Real functions

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Contents

Part I

Winter semester

Chapter 1

Differentiation of measures

1.1 Covering theorems

Covering theorems provide a tool which enables us to infer global properties from local ones in the context of measure theory.

Vitali theorem

Definition. Let $A \subset \mathbf{R}^n$. We say that a system $\mathcal{V}$ consisting of closed balls from $\mathbf{R}^n$ forms **Vitali cover of** A, if

$$\forall x \in A\, \forall \varepsilon > 0\, \exists B \in \mathcal{V} \colon x \in B \ \wedge\ \operatorname{diam} B < \varepsilon.$$

Notation.

- $\lambda_n \ldots$ Lebesgue measure on $\mathbf{R}^n$
- $\lambda_n^* \ldots$ outer Lebesgue measure on $\mathbf{R}^n$
- If $B \subset \mathbf{R}^n$ is a ball and $\alpha > 0$, then $\alpha \star B$ denotes the ball, which is concentric with B and with α-times greater radius than B.

Theorem 1.1 (Vitali)**.** *Let $A \subset \mathbf{R}^n$ and $\mathcal{V}$ be a system of closed balls forming a Vitali cover of A. Then there exists a countable disjoint subsystem $\mathcal{A} \subset \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.*

Proof. First assume that A is bounded. Take an open bounded set $G \subset \mathbf{R}^n$ with $A \subset G$. Set

$$\mathcal{V}^* = \{B \in \mathcal{V};\ B \subset G\}.$$

The system $\mathcal{V}^*$ is a Vitali cover of A again. If there exists a finite disjoint subsystem $\mathcal{V}^*$ covering A, we are done. So assume

($\star$) there is no finite disjoint subsystem of $\mathcal{V}^*$ covering A.

1st step. We set

$$s_1 = \sup\{\operatorname{diam} B; B \in \mathcal{V}^*\}$$

and choose a ball $B_1 \in \mathcal{V}^*$ such that $\operatorname{diam} B_1 > s_1/2$. We know that $\mathcal{V}^* \neq \emptyset$ and $s_1 \leq \operatorname{diam} G < \infty$.

k**-th step.** Suppose that we have already chosen balls $B_1, \ldots, B_{k-1}$. We set

$$s_k = \sup\Big\{\operatorname{diam} B;\ B \in \mathcal{V}^* \ \wedge\ B \cap \bigcup_{i=1}^{k-1} B_i = \emptyset\Big\}.$$

The supremum is considered for a nonempty set since the set $\bigcup_{i=1}^{k-1} B_i$ is closed, which by ($\star$) does not cover A, and $\mathcal{V}^*$ is a Vitali cover of A. We choose a ball $B_k \in \mathcal{V}^*$ such that $B_k \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$ and $\operatorname{diam} B_k > s_k/2$.

This finishes the construction of the sequence $(B_k)_{k=1}^{\infty}$. Set $\mathcal{A} = \{B_k;\ k \in \mathbf{N}\}$. We verify that $\mathcal{A}$ is the desired system.

- *$\mathcal{A}$ is countable.* This follows immediately from the construction.
- *$\mathcal{A}$ is disjoint.* This follows from the construction.
- *It holds $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.* We have

$$\sum_{i=1}^{\infty} \lambda_n(B_i) = \lambda_n\Big(\bigcup_{i=1}^{\infty} B_i\Big) \leq \lambda_n(G) < \infty.$$

Thus the series $\sum_{i=1}^{\infty} \lambda_n(B_i)$ is convergent, therefore $\lim_i \lambda_n(B_i) = 0$. Using the fact that B_i, $i \in \mathbf{N}$, are balls we also have $\lim_i \operatorname{diam} B_i = 0$. We know that $2 \operatorname{diam} B_i > s_i$, consequently $\lim_i s_i = 0$.

We show that

$$\forall x \in A \setminus \bigcup \mathcal{A}\ \forall i \in \mathbf{N}\ \exists j \in \mathbf{N}, j > i :\ x \in 5 \star B_j.$$

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbf{N}$. Denote $\delta = \operatorname{dist}(x, \bigcup_{k=1}^{i} B_k)$. It holds $\delta > 0$ and there exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\operatorname{diam} B < \delta$. Then we have $B \cap \bigcup_{k=1}^{i} B_k = \emptyset$. Thus we have $\operatorname{diam} B > s_p$ for some $p \in \mathbf{N}$ since $\lim_i s_i = 0$. Therefore there exists $j > i$ with $B_j \cap B \neq \emptyset$. Let j be the smallest number with this property. Then we have $s_j \geq \operatorname{diam} B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \emptyset$. Further we have $\operatorname{diam} B_j > s_j/2 \geq \frac{1}{2} \operatorname{diam} B$. Together we have $2 \operatorname{diam} B_j \geq \operatorname{diam} B$. This implies $x \in B \subset 5 \star B_j$.

For any $i \in \mathbf{N}$ we have

$$\lambda_n^*(A \setminus \bigcup \mathcal{A}) \leq \lambda_n\big(\bigcup_{j=i}^{\infty} 5 \star B_j\big) \leq \sum_{j=i}^{\infty} \lambda_n(5 \star B_j) = 5^n \sum_{j=i}^{\infty} \lambda_n(B_j).$$

Using $\lim_{i\to\infty}\sum_{j=i}^{\infty}\lambda_n(B_j)=0$ we get $\lambda_n^*(A\setminus\bigcup\mathcal{A})=0$, and therefore $\lambda_n(A\setminus\bigcup\mathcal{A})=0$.

