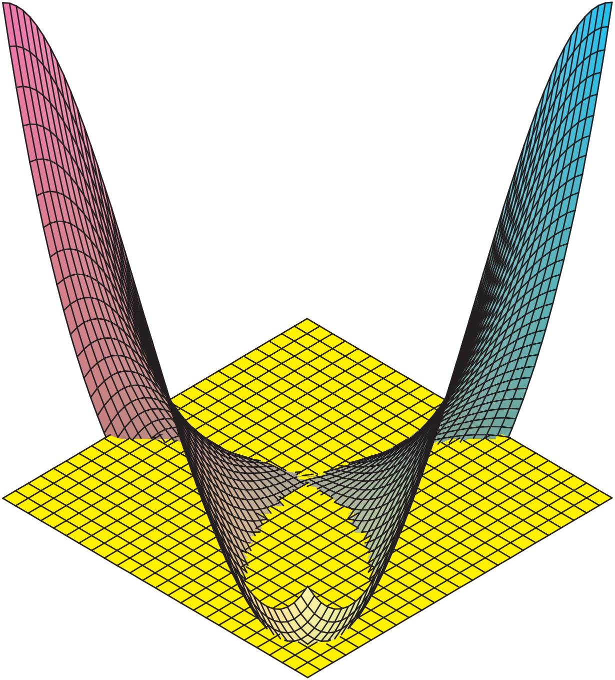
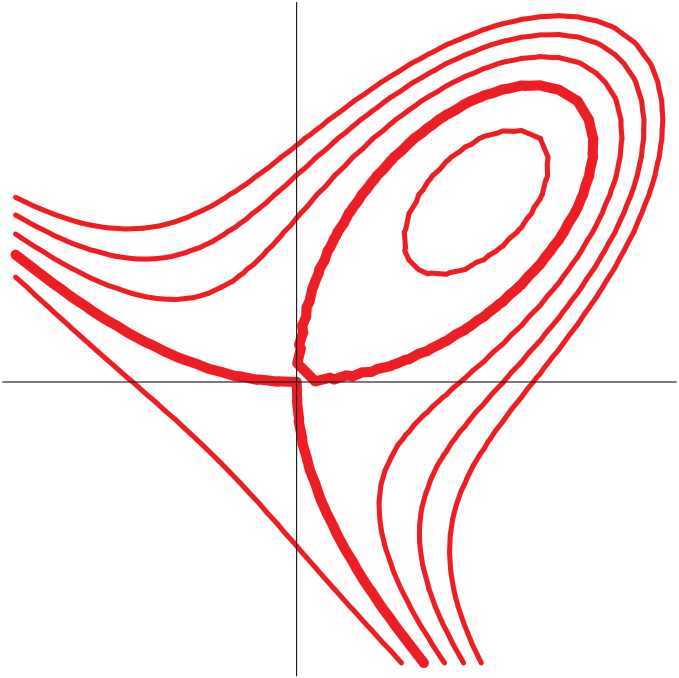
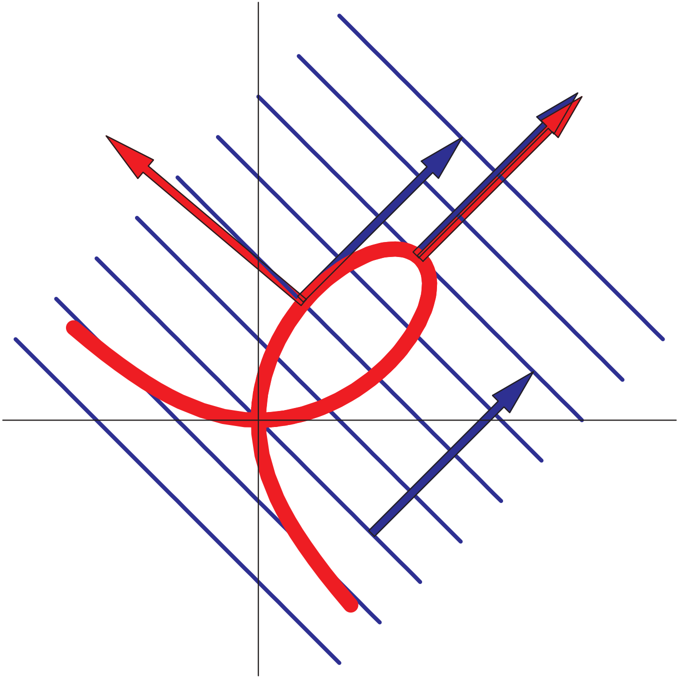
1. $\nabla g(\tilde{x}, \tilde{y}) = \boldsymbol{o}$,
2. *there exists $\lambda \in \mathbf{R}$ satisfying*

$$\frac{\partial f}{\partial x}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial x}(\tilde{x}, \tilde{y}) = 0,$$
$$\frac{\partial f}{\partial y}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial y}(\tilde{x}, \tilde{y}) = 0.$$







Theorem 5.19 (Lagrange multiplier theorem)

Let $m, n \in \mathbf{N}$, $m < n$, $G \subset \mathbf{R}^n$ be an open set, $f, g_1, \dots, g_m \in \mathcal{C}^1(G)$,

$$M = \{\mathbf{z} \in G;\ g_1(\mathbf{z}) = 0, g_2(\mathbf{z}) = 0, \dots, g_m(\mathbf{z}) = 0\}$$

and let $\tilde{\mathbf{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

Theorem 5.19 (Lagrange multiplier theorem)

Let $m, n \in \mathbf{N}$, $m < n$, $G \subset \mathbf{R}^n$ be an open set, $f, g_1, \dots, g_m \in \mathcal{C}^1(G)$,

$$M = \{\boldsymbol{z} \in G;\ g_1(\boldsymbol{z}) = 0, g_2(\boldsymbol{z}) = 0, \dots, g_m(\boldsymbol{z}) = 0\}$$

and let $\tilde{\boldsymbol{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

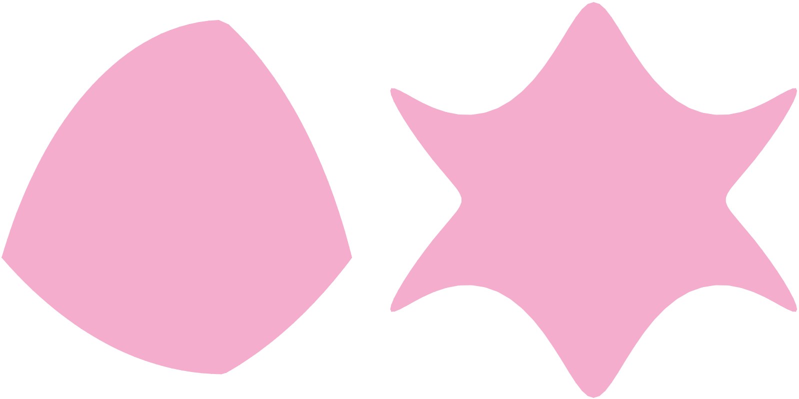
1. *the vectors*

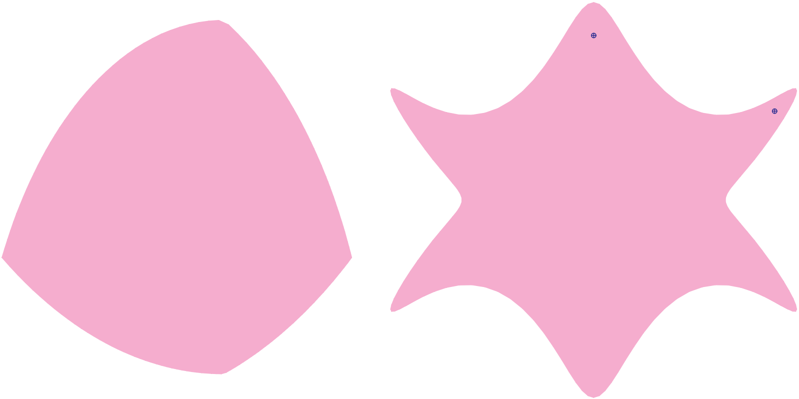
$$\nabla g_1(\tilde{\boldsymbol{z}}), \nabla g_2(\tilde{\boldsymbol{z}}), \dots, \nabla g_m(\tilde{\boldsymbol{z}})$$

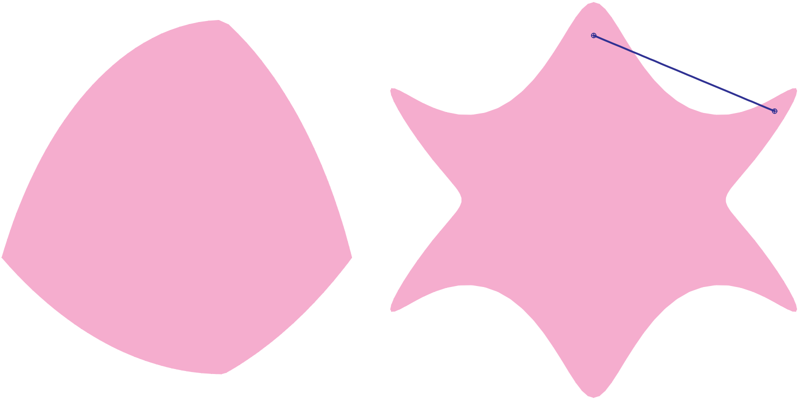
   are linearly dependent,

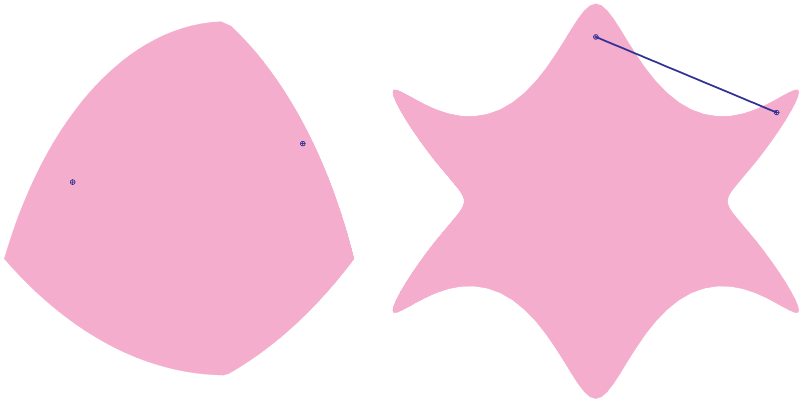
2. *there exist $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbf{R}$ satisfying*

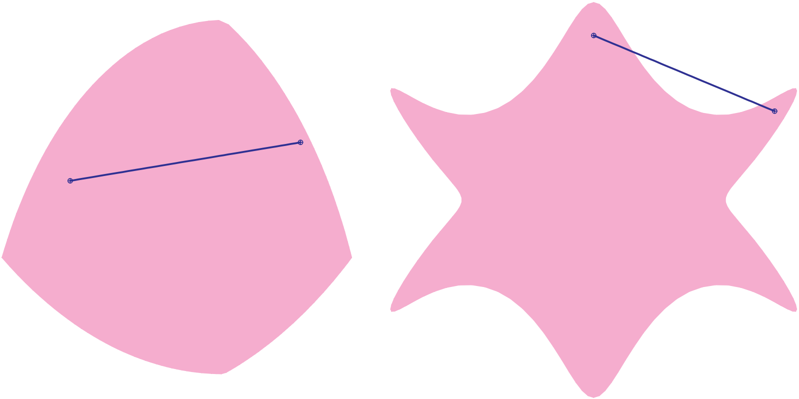
$$\nabla f(\tilde{\boldsymbol{z}}) + \lambda_1 \nabla g_1(\tilde{\boldsymbol{z}}) + \lambda_2 \nabla g_2(\tilde{\boldsymbol{z}}) + \dots + \lambda_m \nabla g_m(\tilde{\boldsymbol{z}}) = \boldsymbol{o}.$$

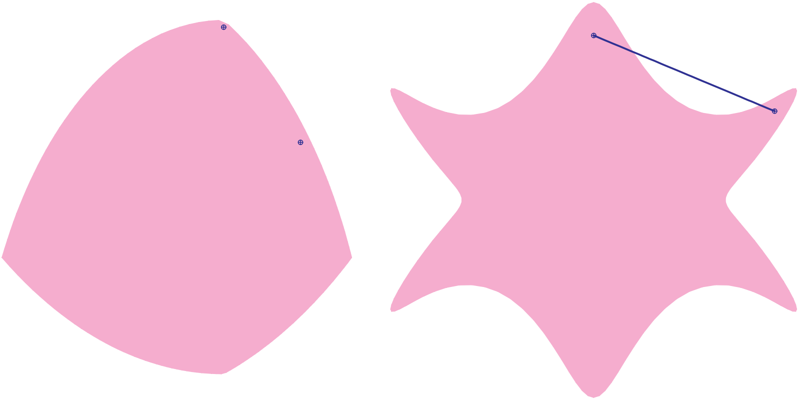


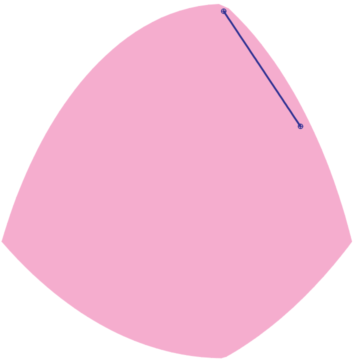


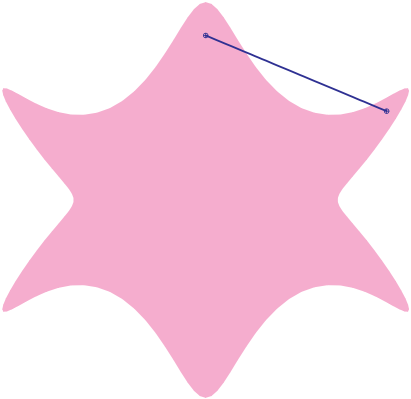


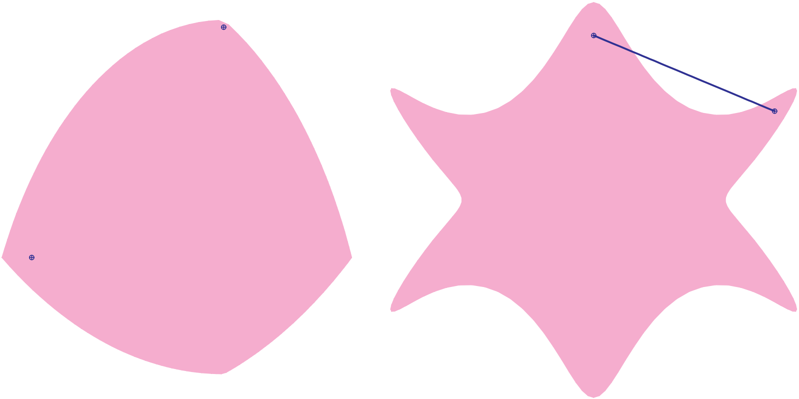


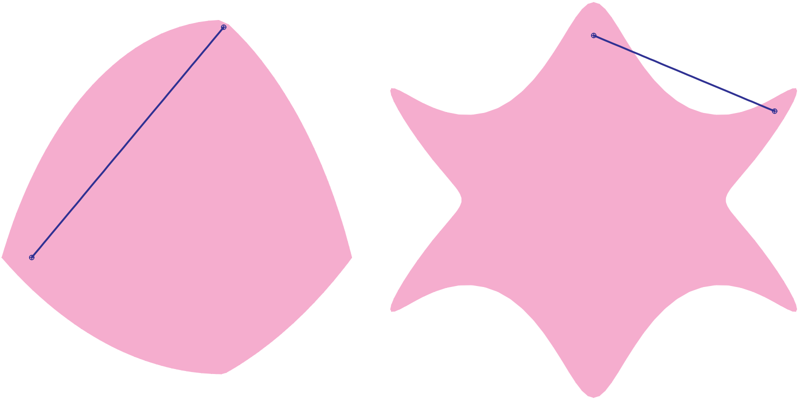




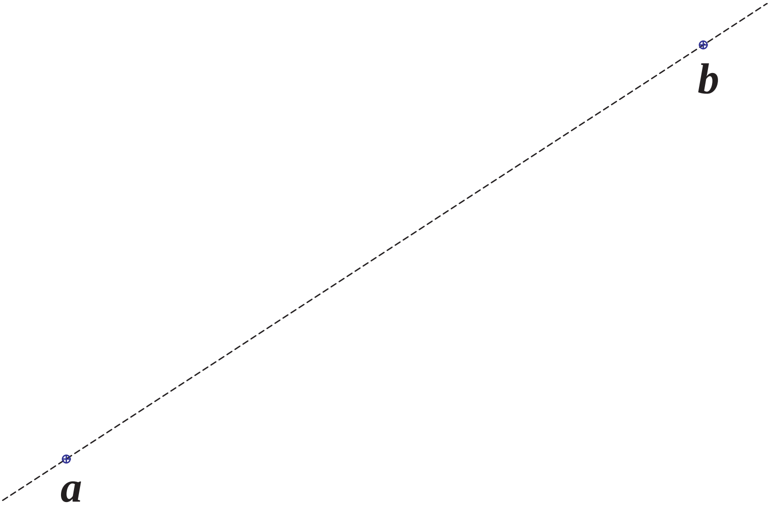


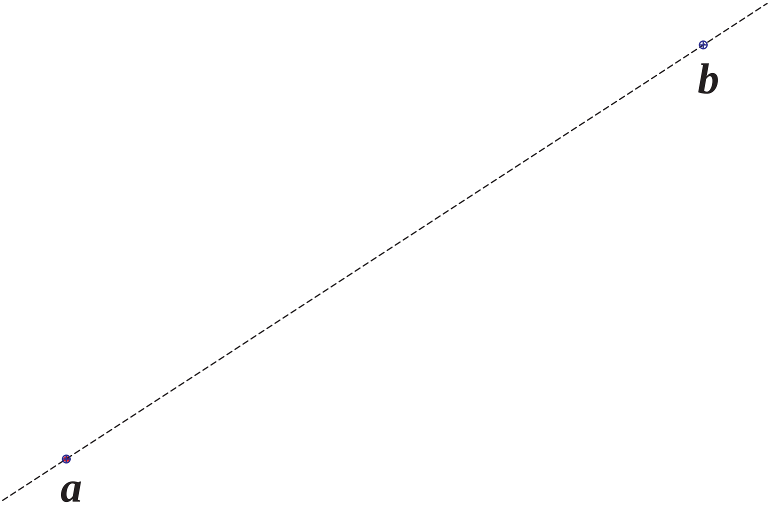




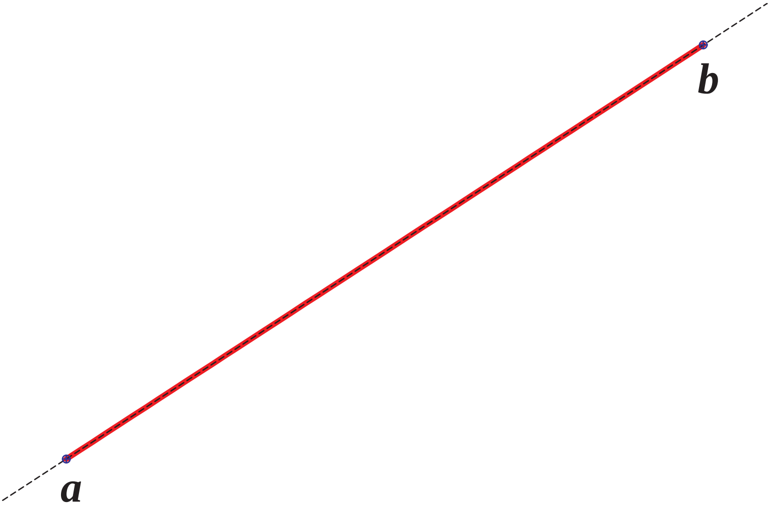



b
a

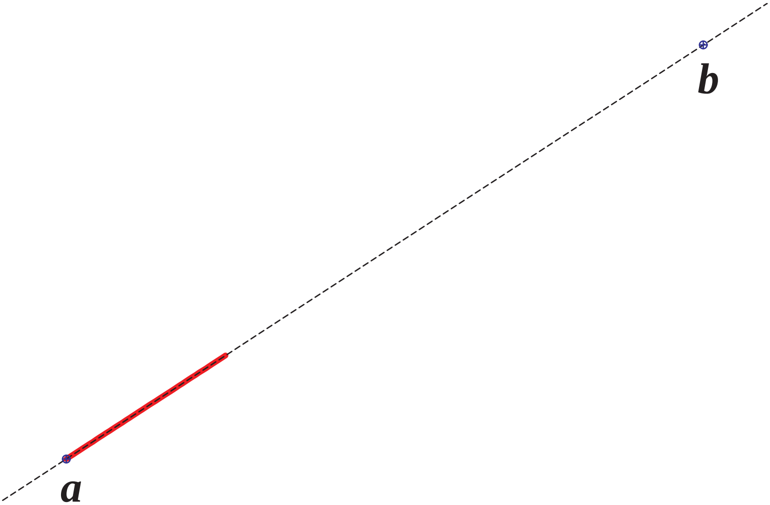

b
a



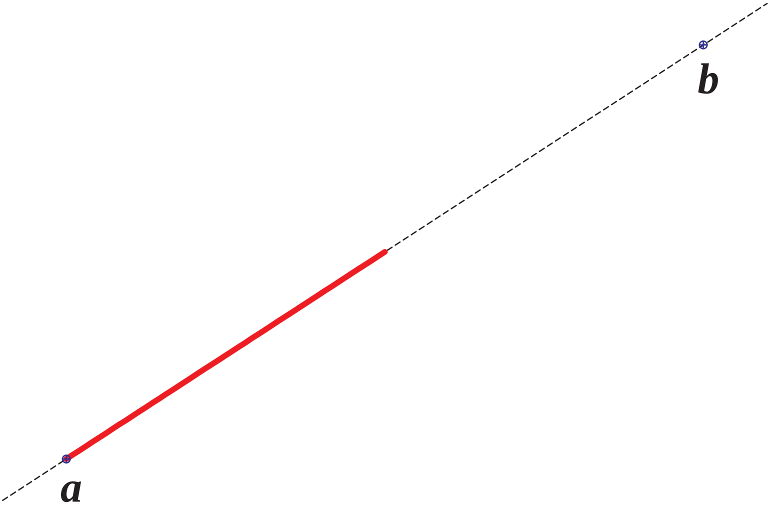
$$\boldsymbol{a} = 1 \cdot \boldsymbol{a} + 0 \cdot \boldsymbol{b} = \boldsymbol{a} + 0 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