Now we assume that the set A is a general subset of $\mathbf{R}^n$. Let $(G_j)_{j=1}^{\infty}$ be a sequence of bounded disjoint open sets such that $\lambda_n(\mathbf{R}^n\setminus\bigcup_{j=1}^{\infty}G_j)=0$. Denote

$$\mathcal{V}_j^*=\{B\in\mathcal{V};\ B\subset G_j\}.$$

The system $\mathcal{V}_j^*$ forms a Vitali cover of the bounded set $G_j\cap A$. Using the previous part of the construction we find a countable disjoint system $\mathcal{A}_j\subset\mathcal{V}_j^*$ with $\lambda_n\big((G_j\cap A)\setminus\bigcup\mathcal{A}_j\big)=0$. Now we set $\mathcal{A}=\bigcup_j\mathcal{A}_j$. □

Definition. We say that a measure μ on $\mathbf{R}^n$ satisfies **Vitali theorem**, if for every $M\subset\mathbf{R}^n$ and every Vitali cover $\mathcal{V}$ of M there exists countable disjoint cover $\mathcal{A}\subset\mathcal{V}$ such that $\mu\big(M\setminus\bigcup\mathcal{A}\big)=0$.

Remark. (1) By Theorem 1.1 λ_n satisfies Vitali theorem.

(2) If μ satisfies Vitali theorem and $\nu\ll\mu$, then ν satisfies Vitali theorem.

Remark. If μ is the Borel measure on $\mathbf{R}^2$ such that $\mu(A)=\lambda_1\big(A\cap(\mathbf{R}\times\{0\})\big)$ for any $A\subset\mathbf{R}^2$ Borel, then Vitali theorem does not hold for μ.

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Theorem 1.2. *Let $E\subset\mathbf{R}^n$ be measurable and $\mathcal{S}$ be a finite system of closed balls covering E. Then there exists a disjoint system $\mathcal{L}\subset\mathcal{S}$ such that $\lambda_n(E)\le 3^n\sum_{B\in\mathcal{L}}\lambda_n(B)$.*

Proof. Without any loss of generality we may assume that $\mathcal{S}$ is nonempty. Choose $B_1\in\mathcal{S}$ with maximal radius among balls in $\mathcal{S}$. Suppose that we have already constructed $B_1,\dots,B_{k-1}$. If possible, choose $B_k\in\mathcal{S}$ disjoint with $\bigcup_{i<k}B_i$ and with maximal radius among balls in $\mathcal{S}$ satisfying this property. We construct a finite sequence of closed balls $B_1,\dots,B_N$ and set $\mathcal{L}=\{B_1,\dots,B_N\}$. We have $E\subset\bigcup_{B\in\mathcal{L}}3\star B$. To this end consider $x\in E$. Then there exists $B\in\mathcal{S}$ with $x\in B$. We find minimal k such that $B\cap B_k\neq\emptyset$. Then we have $\operatorname{radius}(B)\le\operatorname{radius}(B_k)$. This implies that $x\in B\subset 3\star B_k$.

Then we have

$$\lambda_n(E)\le\lambda_n\Big(\bigcup_{B\in\mathcal{L}}3\star B\Big)\le\sum_{B\in\mathcal{L}}\lambda_n(3\star B)=3^n\sum_{B\in\mathcal{L}}\lambda_n(B).$$

□

Besicovitch theorem

Theorem 1.3 (Besicovitch [**?**]). *For each $n\in\mathbf{N}$ there exists $N\in\mathbf{N}$ with the following property. If $A\subset\mathbf{R}^n$ and $\Delta\colon A\to(0,\infty)$ is a bounded function, then there exist sets $A_1,\dots,A_N$ such that*

- $\big\{\overline{B}(x,\Delta(x));\ x\in A_i\big\}$ *is disjoint for every* $i\in\{1,\dots,N\}$,
- $A\subset\bigcup\big\{\overline{B}(x,\Delta(x));\ x\in\bigcup_{i=1}^N A_i\big\}$.

Proof. The case of a bounded set A. Let $R=\sup_A\Delta$. Choose $B_1:=\overline{B}(a_1,r_1)$ such that $a_1\in A$ and $r_1:=\Delta(a_1)>\frac{3}{4}R$. Assume that we have already chosen balls $B_1,\dots,B_{j-1}$ where $j\geq 2$. If

$$F_j:=A\setminus\bigcup_{i=1}^{j-1}\overline{B}(a_i,r_i)=\emptyset,$$

then the process stops and we set $J=j$. If $F_j\neq\emptyset$, we continue by choosing $B_j:=\overline{B}(a_j,r_j)$ such that $a_j\in F_j$ and

$$r_j:=\Delta(a_j)>\tfrac{3}{4}\sup_{F_j}\Delta. \tag{1.1}$$

If $F_j\neq\emptyset$ for all j, then we set $J=\infty$. In this case $\lim_{j\to\infty}r_j=0$ because A is bounded and the inequalities

$$\|a_i-a_j\|\geq r_i=\frac{1}{3}r_i+\frac{2}{3}r_i>\frac{1}{3}r_i+\frac{1}{2}r_j>\frac{1}{3}r_i+\frac{1}{3}r_j$$

for $i<j<J$ imply that

$$\big\{\tfrac{1}{3}\star B_j;\ j<J\big\}\text{ is a disjoint family.} \tag{1.2}$$

In case $J<\infty$, we have $A\subset\bigcup_{j<J}B_j$. This is also true in the case $J=\infty$. Otherwise there exist $a\in\bigcap_{j=1}^{\infty}F_j$ and $j_0\in\mathbf{N}$ with $r_{j_0}\leq\frac{3}{4}\Delta(a)$, contradicting the choice of r_{j_0}.