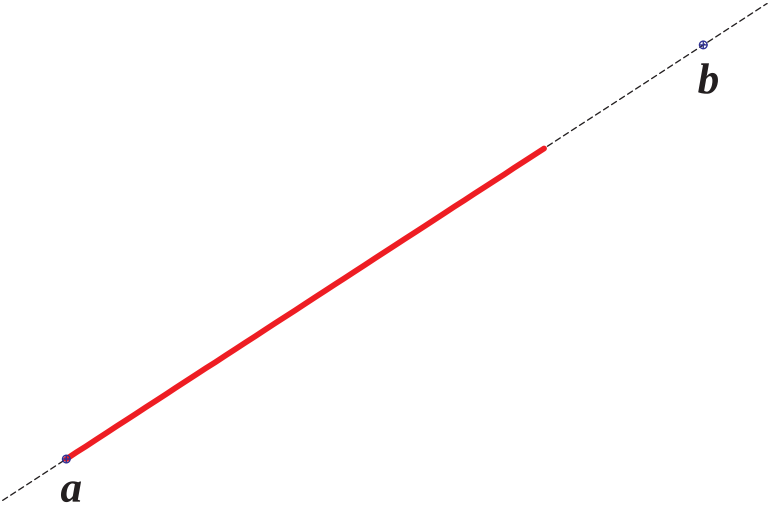
$$\boldsymbol{b} = 0 \cdot \boldsymbol{a} + 1 \cdot \boldsymbol{b} = \boldsymbol{a} + 1 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



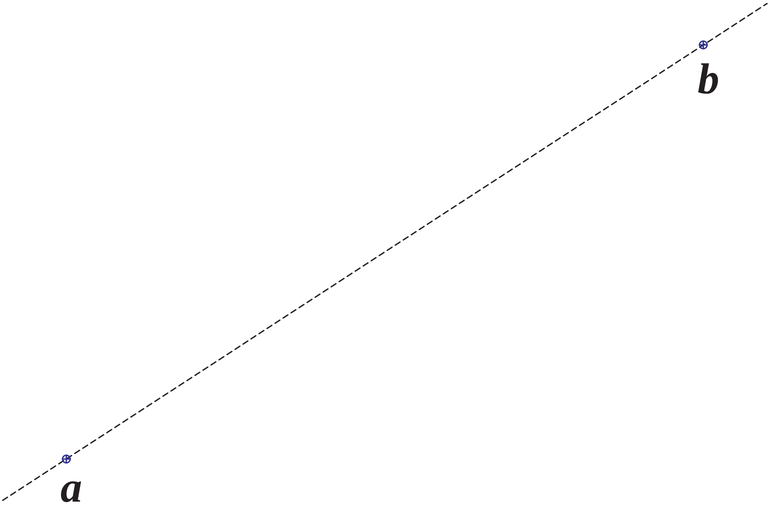
$$\frac{3}{4} \cdot \boldsymbol{a} + \frac{1}{4} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{1}{4} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$\frac{1}{2} \cdot \boldsymbol{a} + \frac{1}{2} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{1}{2} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$\frac{1}{4} \cdot \boldsymbol{a} + \frac{3}{4} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{3}{4} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$t \cdot \boldsymbol{a} + (1 - t) \cdot \boldsymbol{b} = \boldsymbol{a} + (1 - t) \cdot (\boldsymbol{b} - \boldsymbol{a})$$

Definition

Let $M \subset \mathbf{R}^n$. We say that M is convex, if we have

$$\forall \boldsymbol{x}, \boldsymbol{y} \in M \ \forall t \in \langle 0, 1 \rangle : t\boldsymbol{x} + (1 - t)\boldsymbol{y} \in M.$$

Definition

Let $M \subset \mathbf{R}^n$ be a convex set and a function f be defined on M. We say that f is

- concave on M, if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in \langle 0, 1 \rangle \colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}),$$

- strictly concave on M, if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b} \, \forall t \in (0, 1) \colon \\ f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}).$$

Theorem 5.20

Let a function f be concave on an open convex set $G \subset \mathbf{R}^n$. Then f is continuous on G.

Theorem 5.20

Let a function f be concave on an open convex set $G \subset \mathbf{R}^n$. Then f is continuous on G.

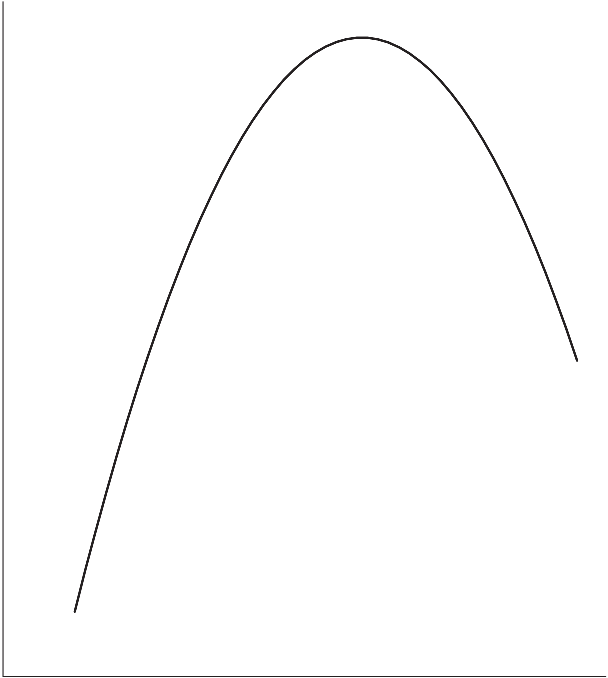
Theorem 5.21

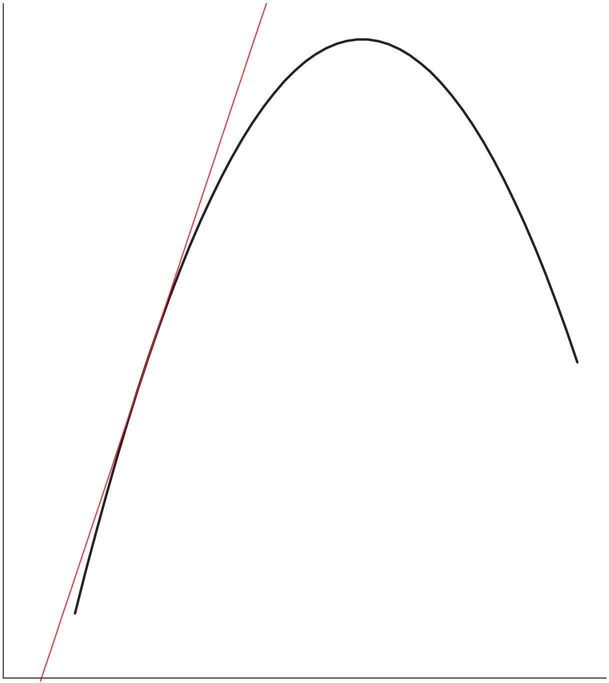
Let a function f be concave on a convex set $M \subset \mathbf{R}^n$. Then for each $\alpha \in \mathbf{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.

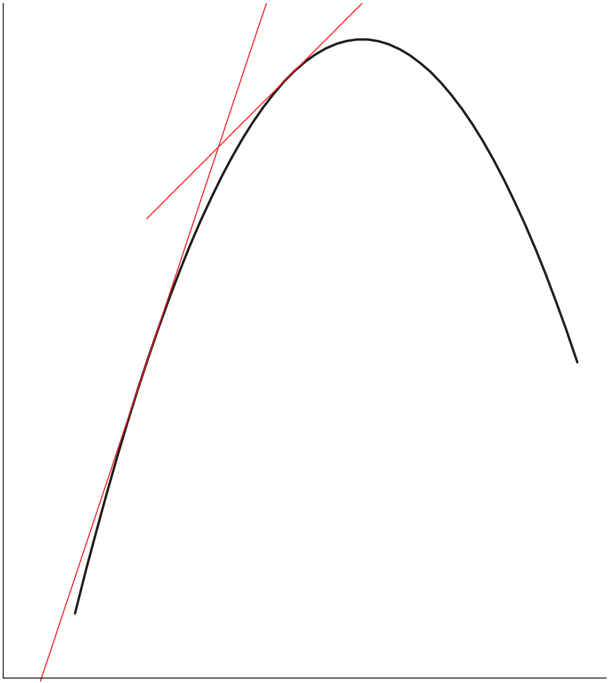
Theorem 5.22 (characterization of concave functions of the class $\mathcal{C}^1$)

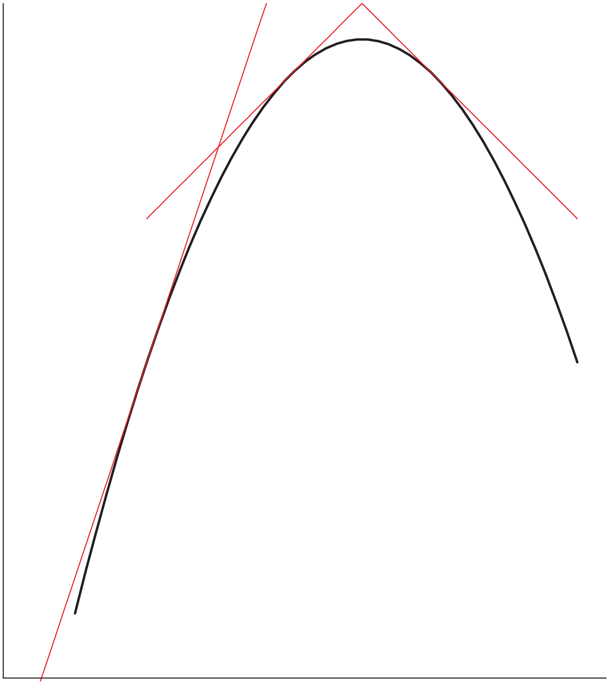
Let $G \subset \mathbf{R}^n$ be a convex open set and $f \in \mathcal{C}^1(G)$. Then the function f is convex on G if and only if we have

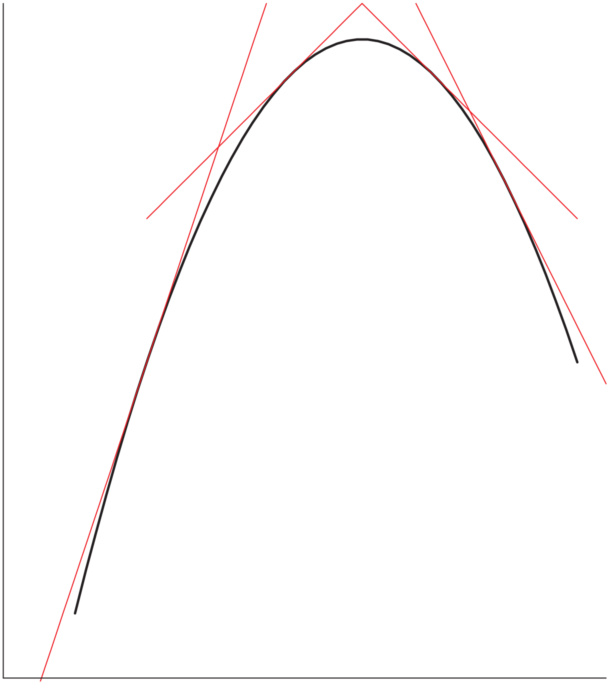
$$\forall \boldsymbol{x}, \boldsymbol{y} \in G\colon f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$











Corollary 5.23

Let $G \subset \mathbf{R}^n$ be a convex open set and $f \in \mathcal{C}^1(G)$ be concave on G. If a point $\boldsymbol{a} \in G$ is a stationary point of f, then $\boldsymbol{a}$ is a point of maximum of f with respect to G.

Theorem 5.24 (characterization of strictly concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbf{R}^n$ be a convex open set and $f \in \mathcal{C}^1(G)$. Then the function f is strictly concave on G if and only if we have

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G, \boldsymbol{x} \neq \boldsymbol{y} \colon f(\boldsymbol{y}) < f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Definition

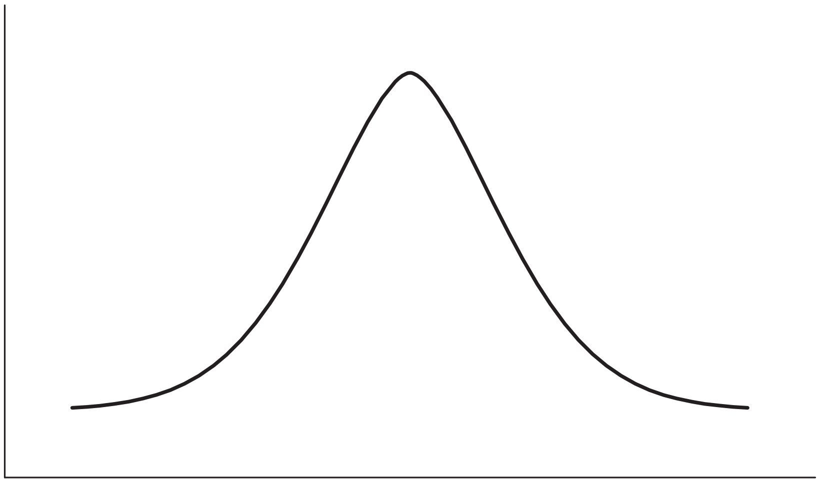
Let $M \subset \mathbf{R}^n$ be a convex set and f be defined on M. We say that f is

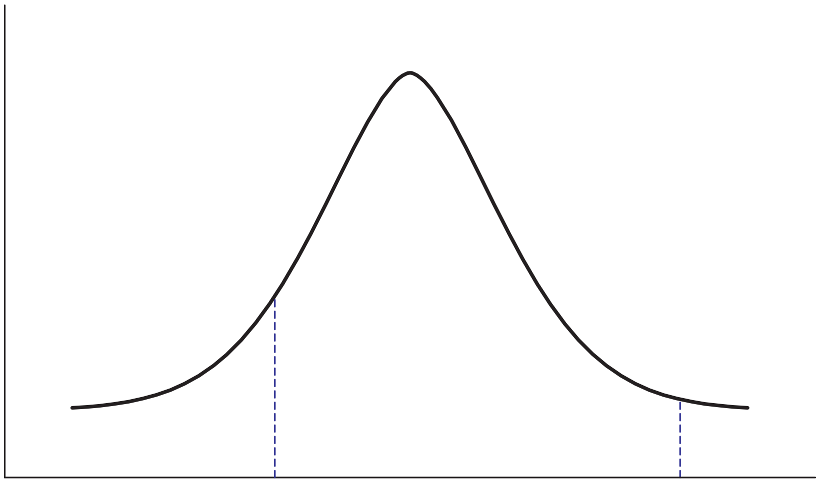
- quasiconcave on M, if

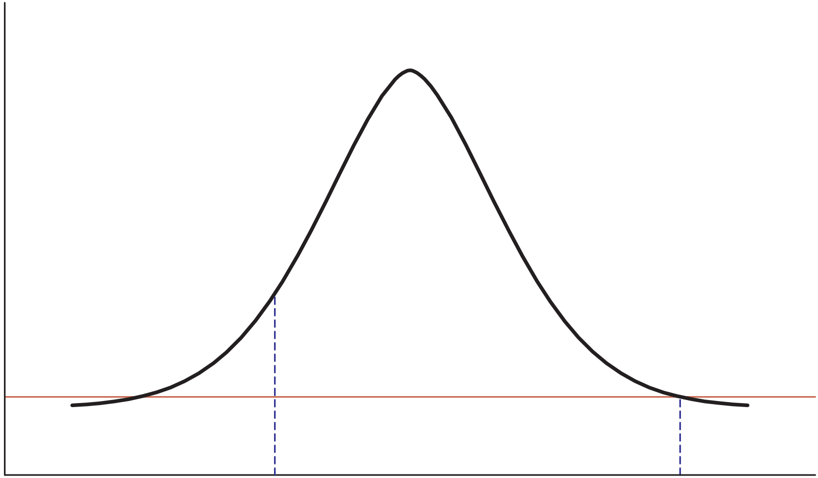
$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \; \forall t \in [0,1]\colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},$$

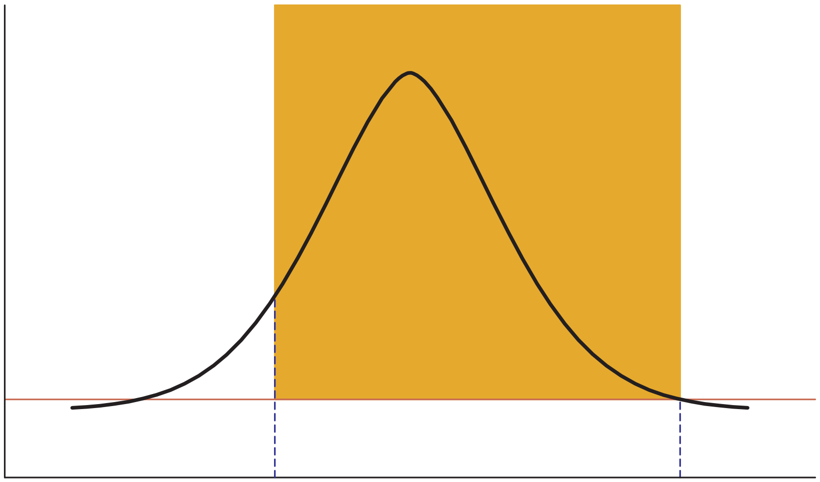
- strictly quasiconcave on M, if

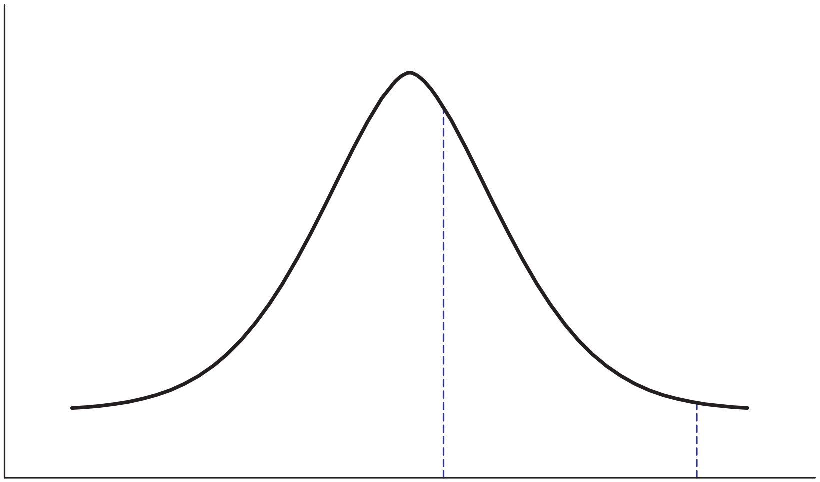
$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \; \forall t \in (0,1)\colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > \min\{f(\boldsymbol{a}), f(\boldsymbol{b}$$

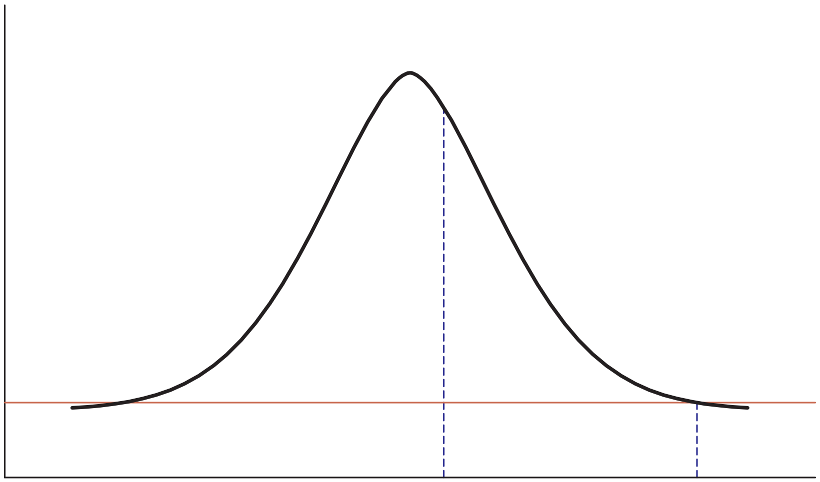


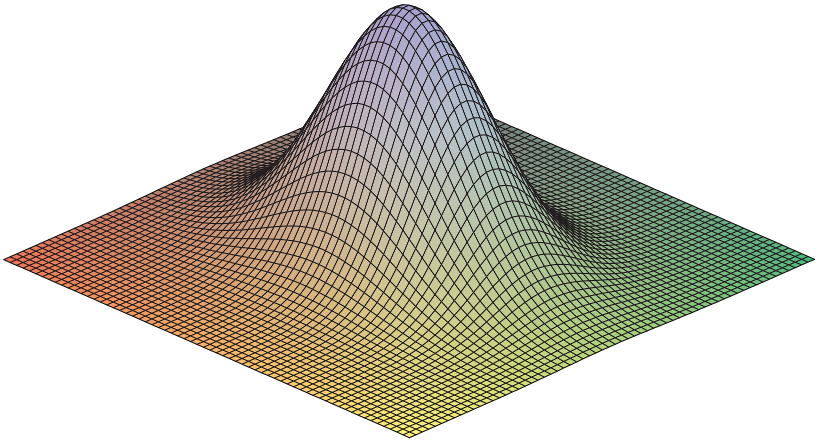












Remark

Let $M \subset \mathbf{R}^n$ be a convex set and f be a function defined on M.

Remark

Let $M \subset \mathbf{R}^n$ be a convex set and f be a function defined on M.

- Let f be concave on M. Then f is quasiconcave on M.
- Let f be strictly concave on M. Then f is strictly quasiconcave on M.

Theorem 5.25 (on uniqueness of extremum)

Let f be a strictly quasiconcave function on a convex set $M \subset \mathbf{R}^n$. Then there exists at most one point of maximum of f.

Theorem 5.25 (on uniqueness of extremum)

Let f be a strictly quasiconcave function on a convex set $M \subset \mathbf{R}^n$. Then there exists at most one point of maximum of f.