Fix $k<J$. We set $I=\{i<k;\ B_i\cap B_k\neq\emptyset\}$. We now prove that there exists $M\in\mathbf{N}$ depending only on n which estimates $|I|$. To this end we split I into I_1 and I_2 and we estimate their cardinality separately.

$$I_1=\{i<k;\ B_i\cap B_k\neq\emptyset, r_i<10r_k\},$$
$$I_2=\{i<k;\ B_i\cap B_k\neq\emptyset, r_i\geq 10r_k\}.$$

The estimate of $|I_1|$. We have $\frac{1}{3}\star B_i\subset 15\star B_k$ for every $i\in I_1$. Indeed, if $x\in\frac{1}{3}\star B_i$, then

$$\|x-a_k\|\leq\|x-a_i\|+\|a_i-a_k\|\leq\frac{10}{3}r_k+r_i+r_k\leq\frac{43}{3}r_k<15r_k.$$

Hence, there are at most 60^n elements of I_1, because for any $i\in I_1$ we have

$$\lambda_n(\tfrac{1}{3}\star B_i)=\lambda_n(\overline{B}(0,1))\cdot\big(\tfrac{1}{3}r_i\big)^n>\lambda_n(\overline{B}(0,1))\cdot\big(\tfrac{1}{4}r_k\big)^n=\frac{1}{60^n}\lambda_n(15\star B_k).$$

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The estimate of $|I_2|$. Denote $b_i = a_i - a_k$. An elementary mesh-like construction gives a family $\{Q_m;\ 1 \le m \le (22n)^n\}$ of closed cubes with edge length $1/(11n)$ (so that $\operatorname{diam} Q_m \le 1/11$), which cover $[-1,1]^n$ and thus in particular the unit sphere. We claim that for each $1 \le m \le (22n)^n$ there is at most one $i \in I_2$ such that $b_i/\|b_i\| \in Q_m$, which estimates the cardinality of I_2.

If the claim were not valid, then there would exist $i, j \in I_2, i < j$, such that

$$\left\| \frac{b_i}{\|b_i\|} - \frac{b_j}{\|b_j\|} \right\| \le \frac{1}{11}.$$

Notice that

$$r_i < \|b_i\| < r_i + r_k \quad \text{and} \quad r_j < \|b_j\| < r_j + r_k, \tag{1.3}$$

as the balls B_i, B_j intersect B_k but does not contain a_k. Hence

$$\big| \|b_i\| - \|b_j\| \big| \le |r_i - r_j| + r_k \le |r_i - r_j| + \frac{1}{10} r_j.$$

and

$$\|b_j\| \le r_j + r_k \le r_j + \frac{1}{10} r_j = \frac{11}{10} r_j. \tag{1.4}$$

We have

$$\begin{aligned}
\|a_i - a_j\| = \|b_i - b_j\| &\le \left\| b_i - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - b_j \right\| \\
&= \left\| \frac{\|b_i\| b_i}{\|b_i\|} - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - \frac{\|b_j\|}{\|b_j\|} b_j \right\| \\
&\le \big| \|b_i\| - \|b_j\| \big| + \frac{1}{11} \|b_j\| \\
&\le |r_i - r_j| + \frac{1}{10} r_j + \frac{1}{10} r_j \qquad \text{(using (1.3) and (1.4))} \\
&\le \begin{cases} r_i - \frac{4}{5} r_j < r_i & \text{if } r_i > r_j, \\ -r_i + \frac{6}{5} r_j \le -r_i + \frac{8}{5} r_i < r_i & \text{if } r_i \le r_j. \end{cases}
\end{aligned}$$

In the last inequality we have used that $i < j$ and thus $r_j < \frac{4}{3} r_i$ by (1.1). We arrived at a contradiction as $i < j$ and thus $a_j \notin B_i$. Hence $|I_2| \le (22n)^n$.

Thus it is sufficient to choose $M > 60^n + (22n)^n$.

Choice of $A_1, \ldots, A_M$. For each $k \in \mathbf{N}$ we define $\lambda_k \in \{1, 2, \ldots, M\}$ such that $\lambda_k = k$ whenever $k \le M$ and for $k > M$ we define λ_k inductively as follows. There is $\lambda_k \in \{1, \ldots, M\}$ such that

$$B_k \cap \bigcup \{B_i;\ i < k, \lambda_i = \lambda_k\} = \emptyset.$$

Now we set $A_j = \{a_i;\ \lambda_i = j\}$, $j = 1, \ldots, M$.

The case of a general set A. For each $l \in \mathbf{N}$ apply the previously obtained result with A replaced by

$$A^l = A \cap \{x;\ 3(l-1)R \le \|x\| < 3lR\},$$

and denote resulting sets as $A_i^l,\ i=1,\dots,M$. Then we set

$$A_i = \bigcup_{l \text{ is odd}} A_i^l, \qquad A_{M+i} = \bigcup_{l \text{ is even}} A_i^l, \qquad i=1,\dots,M.$$

Then we constructed $N := 2M$ subsets which have the required properties. □

Definition. Let P be a locally compact space and $\mathcal{S}$ be a σ-algebra of subsets of P. We say that μ is a **Radon measure** on $(P,\mathcal{S})$ if

(a) $\mathcal{S}$ contains all Borel subsets of P,

(b) $\mu(K) < \infty$ for every compact set $K \subset P$,

(c) $\mu(G) = \sup\{\mu(K);\ K \subset G$ is compact$\}$ for every open set $G \subset P$,

(d) $\mu(A) = \inf\{\mu(G);\ A \subset G, G$ is open$\}$ for every $A \in \mathcal{S}$,

(e) μ is complete.

Definition. Let μ be a measure on X. **Outer measure corresponding** to μ is defined by

$$\mu^*(A) = \inf\{\mu(B);\ A \subset B, B \text{ is } \mu\text{-measurable}\}.$$

Remark. Let μ be a Radon measure on $(\mathbf{R}^n,\mathcal{S})$ and $A \in \mathcal{S}$. Then there exist a Borel set $B \subset \mathbf{R}^n$ such that $A \subset B$ and $\mu(B \setminus A) = 0$. If ν is a Radon measure on $(\mathbf{R}^n,\mathcal{S}')$ with $\nu \ll \mu$, then $\mathcal{S} \subset \mathcal{S}'$.

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Lemma 1.4. *Let μ be a measure on X and $\{A_j\}_{j=1}^\infty$ be an increasing sequence of subset of X. Then $\lim \mu^*(A_j) = \mu^*\big(\bigcup_{j=1}^\infty A_j\big)$.*

Proof. For every $j \in \mathbf{N}$ find a μ-measurable set B_j with $A_j \subset B_j$ and $\mu^*(A_j) = \mu(B_j)$. We set $M_k = \bigcap_{j=1}^k A_j$. Then M_k is μ-measurable $A_k \subset M_k$, and $\mu(M_k) = \mu^*(A_k)$ for every $k \in \mathbf{N}$. Moreover, $\{M_k\}$ is nondecreasing sequence of sets. Then we have

$$\lim_{k\to\infty} \mu^*(A_k) = \lim_{k\to\infty} \mu(M_k) = \mu\Big(\bigcup_{k=1}^\infty M_k\Big) \geq \mu^*\Big(\bigcup_{k=1}^\infty A_k\Big) \geq \lim_{k\to\infty} \mu^*(A_k)$$

and we are done. □

Theorem 1.5. *Let μ be a Radon measure on $\mathbf{R}^n$ and $\mathcal{F}$ be a system of closed balls in $\mathbf{R}^n$. Let A denote the set of centers of the balls in $\mathcal{F}$. Assume $\inf\{r;\ B(a,r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then there exists a countable disjoint system $\mathcal{G} \subset \mathcal{F}$ such that $\mu\big(A \setminus \bigcup \mathcal{G}\big) = 0$.*

Proof. The case $\mu^*(A) < \infty$. Let N be the natural number from Theorem 1.3. Fix θ such that $1 - \frac{1}{N} < \theta < 1$.

Claim. Let $U \subset \mathbf{R}^n$ be an open set. There exists a disjoint finite system $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset U$ and

$$\mu^*\big((A \cap U) \setminus \bigcup \mathcal{H}\big) \leq \theta \mu^*(A \cap U). \tag{1.5}$$

Proof of Claim. We may assume that $\mu^*(A \cap U) > 0$. Let $\mathcal{F}_1 = \{B \in \mathcal{F};\ \operatorname{diam} B < 1, B \subset U\}$. By Theorem 1.3 there exist disjoint families $\mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}_1$ such that

$$A \cap U \subset \bigcup_{i=1}^{N} \bigcup \mathcal{G}_i.$$

Thus

$$\mu^*(A \cap U) \leq \sum_{i=1}^{N} \mu^*\big(A \cap U \cap \bigcup \mathcal{G}_i\big).$$

Consequently, there exists an integer $1 \leq j \leq N$ for which

$$\mu^*\big(A \cap U \cap \bigcup \mathcal{G}_j\big) \geq \frac{1}{N} \mu^*(A \cap U) > (1 - \theta) \mu^*(A \cap U).$$

Using Lemma 1.4 we find a finite system $\mathcal{H} \subset \mathcal{G}_j$ such that

$$\mu^*\big(A \cap U \cap \bigcup \mathcal{H}\big) > (1 - \theta) \mu^*(A \cap U).$$

The set $\bigcup \mathcal{H}$ is μ-measurable and therefore

$$\begin{aligned} \mu^*(A \cap U) &= \mu^*\big(A \cap U \cap \bigcup \mathcal{H}\big) + \mu^*\big(A \cap U \setminus \bigcup \mathcal{H}\big) \\ &\geq (1 - \theta) \mu^*(A \cap U) + \mu^*\big(A \cap U \setminus \bigcup \mathcal{H}\big). \end{aligned}$$

This gives (1.5). □

Set $U_1 = \mathbf{R}^n$. Using Claim we find a disjoint finite system $\mathcal{H}_1 \subset \mathcal{F}$ such that $\bigcup \mathcal{H}_1 \subset U_1$ and

$$\mu^*\big((A \cap U_1) \setminus \bigcup \mathcal{H}_1\big) \leq \theta \mu^*(A \cap U_1).$$

Continuing by induction we obtain a sequence of open set (U_j) and finite disjoint finite systems $(\mathcal{H}_j)$ such that $U_{j+1} = U_j \setminus \bigcup \mathcal{H}_j$, $\mathcal{H}_j \subset \mathcal{F}$, $\bigcup \mathcal{H}_j \subset U_j$, and

$$\mu(A \cap U_{j+1}) = \mu^*\big((A \cap U_j) \setminus \bigcup \mathcal{H}_j\big) \leq \theta \mu^*(A \cap U_j)$$

for every $j \in \mathbf{N}$. Together we have

$$\mu^*\big(A \cap U_{j+1}\big) \leq \theta^j \mu^*(A)$$

for every $j \in \mathbf{N}$. Since $\mu^*(A) < \infty$ we get $\mu^*\big(A \setminus \bigcup_{j=1}^{\infty} \bigcup \mathcal{H}_j\big) = 0$. Thus we set $\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$ and we are done.