Corollary 5.26

Let $M \subset \mathbf{R}^n$ be a convex, bounded, closed and nonempty set. Let f be continuous and strictly quasiconcave function on M. Then f attains its maximum on M in a unique point.

Theorem 5.27 (characterization of quasiconcave functions via level sets)

Let $M \subset \mathbf{R}^n$ be a convex set and f be defined on M. The function f is quasiconcave on M if and only if for each $\alpha \in \mathbf{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.

Definition

The scheme

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbf{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is called a matrix of the type $m \times n$. We write $(a_{ij})_{\substack{i=1..m\\ j=1..n}}$.

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Definition

Let

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, $i \in \{1, 2, \dots, m\}$, is called *i-th row* of the matrix $\mathbb{A}$.

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Definition

We say that two matrices are equal, if they are of the same type and the corresponding elements are equal, i.e., if $\mathbb{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$ and $\mathbb{B} = (b_{uv})_{\substack{u=1..r \\ v=1..s}}$, then $\mathbb{A} = \mathbb{B}$ if and only if $m = r$, $n = s$ and $a_{ij} = b_{ij}$ for every $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$.

Definition

Let $\mathbb{A}, \mathbb{B} \in M(m \times n)$, $\mathbb{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\mathbb{B} = (b_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\lambda \in \mathbf{R}$.

The sum of $\mathbb{A}$ and $\mathbb{B}$ is defined by

$$\mathbb{A} + \mathbb{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

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Product of a real number λ and the matrix $\mathbb{A}$ is defined by

$$\lambda \mathbb{A} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}.$$

Proposition 6.1 (basic properties)

- $\forall \mathbb{A}, \mathbb{B}, \mathbb{C} \in M(m \times n)\colon \mathbb{A} + (\mathbb{B} + \mathbb{C}) = (\mathbb{A} + \mathbb{B}) + \mathbb{C}$, *(associativity)*
- $\forall \mathbb{A}, \mathbb{B} \in M(m \times n)\colon \mathbb{A} + \mathbb{B} = \mathbb{B} + \mathbb{A}$, *(commutativity)*
- $\exists! \mathbb{O} \in M(m \times n)\ \forall \mathbb{A} \in M(m \times n)\colon \mathbb{A} + \mathbb{O} = \mathbb{A}$, *(existence of the zero element)*
- $\forall \mathbb{A} \in M(m \times n)\ \exists \mathbb{C}_{\mathbb{A}} \in M(m \times n)\colon \mathbb{A} + \mathbb{C}_{\mathbb{A}} = \mathbb{O}$,
- $\forall \mathbb{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbf{R}\colon (\lambda\mu)\mathbb{A} = \lambda(\mu\mathbb{A})$,
- $\forall \mathbb{A} \in M(m \times n)\colon 1 \cdot \mathbb{A} = \mathbb{A}$,
- $\forall \mathbb{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbf{R}\colon (\lambda + \mu)\mathbb{A} = \lambda\mathbb{A} + \mu\mathbb{A}$,
- $\forall \mathbb{A}, \mathbb{B} \in M(m \times n)\ \forall \lambda \in \mathbf{R}\colon \lambda(\mathbb{A} + \mathbb{B}) = \lambda\mathbb{A} + \lambda\mathbb{B}$.

Definition

Let $\mathbb{A} \in M(m \times n)$, $\mathbb{A} = (a_{is})_{\substack{i=1..m \\ s=1..n}}$, $\mathbb{B} \in M(n \times k)$, $\mathbb{B} = (b_{sj})_{\substack{s=1..n \\ j=1..k}}$. Then the product of matrices $\mathbb{A}$ and $\mathbb{B}$ is defined as $\mathbb{A}\mathbb{B} \in M(m \times k)$, $\mathbb{A}\mathbb{B} = (c_{ij})_{\substack{i=1..m \\ j=1..k}}$, where

$$c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}.$$

Theorem 6.2 (properties of matrix multiplication)

Let $m, n, k, l \in \mathbf{N}$. Then we have:

(i) $\forall \mathbb{A} \in M(m \times n)\ \forall \mathbb{B} \in M(n \times k)\ \forall \mathbb{C} \in M(k \times l)\colon \mathbb{A}(\mathbb{B}\mathbb{C}) = (\mathbb{A}\mathbb{B})\mathbb{C}$, *(associativity)*

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(ii) $\forall \mathbb{A} \in M(m \times n)\ \forall \mathbb{B}, \mathbb{C} \in M(n \times k)\colon \mathbb{A}(\mathbb{B} + \mathbb{C}) = \mathbb{A}\mathbb{B} + \mathbb{A}\mathbb{C}$, *(left distributivity)*

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(iv) $\exists! \mathbb{I} \in M(n \times n)\ \forall \mathbb{A} \in M(n \times n)\colon \mathbb{I}\mathbb{A} = \mathbb{A}\mathbb{I} = \mathbb{A}$. *(identity matrix* $\mathbb{I}$*)*

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Remark

Warning! Matrix multiplication is not commutative.

Definition

Transpose matrix for a matrix

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is defined by

$$\mathbb{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e., if $\mathbb{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, then $\mathbb{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, where $b_{uv} = a_{vu}$ for each $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

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We have

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Definition

Let $\mathbb{A} \in M(n \times n)$. We say that $\mathbb{A}$ is regular matrix, if there exists $\mathbb{B} \in M(n \times n)$ such that

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Remark

A matrix $\mathbb{A} \in M(n \times n)$ is regular, if and only if $\mathbb{A}$ has its inverse matrix.

Theorem 6.4 (regularity and matrix operations)

Let $\mathbb{A}, \mathbb{B} \in M(n \times n)$ *be regular. Then we have:*

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(iii) $\mathbb{A}\mathbb{B}$ *is regular and* $(\mathbb{A}\mathbb{B})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}$.

Definition

Let $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbf{R}^n$ be vectors. Linear combination of vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k$ is an expression $\lambda_1 \mathbf{v}^1 + \cdots + \lambda_k \mathbf{v}^k$, where $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$.

Definition

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Definition

We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ are linearly dependent, if there exists their nontrivial linear combination, which is equal to the zero vector.

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We say that vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ are linearly independent, if they are not linearly dependent, i.e., if $\lambda_1, \dots, \lambda_k \in \mathbf{R}$ satisfy $\lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k = \boldsymbol{o}$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Definition

Let $\mathbb{A} \in M(m \times n)$. Rank of the matrix $\mathbb{A}$ is the maximal number of linearly independent row vectors of $\mathbb{A}$. Rank of $\mathbb{A}$ is denoted by $\text{rk}(\mathbb{A})$.

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We say that $\mathbb{A} \in M(m \times n)$ is in the row echelon form, if for each $i \in \{2, \ldots, m\}$ we have, that i-th row of $\mathbb{A}$ is a zero vector or the number of zeros at the beginning of the row is bigger than the number of zeros at the beginning of $(i-1)$-st row.

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Remark

The rank of row echelon matrix $\mathbb{A}$ is equal to the number of nonzero rows of $\mathbb{A}$.

Definition

Elementary row transformations of the matrix $\mathbb{A}$ are defined as:

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Definition

Transformation is defined as a finite sequence of elementary row transformation. If the matrix $\mathbb{B} \in M(m \times n)$ was created from $\mathbb{A} \in M(m \times n)$ applying a transformation T to $\mathbb{A}$, then this fact is denoted by $\mathbb{A} \overset{T}{\rightsquigarrow} \mathbb{B}$.

Theorem 6.5 (properties of transformation)

(i) *Let $\mathbb{A} \in M(m \times n)$. Then there exists a transformation transforming $\mathbb{A}$ to a row echelon matrix.*

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(ii) *Let* T_1 *be a transformation applicable to matrices of the type* $m \times n$*. Then there exists a transformation* T_2 *applicable to matrices of the type* $m \times n$ *such that if* $\mathbb{A} \overset{T_1}{\rightsquigarrow} \mathbb{B}$ *for some* $\mathbb{A}, \mathbb{B} \in M(m \times n)$*, then* $\mathbb{B} \overset{T_2}{\rightsquigarrow} \mathbb{A}$*.*

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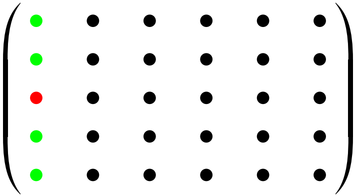
(iii) *Let* $\mathbb{A}, \mathbb{B} \in M(m \times n)$ *and there exist a transformation* T *such that* $\mathbb{A} \overset{T}{\rightsquigarrow} \mathbb{B}$. *Then* $\operatorname{rk}(\mathbb{A}) = \operatorname{rk}(\mathbb{B})$.

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \color{green}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \color{green}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \color{green}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \color{green}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \color{green}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$



$$\begin{pmatrix} {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \color{green}\bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \color{green}\bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \color{green}\bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \color{green}\bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{red}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{red}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \end{pmatrix}$$

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$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \textcolor{red}{\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & \textcolor{green}{\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & \textcolor{green}{\bullet} & \bullet & \bullet \end{pmatrix}$$

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$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet \end{pmatrix}$$

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$$
\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & 0 & \bullet & \bullet & \bullet \\
0 & 0 & 0 & 0 & 0 & {\color{red}\bullet} \\
0 & 0 & 0 & 0 & 0 & {\color{green}\bullet}
\end{pmatrix}
$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Theorem 6.6 (multiplication and transformation)

Let $\mathbb{A} \in M(m \times k)$, $\mathbb{B} \in M(k \times n)$, $\mathbb{C} \in M(m \times n)$ *and we have* $\mathbb{A}\mathbb{B} = \mathbb{C}$. *Let* T *be a transformation and* $\mathbb{A} \overset{T}{\rightsquigarrow} \mathbb{A}'$ *and* $\mathbb{C} \overset{T}{\rightsquigarrow} \mathbb{C}'$. *Then we have* $\mathbb{A}'\mathbb{B} = \mathbb{C}'$.

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Lemma 6.7

Let $\mathbb{A} \in M(n \times n)$ *and* $\mathrm{rk}(\mathbb{A}) = n$. *Then there exists a transformation transforming* $\mathbb{A}$ *to* $\mathbb{I}$.

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Lemma 6.7

Let $\mathbb{A} \in M(n \times n)$ *and* $\operatorname{rk}(\mathbb{A}) = n$. *Then there exists a transformation transforming* $\mathbb{A}$ *to* $\mathbb{I}$.

Theorem 6.8

Let $\mathbb{A} \in M(n \times n)$. *Then* $\mathbb{A}$ *is regular if and only if* $\operatorname{rk}(\mathbb{A}) = n$.

Definition

Let $\mathbb{A} \in M(n \times n)$. The symbol $\mathbb{A}_{ij}$ denotes the matrix of the type $(n-1) \times (n-1)$, which is created from $\mathbb{A}$ omitting i-th row and j-th column.

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$$\mathbb{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i,1} & \dots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \dots & a_{i,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

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Definition

Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$. Determinant of the matrix $\mathbb{A}$ is defined by

$$\det \mathbb{A} = \begin{cases} a_{11} & \text{then } n = 1, \\ \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \mathbb{A}_{i1} & \text{then } n > 1. \end{cases}$$

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For $\det \mathbb{A}$ we will use also the symbol

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Theorem 6.9

Let $j, n \in \mathbf{N}$, $j \leq n$, and matrices $\mathbb{A}, \mathbb{B}, \mathbb{C} \in M(n \times n)$ coincide at each row except j-th row. Let j-th row of $\mathbb{A}$ be equal to the sum of j-th rows of $\mathbb{B}$ and $\mathbb{C}$. Then we have $\det \mathbb{A} = \det \mathbb{B} + \det \mathbb{C}$.