The general case. We find a sequence of bounded disjoint open sets $(G_j)_{j=1}^{\infty}$ such that $\mu\big(\mathbf{R}^n \setminus \bigcup_{j=1}^{\infty} G_j\big) = 0$. Then $\mu(G_j) < \infty$ for every $j \in \mathbf{N}$ and we proceed as in the proof of Theorem 1.1 □

1.2 Differentiation of measures

Notation. The symbol $\mathcal{B}$ stands for the family of all closed balls in $\mathbf{R}^n$.

Definition. Let ν and μ are measures on $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. Then we define

- **upper derivative of ν with respect to μ at x** by

$$\overline{D}(\nu, \mu, x) = \lim_{r\to 0+} \left(\sup\{\nu(B)/\mu(B);\ x \in B,\ B \in \mathcal{B},\ \operatorname{diam} B < r\}\right),$$

  if the term at the right side is defined,

- **lower derivative of ν with respect to μ at x** by

$$\underline{D}(\nu, \mu, x) = \lim_{r\to 0+} \left(\inf\{\nu(B)/\mu(B);\ x \in B,\ B \in \mathcal{B},\ \operatorname{diam} B < r\}\right),$$

  if the term at the right side is defined,

- **derivative of ν with respect to μ at x** (denoting $D(\nu, \mu, x)$) as the common value of $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$, if it is defined.

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Remark. The value $\overline{D}(\nu, \mu, x)$ ($\underline{D}(\nu, \mu, x)$) is well defined if and only if

$$\forall B \in \mathcal{B},\ x \in B\colon \mu(B) > 0.$$

Theorem 1.6. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfy Vitali theorem. Then $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$ exist μ-a.e.*

Proof. Denote

$$\begin{aligned} M &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) \text{ is not defined}\},\\ \mathcal{V} &= \{B \in \mathcal{B};\ \mu(B) = 0\}. \end{aligned}$$

The family $\mathcal{V}$ is a Vitali cover of M. We find a countable disjoint system $\mathcal{A} \subset \mathcal{V}$ such that $\mu(M \setminus \bigcup \mathcal{A}) = 0$. The we have

$$\mu\Big(\bigcup \mathcal{A}\Big) = \sum_{B\in\mathcal{A}} \mu(B) = 0,$$

therefore $\mu(M) = 0$.

The proof for $\underline{D}(\nu, \mu, x)$ is analogous. □

Theorem 1.7. *Let ν and μ be Radon measures on $\mathbf{R}^n$, μ satisfy Vitali theorem, $c \in (0, \infty)$, and $M \subset \mathbf{R}^n$.*

(i) *If for every $x \in M$ we have $\overline{D}(\nu, \mu, x) > c$, then $\nu^*(M) \geq c\mu^*(M)$.*

(ii) *If for every $x \in M$ we have $\underline{D}(\nu, \mu, x) < c$, then there exists $H \subset M$ such that $\mu(M \setminus H) = 0$ and $\nu^*(H) \leq c\mu^*(M)$.*

Proof. (i) Choose $\varepsilon > 0$. There exists an open set $G \subset \mathbf{R}^n$ with $M \subset G$ and $\nu(G) \leq \nu^*(M) + \varepsilon$. Set

$$\mathcal{V} = \{B \in \mathcal{B};\ B \subset G, \nu(B) > c\mu(B)\}.$$

The family $\mathcal{V}$ is a Vitali cover of M. There exists a disjoint countable subfamily $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$. Then we have

$$\begin{aligned} \nu^*(M) + \varepsilon \geq \nu(G) \geq \nu\big(\bigcup \mathcal{A}\big) &= \sum_{B \in \mathcal{A}} \nu(B) \\ &\geq \sum_{B \in \mathcal{A}} c\mu(B) = c\mu\big(\bigcup \mathcal{A}\big) \geq c\mu^*(M). \end{aligned}$$

Taking $\varepsilon \to 0+$ we get the desired inequality.

(ii) Choose $k \in \mathbf{N}$. There exists an open set $G_k \subset \mathbf{R}^n$ such that $M \subset G_k$ and $\mu(G_k) \leq \mu^*(M) + 1/k$. Set

$$\mathcal{V}_k = \{B \in \mathcal{B};\ B \subset G_k, \nu(B) < c\mu(B)\}.$$

The system $\mathcal{V}_k$ is a Vitali cover of M. Thus there exists a countable disjoint subfamily $\mathcal{A}_k \subset \mathcal{V}_k$ such that $\mu(M \setminus \bigcup \mathcal{A}_k) = 0$. Set $H_k = M \cap \bigcup \mathcal{A}_k$. Then $\mu(M \setminus H_k) = 0$, $H_k \subset M$ and we have

$$\begin{aligned} \nu^*(H_k) \leq \nu\big(\bigcup \mathcal{A}_k\big) = \sum_{B \in \mathcal{A}} \nu(B) \leq c \sum_{B \in \mathcal{A}} \mu(B) = c\mu\big(\bigcup \mathcal{A}\big) \\ \leq c\mu(G_k) \leq c\big(\mu^*(M) + \tfrac{1}{k}\big). \end{aligned}$$

Now we set $H = \bigcap_{k=1}^{\infty} H_k$. Then we have $\nu^*(H) \leq c\mu^*(M)$ and

$$\mu(M \setminus H) = \mu^*(M \setminus H) \leq \sum_{k=1}^{\infty} \mu^*(M \setminus H_k) = 0.$$

□

Theorem 1.8. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfies Vitali theorem. Then $D(\nu, \mu, x)$ is finite μ-a.e.*

Proof. Denote

$$\begin{aligned} D &= \{x \in \mathbf{R}^n;\ D(\nu, \mu, x) \in \langle 0, \infty)\}, \\ N_1 &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) \text{ is not defined}\}, \\ N_2 &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu, \mu, x) \text{ is not defined}\}, \\ N_3 &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) = \infty\}, \\ N_4 &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu, \mu, x) < \overline{D}(\nu, \mu, x)\}. \end{aligned}$$

Then we have

- $D = \mathbf{R}^n \setminus (N_1 \cup N_2 \cup N_3 \cup N_4)$,
- $\mu(N_1) = \mu(N_2) = 0$ (Theorem 1.6).