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$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1+v_1 & \dots & u_n+v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1 & \dots & u_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ v_1 & \dots & v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Theorem 6.10 (determinant and transformation)

Let $\mathbb{A}, \mathbb{A}' \in M(n \times n)$*.*

(i) *Let* $\mathbb{A}'$ *be created from* $\mathbb{A}$ *such that we interchanged two rows in* $\mathbb{A}$ *(i.e., we applied an elementary row transformation of the first kind). Then we have* $\det \mathbb{A}' = -\det \mathbb{A}$*.*

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(ii) *Let* $\mathbb{A}'$ *be created from* $\mathbb{A}$ *such that a row in* $\mathbb{A}$ *is multiplied by* $\lambda \in \mathbf{R}$. *Then we have* $\det \mathbb{A}' = \lambda \det \mathbb{A}$.

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(iii) *Let* $\mathbb{A}'$ *be created from* $\mathbb{A}$ *such that we added a row of* $\mathbb{A}$ *to another row of* $\mathbb{A}$ *(i.e., we applied an elementary row transformation of the third kind). Then we have* $\det \mathbb{A}' = \det \mathbb{A}$*.*

Corollary 6.11

Let $\mathbb{A}, \mathbb{A}' \in M(n \times n)$ *and* $\mathbb{A}'$ *be created from* $\mathbb{A}$ *applying a transformation. Then* $\det \mathbb{A}' \neq 0$ *if and only if* $\det \mathbb{A} \neq 0$.

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Theorem 6.12 (determinant and transposition)

Let $\mathbb{A} \in M(n \times n)$*. Then we have* $\det \mathbb{A}^T = \det \mathbb{A}$*.*

Corollary 6.11

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Theorem 6.12 (determinant and transposition)

Let $\mathbb{A} \in M(n \times n)$*. Then we have* $\det \mathbb{A}^T = \det \mathbb{A}$*.*

Theorem 6.13 (determinant of product)

Let $\mathbb{A}, \mathbb{B} \in M(n \times n)$*. Then we have*

$$\det \mathbb{A}\mathbb{B} = \det \mathbb{A} \cdot \det \mathbb{B}.$$

Theorem 6.14

Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$, $k \in \{1, \ldots, n\}$. *Then*

$$\det \mathbb{A} = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det \mathbb{A}_{ik},$$

$$\det \mathbb{A} = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det \mathbb{A}_{kj}.$$

Definition

Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$. We say that $\mathbb{A}$ is upper triangular matrix if we have $a_{ij} = 0$ for $i > j$, $i, j \in \{1, \ldots, n\}$.

Definition

Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$. We say that $\mathbb{A}$ is upper triangular matrix if we have $a_{ij} = 0$ for $i > j$, $i, j \in \{1, \ldots, n\}$. We say that $\mathbb{A}$ is lower triangular matrix, if we have $a_{ij} = 0$ for $i < j$, $i, j \in \{1, \ldots, n\}$.

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Theorem 6.15

Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$ is upper (lower, respectively) triangular matrix. Then we have

$$\det \mathbb{A} = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}.$$

Theorem 6.16

Let $\mathbb{A} \in M(n \times n)$*. Then* $\mathbb{A}$ *is regular if and only if* $\det \mathbb{A} \neq 0$*.*

The system of n equations with n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{S}$$

Matrix form

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\mathbb{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ is called matrix of the system, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ vector of the right side and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ vectors of unknowns.

Theorem 6.17

Let $\mathbb{A} \in M(n \times n)$. Then the following are equivalent.

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Let $\mathbb{A} \in M(n \times n)$. Then the following are equivalent.

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(ii) *The system* (S) *have for each* $\boldsymbol{b}$ *a unique solution.*

(iii) *The system* (S) *have for each* $\boldsymbol{b}$ *at least one solution.*

Theorem 6.18 (Cramer's rule)

Let $\mathbb{A} \in M(n \times n)$ be a regular matrix, $\boldsymbol{b} \in M(n \times 1)$, $\boldsymbol{x} \in M(n \times 1)$, and $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$. Then

$$x_j = \frac{\begin{vmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{vmatrix}}{\det \mathbb{A}}$$

for $j = 1, \dots, n$.

System of m equations with n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{S'}$$

Matrix notation

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\mathbb{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in M(m \times n)$,

$\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M(m \times 1)$ a $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M(n \times 1)$.

Definition

The matrix

$$(\mathbb{A}|\boldsymbol{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

is called extended matrix of the system (S').

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is called extended matrix of the system (S').

Theorem 6.19

The system (S') *has a solution if and only if the matrix has the same rank as the extended matrix of the system.*

Definition

We say that a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is linear if

(i) $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{R}^n\colon f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$,

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(ii) $\forall \lambda \in \mathbf{R}\ \forall \boldsymbol{u} \in \mathbf{R}^n\colon f(\lambda \boldsymbol{u}) = \lambda f(\boldsymbol{u})$.

Definition

Let $i \in \{1, \ldots, n\}$. The vector

$$\boldsymbol{e}^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ldots \text{ } i\text{-th coordinate}$$

is called i-th canonical vector of the space $\mathbf{R}^n$.

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is called i-th canonical vector of the space $\mathbf{R}^n$. The set $\{\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n\}$ of all canonical vectors in $\mathbf{R}^n$ is called canonical basis of the space $\mathbf{R}^n$.

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The properties of canonical vectors:

(i) $\forall \boldsymbol{x} \in \mathbf{R}^n \ \exists \lambda_1, \dots, \lambda_n \in \mathbf{R}\colon \boldsymbol{x} = \lambda_1 \boldsymbol{e}^1 + \dots + \lambda_n \boldsymbol{e}^n$,

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(ii) the vectors $\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$ are linearly independent.

Theorem 6.20 (representation of linear mappings)

The mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is linear if and only if there exists a matrix $\mathbb{A} \in M(m \times n)$ such that

$$\forall \boldsymbol{u} \in \mathbf{R}^n\colon f(\boldsymbol{u}) = \mathbb{A}\boldsymbol{u} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Theorem 6.21

Let a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be linear. Then the following are equivalent.

(i) *The mapping f is a bijection (i.e., f is an injective mapping $\mathbf{R}^n$ onto $\mathbf{R}^n$).*

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Theorem 6.22

Let $f\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear mapping represented by matrix $\mathbb{A} \in M(m \times n)$ a $g\colon \mathbf{R}^m \to \mathbf{R}^k$ be a linear mapping represented by a matrix $\mathbb{B} \in M(k \times m)$. Then the composed mapping $g \circ f\colon \mathbf{R}^n \to \mathbf{R}^k$ is linear and is represented by the matrix $\mathbb{B}\mathbb{A}$.

Infinite series

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The number s_m is called m-th partial sum of the series $\sum_{n=1}^{\infty} a_n$.

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The number s_m is called *m*-th partial sum of the series $\sum_{n=1}^{\infty} a_n$. The element a_n is called *n*-th member of the series $\sum_{n=1}^{\infty} a_n$. The sum of infinite series $\sum_{n=1}^{\infty} a_n$ is defined as the limit of the sequence $\{s_m\}$, if such a limit exists. The sum of the series is denoted by the symbol $\sum_{n=1}^{\infty} a_n$. We say that a series converges, if its sum is a real number. In the opposite case, we say that the series diverges.

Theorem 7.1 (necessary condition)

If a series $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim a_n = 0$.

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Remark

Suppose that $\alpha \in \mathbf{R}$ and a series $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=1}^{\infty} \alpha a_n$ converges and it holds $\sum_{n=1}^{\infty} \alpha a_n = \alpha \sum_{n=1}^{\infty} a_n$.

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Suppose that $\alpha \in \mathbf{R}$ and a series $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=1}^{\infty} \alpha a_n$ converges and it holds $\sum_{n=1}^{\infty} \alpha a_n = \alpha \sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then the series $\sum_{n=1}^{\infty}(a_n + b_n)$ convergens and if holds $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Theorem 7.2

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series satisfying $0 \leq a_n \leq b_n$ for each $n \in \mathbf{N}$.

(i) *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Theorem 7.2

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be series satisfying* $0 \leq a_n \leq b_n$ *for each* $n \in \mathbf{N}$*.*

(i) *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

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We say that $\sum_{n=1}^{\infty} a_n$ is absolute convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

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Remark

Let $|a_n| \leq b_n$ for each $n \in \mathbf{N}$. If the series $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 7.4 (limit test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative members.

(i) *Let*

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists proper. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

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(ii) *Let*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, +\infty).$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 7.5 (Cauchy test)

Let $\sum_{n=1}^{\infty} a_n$ be a series. The we have

(i) *If* $\lim \sqrt[n]{|a_n|} < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

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Theorem 7.6 (d'Alembert test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero members. Then we have

(i) *If* $\lim |a_{n+1}/a_n| < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *absolutely convergent.*

Theorem 7.6 (d'Alembert test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero members. Then we have

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(ii) *If* $\lim |a_{n+1}/a_n| > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 7.7

Let $\alpha \in \mathbf{R}$. The series $\sum_{n=1}^{\infty} 1/n^{\alpha}$ converges if and only if $\alpha > 1$.

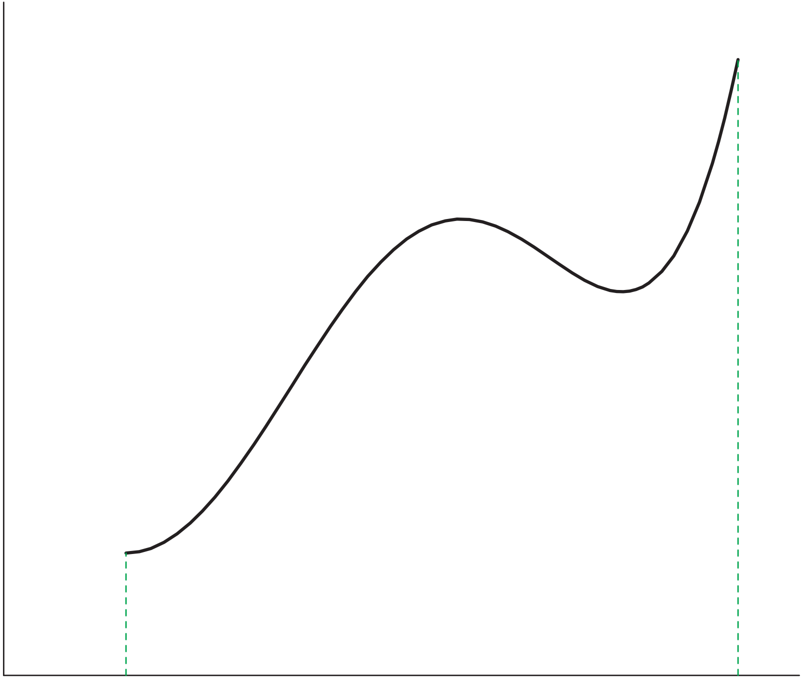
Theorem 7.8 (Leibniz)