Further we define

$$\begin{aligned} A_k &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) > k\}, \\ A(r,s) &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu, \mu, x) < s < r < \overline{D}(\nu, \mu, x)\}, \quad s, r \in \mathbf{Q}^+, s < r. \end{aligned}$$

The we have

$$\begin{aligned} N_3 &= \bigcap_{k=1}^{\infty} A_k, \\ N_4 &= \bigcup \{A(r,s);\ r, s \in \mathbf{Q}^+, s < r\}. \end{aligned}$$

We show $\mu(N_3) = 0$. Choose $Q \subset N_3$ bounded. By Theorem 1.7(i) we have

$$k\mu^*(Q) \le \nu^*(Q) < \infty$$

for every $k \in \mathbf{N}$. Therefore $\mu^*(Q) = 0$ and thus also $\mu(N_3) = 0$, since N_3 is a countable union of bounded sets.

We show $\mu(N_4) = 0$. It is sufficient to show $\mu(A(r,s)) = 0$ for every $s, r \in \mathbf{Q}^+, s < r$. Choose $Q \subset A(r,s)$ bounded. By Theorem 1.7(ii) there exists $H \subset Q$ such that $\mu(Q \setminus H) = 0$ and $\nu^*(H) \le s\mu^*(Q)$. By Theorem 1.7(i) we have $r\mu^*(H) \le \nu^*(H)$. We may conclude

$$r\mu^*(Q) = r\mu^*(H) \le \nu^*(H) \le s\mu^*(Q) < \infty.$$

Since $r > s > 0$, we have $\mu^*(Q) = 0$. This implies $\mu(A(r,s)) = 0$. □

Lemma 1.9. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfies Vitali theorem. Then the mappings $x \mapsto \overline{D}(\nu, \mu, x)$, $x \mapsto \underline{D}(\nu, \mu, x)$ are μ-measurable.*

Proof. We start with the following observation.

The set

$$M(r, \alpha) = \left\{x \in \mathbf{R}^n;\ \exists B \in \mathcal{B}\colon\ \operatorname{diam} B < r \wedge x \in B \wedge \tfrac{\nu(B)}{\mu(B)} < \alpha\right\}$$

is open for every $r > 0$ and $\alpha \in \mathbf{R}$.

If $x \in M(r, \alpha)$, then there exist $y \in \mathbf{R}^n$ and $s > 0$ with $x \in \overline{B}(y, s)$, $2s < r$,

$$\frac{\nu\big(\overline{B}(y,s)\big)}{\mu\big(\overline{B}(y,s)\big)} < \alpha.$$

We find $s' > s$ such that $2s' < r$, $\nu\big(\overline{B}(y,s')\big)/\mu\big(\overline{B}(y,s')\big) < \alpha$. Now we have $x \in B(y,s') \subset M(r,\alpha)$. This finishes the proof of the observation.

The end of the lecture no. 5, 29. 10. 2024

Denote $D = \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) \text{ exists finite}\}$. The set D is μ-measurable by Theorem 1.8. For every $x \in D$ we have

$$\begin{aligned}
&\underline{D}(\nu,\mu,x) < \alpha \\
&\Leftrightarrow \exists \tau \in \mathbf{Q}, \tau > 0\ \forall r \in \mathbf{Q}, r > 0\ \exists B \in \mathcal{B} \colon\ \operatorname{diam} B < r,\ x \in B, \frac{\nu(B)}{\mu(B)} < \alpha - \tau \\
&\Leftrightarrow \exists \tau \in \mathbf{Q}, \tau > 0\ \forall r \in \mathbf{Q}, r > 0 :\ x \in M(r, \alpha - \tau).
\end{aligned}$$

The set $\{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) < \alpha\}$ is intersection of D with a Borel set. This implies that the mapping $x \mapsto \underline{D}(\nu,\mu,x)$ is μ-measurable.

Measurability of the mapping $x \mapsto \overline{D}(\nu,\mu,x)$ can be proved analogously. □

Theorem 1.10. *Let ν and μ be Radon measures on $\mathbf{R}^n$, μ satisfy Vitali theorem, $\nu \ll \mu$, and $B \subset \mathbf{R}^n$ be μ-measurable. Then we have*

$$\int_B D(\nu,\mu,x)\, d\mu(x) = \nu(B).$$

Proof. Choose $\beta \in \mathbf{R}$, $\beta > 1$. Define

$$\begin{aligned}
B_k &= \{x \in B;\ \beta^k < D(\nu,\mu,x) \le \beta^{k+1}\}, \qquad k \in \mathbf{Z}, \\
N &= \{x \in B;\ D(\nu,\mu,x) = 0\}.
\end{aligned}$$

These sets are μ-measurable by Lemma 1.9. Using Theorem 1.8 we have

$$\mu\Big(B \setminus \big(\bigcup_{k=-\infty}^{\infty} B_k \cup N\big)\Big) = 0.$$

Then we have

$$\begin{aligned}
\int_B D(\nu,\mu,x)\, d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu,\mu,x)\, d\mu(x) \le \sum_{k=-\infty}^{\infty} \beta^{k+1}\mu(B_k) \\
&\overset{\text{Theorem 1.7(i)}}{\le} \sum_{k=-\infty}^{\infty} \beta^{k+1}\beta^{-k}\nu(B_k) \le \beta\nu(B).
\end{aligned}$$

Going $\beta \to 1+$ we get

$$\int_B D(\nu,\mu,x)\, d\mu(x) \le \nu(B).$$

Now let $\beta > 1$ again. Define

$$C_k = \{x \in B;\ \beta^k \le D(\nu, \mu, x) < \beta^{k+1}\}, \qquad k \in \mathbf{Z}.$$

Besides the equality

$$\mu\Big(B \setminus \big(\bigcup_{k=-\infty}^{\infty} C_k \cup N\big)\Big) = 0,$$

we have also $\nu\big(B \setminus (\bigcup_{k=-\infty}^{\infty} C_k \cup N)\big) = 0$, since $\nu \ll \mu$. By Theorem 1.7(ii) and absolute continuity of ν with respect to μ we obtain $\nu^*(Q) \le c\mu^*(Q) < \infty$ for any $c > 0$ and $Q \subset N$ bounded. Similarly as in the proof of Theorem 1.8 we get $\nu(N) = 0$. Then we have

$$\begin{aligned}\int_B D(\nu, \mu, x)\, d\mu(x) &\ge \sum_{k=-\infty}^{\infty} \int_{C_k} D(\nu, \mu, x)\, d\mu(x) \ge \sum_{k=-\infty}^{\infty} \beta^k \mu(C_k) \\ &\overset{\text{Theorem 1.7(ii)}}{\ge} \sum_{k=-\infty}^{\infty} \beta^k \beta^{-(k+1)} \nu(C_k) = \frac{1}{\beta}\nu(B).\end{aligned}$$

Now it follows $\int_B D(\nu, \mu, x)\, d\mu(x) \ge \nu(B)$. □

1.3 Lebesgue points

Definition. Let μ be a Radon measure on $\mathbf{R}^n$. The symbol $\mathcal{L}^1_{loc}(\mu)$ denotes the set of all functions $f\colon \mathbf{R}^n \to \mathbf{C}$, which are μ-measurable and for every $x \in \mathbf{R}^n$ there exists $r > 0$ such that $\int_{B(x,r)} |f(t)|\, d\mu(t) < \infty$.

Definition. Let $f \in \mathcal{L}^1_{loc}(\mu)$. We say that $x \in \mathbf{R}^n$ is **Lebesgue point of f (with respect to μ)**, if it holds

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B}, x \in B, \operatorname{diam} B < \delta \colon \frac{\int_B |f(t) - f(x)|\, d\mu(t)}{\mu(B)} < \varepsilon.$$

Theorem 1.11. *Let μ be a Radon measure on $\mathbf{R}^n$ satisfying Vitali theorem and $f \in \mathcal{L}^1_{loc}(\mu)$. Then μ-a.e. points of f are Lebesgue points.*

Proof. Without any loss of generality we may assume that $\mu(\mathbf{R}^n) < \infty$ and $f \in \mathcal{L}^1(\mu)$. Let (C_k) be a sequence of closed discs in $\mathbf{C}$, which forms a basis of $\mathbf{C}$. We denote

$$g_k(x) := \operatorname{dist}(f(x), C_k), \qquad x \in \mathbf{R}^n.$$

The function g_k is nonnegative μ-measurable function satisfying $g_k \in \mathcal{L}^1(\mu)$. Let $\nu_k = \int g_k\, d\mu$. By Theorem 1.10 we have $D(\nu_k, \mu, x) = g_k(x)$ μ-a.e. Denote

$$P_k = \{x \in f^{-1}(C_k);\ \neg(D(\nu_k, \mu, x) = 0)\}.$$

We have $g_k = 0$ on $f^{-1}(C_k)$, therefore $\mu(P_k) = 0$. We show that every point from $\mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} P_k$ is a Lebesgue point of f.

Let $x \in \mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} P_k$. Choose $\varepsilon > 0$. We find C_k such that $f(x) \in C_k$ and $C_k \subset B(f(x), \varepsilon/2)$. For any $t \in \mathbf{R}^n$ it holds

$$|f(t) - f(x)| \le g_k(t) + \varepsilon.$$

There exists $\delta > 0$ such that

$$\forall B \in \mathcal{B},\ x \in B,\ \operatorname{diam} B < \delta : \ \frac{\int_B g_k(t)\, d\mu(t)}{\mu(B)} < \varepsilon,$$

since $D(\nu_k, \mu, x) = 0$. Take $B \in \mathcal{B}$ with $x \in B$, $\operatorname{diam} B < \delta$ we get

$$\frac{\int_B |f(t) - f(x)|\, d\mu(t)}{\mu(B)} \le \frac{\int_B g_k(t)\, d\mu(t) + \varepsilon\mu(B)}{\mu(B)} < 2\varepsilon.$$

This finishes the proof. $\square$

1.4 Density theorem

Definition. Let μ be a measure on $\mathbf{R}^n$, $A \subset \mathbf{R}^n$ be μ-measurable, and $x \in \mathbf{R}^n$. We say that $c \in [0, 1]$ is **μ-density of the set A at x**, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B},\ x \in B,\ \operatorname{diam} B < \delta \colon \left| \frac{\mu(A \cap B)}{\mu(B)} - c \right| < \varepsilon.$$

We denote $\mathrm{d}_\mu(A, x) = c$.