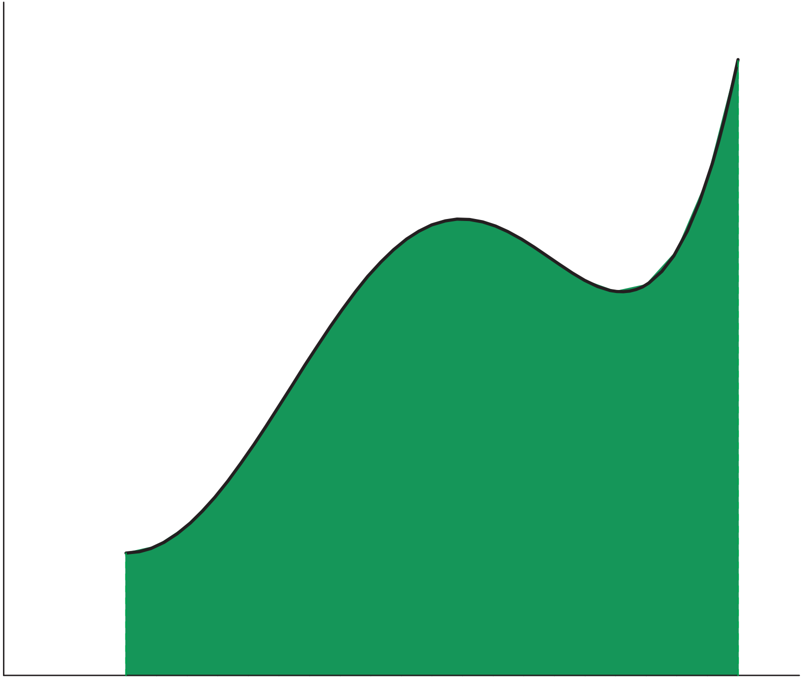
Let $\sum_{n=1}^{\infty}(-1)^n a_n$ *be a series. Assume*

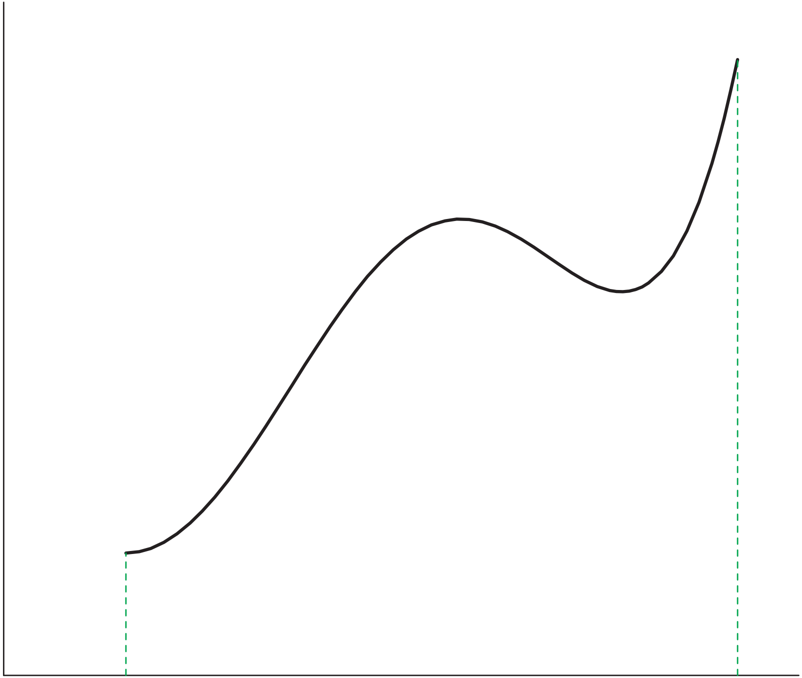
- $a_n \geq a_{n+1} \geq 0$ *for every* $n \in \mathbf{N}$,
- $\lim_{n\to\infty} a_n = 0$.

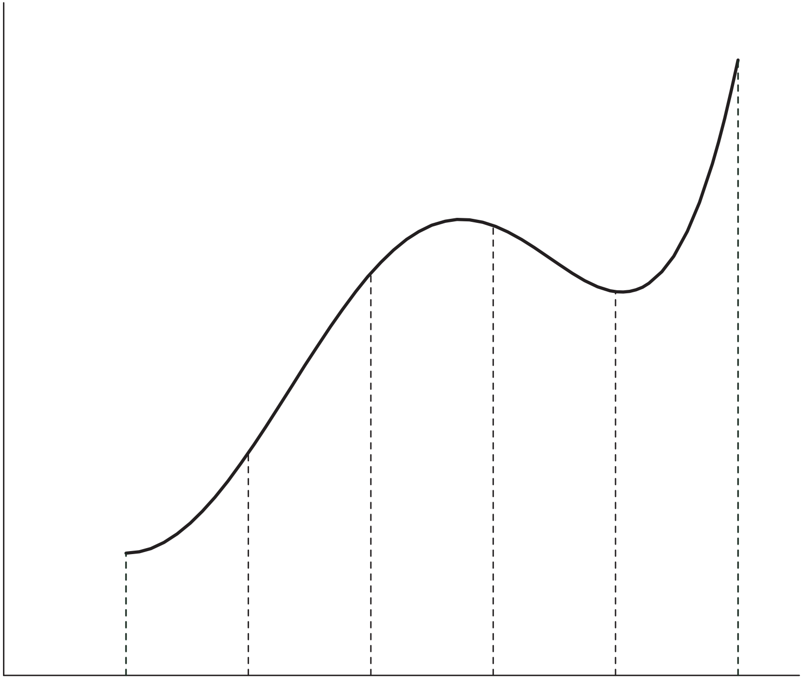
Then $\sum_{n=1}^{\infty}(-1)^n a_n$ *converges.*

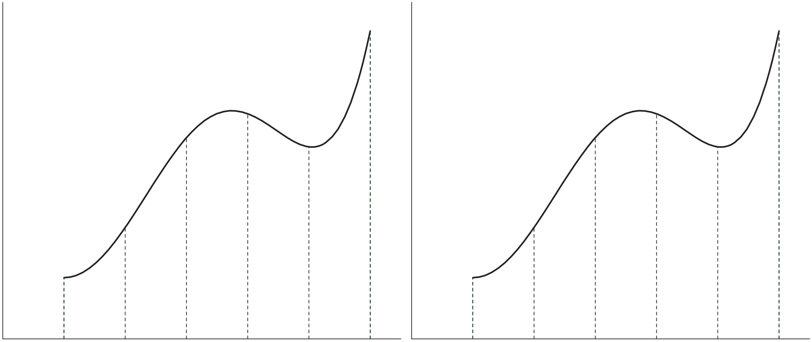
Integrals – Riemann integral

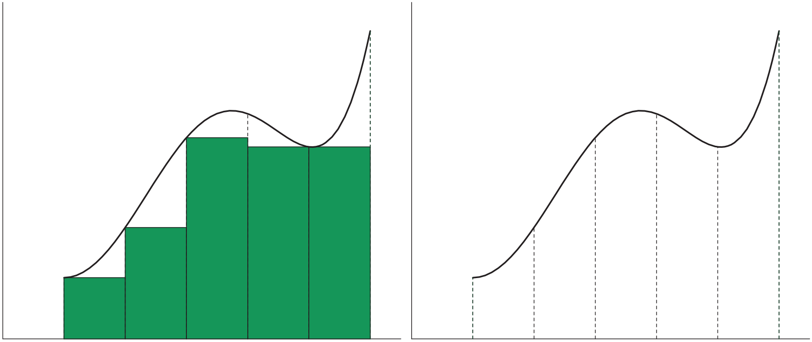


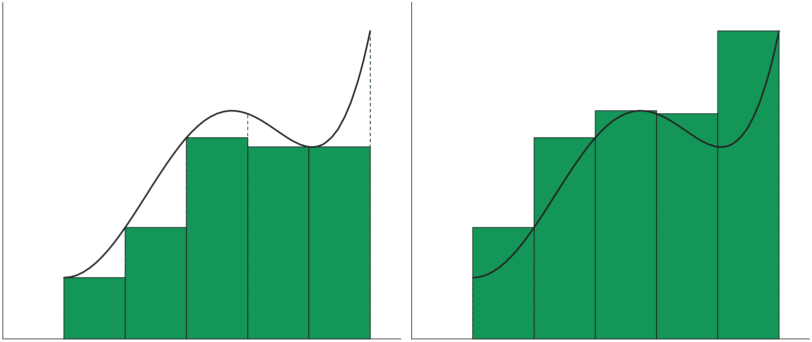


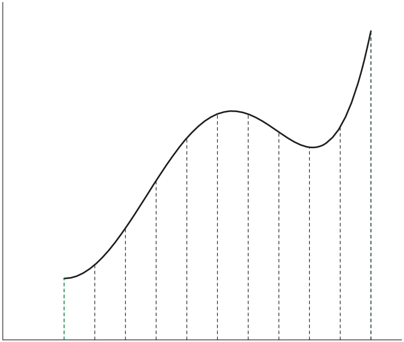


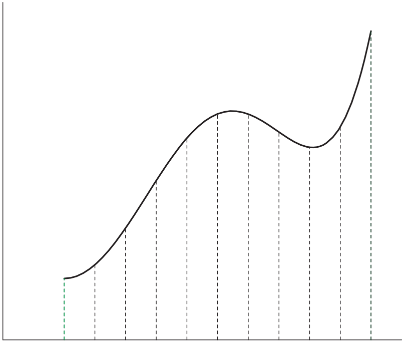


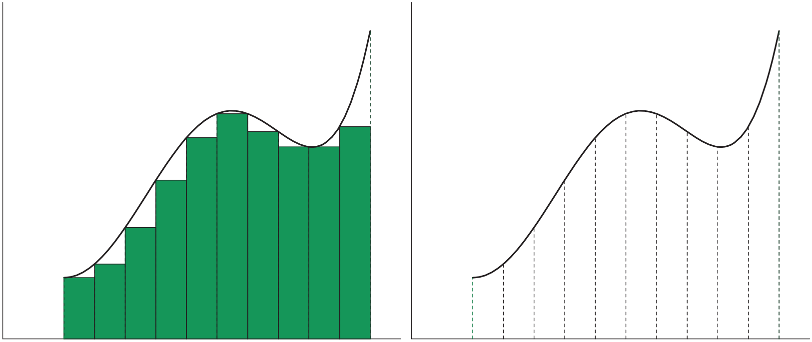


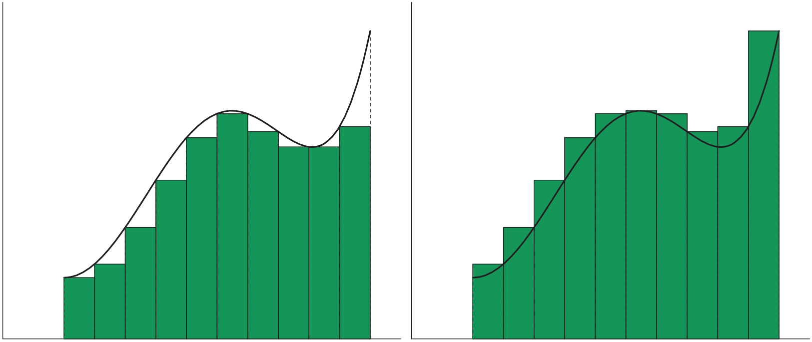












Definition

A finite sequence $\{x_j\}_{j=0}^{n}$ is called a partition of the interval $[a, b]$, if we have

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points $x_0, \ldots, x_n$ are called partition points.

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We say that a partition D' of an interval $[a, b]$ is a refinement of the partition D of the interval $[a, b]$, if each point of D is a partition point of D'.

Definition

Let f be a bounded function on an interval $[a, b]$ and $D = \{x_j\}_{j=0}^{n}$ be a partition of $[a, b]$. We denote

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \text{ where } M_j = \sup\{f(x);\ x \in [x_{j-1}, x_j]\},$$

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$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\big\{\overline{S}(f, D);\ D \text{ is a partition of the interval } [a, b]\big\},$$

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\big\{\underline{S}(f, D);\ D \text{ is a partition of the interval } [a, b]\big\}.$$

Definition

We say that a bounded function f has Riemann integral over the interval $[a, b]$, if $\overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x$.

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Remark

Let D, D' be partitions of the interval $[a, b]$, D' refine D, and let f be a bounded function on the interval $[a, b]$. Then we have

$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$

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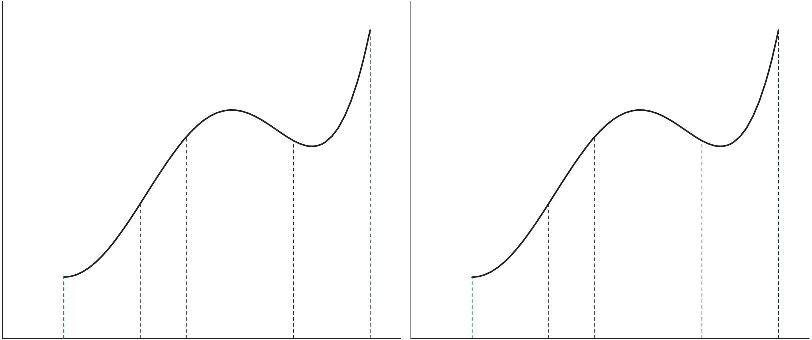

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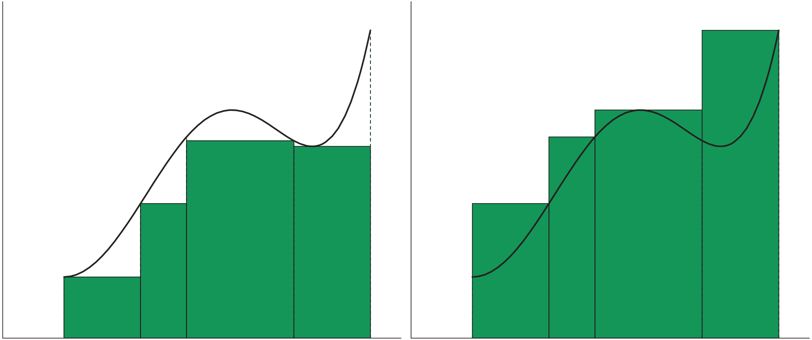

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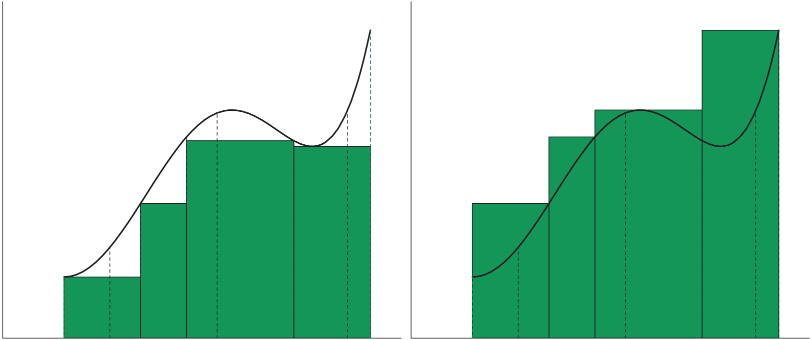

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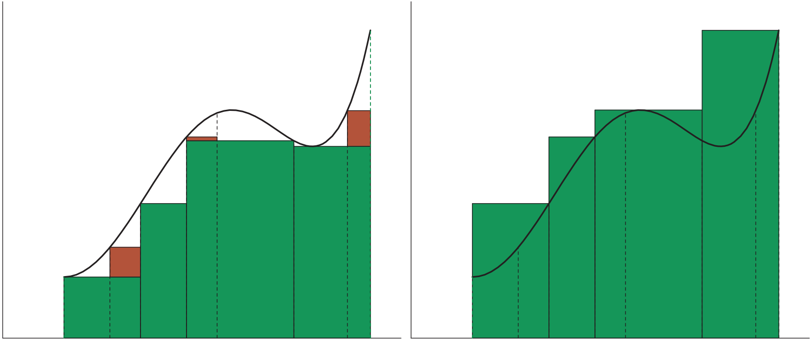

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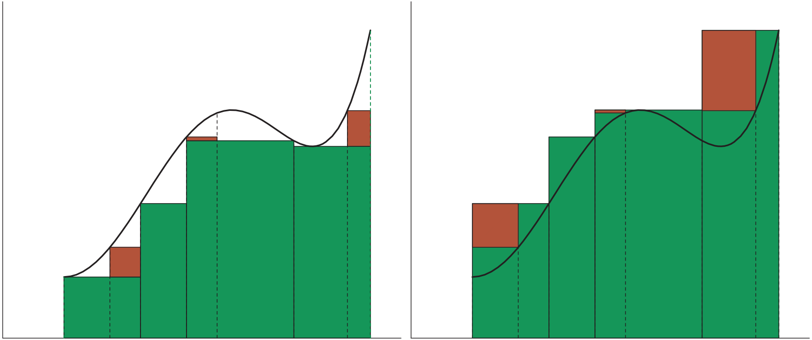

Theorem 8.1

(i) *Let a function f have Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has Riemann integral over $[c, d]$.*

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(ii) *Let $c \in (a, b)$ and a function f have Riemann integral over $[a, c]$ and $[c, b]$. Then f has Riemann integral over $[a, b]$ and we have*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Theorem 8.2

Let f and g be functions with Riemann integral over $[a, b]$ and let $\alpha \in \mathbf{R}$. Then

(i) *the function αf has Riemann integral over $[a, b]$ and it holds*

$$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x,$$

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$$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x,$$

(ii) *the function $f + g$ has Riemann integral over $[a, b]$ and it holds*
$$\int_a^b \big(f(x) + g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$$

Theorem 8.3

Let $a, b \in \mathbf{R}$, $a < b$, and let f and g be functions with Riemann integral over $[a, b]$.

(i) *If $f(x) \geq 0$ for each $x \in [a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x \geq 0.$$

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(ii) *If $f(x) \leq g(x)$ for each $x \in [a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

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$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

(iii) *The function $|f|$ has Riemann integral over $[a, b]$ and it holds*

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Theorem 8.4

Let a function f be continuous on the interval $[a, b]$, $a, b \in \mathbf{R}$. Then f has Riemann integral over $[a, b]$.

Theorem 8.5

Let f be a continuous function on $[a, b]$ and let $c \in [a, b]$. If we denote $F(x) = \int_c^x f(t)\,\mathrm{d}t$ for $x \in (a, b)$, then $F'(x) = f(x)$ for each $x \in (a, b)$.

Primitive function

Definition

Let a function f be defined on an open interval I. We say that a function F is a primitive function of f on I, if for each $x \in I$ there exists $F'(x)$ and $F'(x) = f(x)$.

Theorem 8.6

Let F and G be primitive functions of f on an open interval I. Then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for each $x \in I$.

Theorem 8.7

Let f be a continuous function on an open interval I. Then f has on I a primitive function.

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Theorem 8.8

Let f have on an open interval I a primitive function F, let a function g have on I a primitive function G, and $\alpha, \beta \in \mathbf{R}$. Then the function $\alpha F + \beta G$ is a primitive function of $\alpha f + \beta g$ on I.

Theorem 8.9 (substitution)

(i) Let F be a primitive function of f on (a, b). Let φ be a function defined on an interval (α, β) with values in (a, b) and φ has at each point $t \in (\alpha, \beta)$ proper derivative. Then we have

$$\int f(\varphi(t))\varphi'(t)\,dt \stackrel{c}{=} F(\varphi(t)) \text{ on } (\alpha, \beta).$$

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$$\int f(\varphi(t))\varphi'(t)\, dt \stackrel{c}{=} F(\varphi(t)) \text{ on } (\alpha, \beta).$$

(ii) Let a function φ have at each point of an interval (α, β) nonzero proper derivative and $\varphi((\alpha, \beta)) = (a, b)$. Let f be defined on an interval (a, b) and we have

$$\int f(\varphi(t))\varphi'(t)\, dt \stackrel{c}{=} G(t) \text{ on } (\alpha, \beta).$$

Then we have

$$\int f(x)\, dx \stackrel{c}{=} G(\varphi^{-1}(x)) \text{ on } (a, b).$$

Theorem 8.10 (integration per partes)

Let I be an open interval and let functions f and g be continuous on I. Let F be a primitive function of f on I and G be a primitive function of g on I. Then we have

$$\int g(x)F(x)\,dx = G(x)F(x) - \int G(x)f(x)\,dx \ \text{na } I.$$

Definition

Rational function is a ratio of two polynomials, where the polynomial in denominator is not identically zero.

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Theorem 8.11

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(ii) $Q(x) = a_n(x - x_1)^{p_1} \dots (x - x_k)^{p_k}(x^2 + \alpha_1 x + \beta_1)^{q_1} \dots (x^2 + \alpha_l x + \beta_l)^{q_l}$,

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(v) *the polynomials* $x - x_1,\ x - x_2, \dots, x - x_k,\ x^2 + \alpha_1 x + \beta_1,\ \dots, x^2 + \alpha_l x + \beta_l$ *have no common root,*

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(vi) *the polynomials* $x^2 + \alpha_1 x + \beta_1,\ \dots, x^2 + \alpha_l x + \beta_l$ *have no real root.*

Then there exist unique real numbers $A_1^1, \ldots, A_{p_1}^1$, $\ldots, A_1^k, \ldots, A_{p_k}^k$, B_1^1, $C_1^1, \ldots, B_{q_1}^1$, $C_{q_1}^1, \ldots, B_1^l$, $C_1^l, \ldots, B_{q_l}^l$, $C_{q_l}^l$ such that we have

$$
\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{A_1^1}{(x-x_1)^{p_1}} + \cdots + \frac{A_{p_1}^1}{(x-x_1)} \\
&\quad + \cdots + \frac{A_1^k}{(x-x_k)^{p_k}} + \cdots + \frac{A_{p_k}^k}{x-x_k} \\
&\quad + \frac{B_1^1 x + C_1^1}{(x^2+\alpha_1 x+\beta_1)^{q_1}} + \cdots + \frac{B_{q_1}^1 x + C_{q_1}^1}{x^2+\alpha_1 x+\beta_1} + \ldots \\
&\quad + \frac{B_1^l x + C_1^l}{(x^2+\alpha_l x+\beta_l)^{q_l}} + \cdots + \frac{B_{q_l}^l x + C_{q_l}^l}{x^2+\alpha_l x+\beta_l}.
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\end{aligned}$$