The end of the lecture no. 6, 12. 11. 2024

Theorem 1.12. *Let μ be a Radon measure on $\mathbf{R}^n$ satisfying Vitali theorem and $M \subset \mathbf{R}^n$ be μ-measurable. Then*

- $\mathrm{d}_\mu(M, x) = 1$ *for μ-a.e. $x \in M$,*
- $\mathrm{d}_\mu(M, x) = 0$ *for μ-a.e. $x \in \mathbf{R}^n \setminus M$.*

Proof. Define ν on $\mathbf{R}^n$ by

$$\nu(A) = \mu(A \cap M) \quad \text{for every } A \subset \mathbf{R}^n\ \mu\text{-measurable.}$$

Then we have

- $\mathrm{d}_\mu(M, x) = D(\nu, \mu, x)$, if at least one term is well defined,
- $\nu \ll \mu$,
- $\nu = \int \chi_M \, d\mu$.

By Theorem 1.10 we have $\nu = \int D(\nu, \mu, x)\, d\mu(x)$ therefore $\mathrm{d}_\mu(M, x) = D(\nu, \mu, x) = \chi_M(x)$ μ-a.e. $\square$

1.5 AC and BV functions

Remark. For $a, c, b \in \mathbf{R}, a < c < b$, it holds

- $\mathrm{V}_a^b f = \mathrm{V}_a^c f + \mathrm{V}_c^b f$,
- $|f(b) - f(a)| \leq \mathrm{V}_a^b f$.

Example. Let f be a function with continuous derivative on an interval $[a, b]$. Then $\mathrm{V}_a^b f = \int_a^b |f'(x)|\, dx$.

Remark. Let I be a closed nonempty interval. Then we have

(a) $f, g \in AC(I) \Rightarrow f + g \in AC(I)$,

(b) $f \in AC(I), \alpha \in \mathbf{R} \Rightarrow \alpha f \in AC(I)$.

Theorem 1.13. *Let $f\colon [a, b] \to \mathbf{R}$, $a < b$. Then f is absolutely continuous on $[a, b]$ if and only if f is difference of of two nondecreasing absolutely continuous functions on $[a, b]$.*

Proof. $\Rightarrow$ We denote $v(x) = \mathrm{V}_a^x f$, $x \in [a, b]$. The function v is well defined since $f \in BV([a, x])$, $x \in [a, b]$. For every $x, y \in I := [a, b]$, $x < y$, we have $v(y) - v(x) = V_x^y f$.

The function v is nondecreasing. This is obvious.

The function $v - f$ is nondecreasing. For every $x, y \in I, x < y$ we have

$$\big(v(y) - f(y)\big) - \big(v(x) - f(x)\big) = \big(v(y) - v(x)\big) - \big(f(y) - f(x)\big) = V_x^y f - \big(f(y) - f(x)\big) \geq 0.$$

The function v is absolutely continuous. Choose $\varepsilon > 0$. We find $\delta > 0$ such that

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| < \varepsilon,$$

whenever $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ are points from $I = [a, b]$ with $\sum_{j=1}^m (b_j - a_j) < \delta$. Now assume that we have points $A_1 < B_1 \leq A_2 < B_2 \leq \cdots \leq A_p < B_p$ from I satisfying $\sum_{j=1}^p (B_j - A_j) < \delta$. For each $j \in \{1, \dots, p\}$ we find points

$$A_j = a_1^j < b_1^j = a_2^j < b_2^j = \cdots < b_{m_j}^j = B_j$$

such that

$$v(B_j) - v(A_j) = V_{A_j}^{B_j} f < \sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| + \frac{\varepsilon}{p}.$$

The we have

$$\sum_{j=1}^{p} \sum_{i=1}^{m_j} (b_i^j - a_i^j) = \sum_{j=1}^{p} (B_j - A_j) < \delta$$

and

$$\sum_{j=1}^{p} |v(B_j) - v(A_j)| < \sum_{j=1}^{p} \Big(\sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| + \frac{\varepsilon}{p} \Big) < \varepsilon + \varepsilon = 2\varepsilon$$

Now we can write $f = v - (v - f)$. □

Remark. Let $F\colon \mathbf{R} \to \mathbf{R}$ be nondecreasing function which is continuous at each point from the right. Then there exists a Radon measure ν_F such that F is the distribution function of ν_F, i.e.,

$$\nu_F\big((a, b]\big) = F(b) - F(a), \qquad a, b \in \mathbf{R}, a < b.$$

Lemma 1.14. *Let $f\colon (a, b) \to \mathbf{R}$, $x_0 \in (a, b)$, and $f'(x_0) \in \mathbf{R}$. Then we have*

$$\lim_{\substack{[x_1,x_2]\to[x_0,x_0] \\ x_1 \le x_0 \le x_2, x_1 \ne x_2}} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0).$$

Lemma 1.15. *Let $f\colon (a, b) \to \mathbf{R}$ be nondecreasing on (a, b), $C(f)$ be the set of all points of continuity of f, and $A \in \mathbf{R}$. Then for every $x_0 \in C(f)$ it holds*

$$f'(x_0) = A \Leftrightarrow \lim_{\substack{[x_1,x_2]\to[x_0,x_0] \\ x_1 \le x_0 \le x_2, x_1 \ne x_2 \\ x_1, x_2 \in C(f)}} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = A.$$

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Lemma 1.16. *Let f be a distribution function of a Radon measure μ on $\mathbf{R}$, $x_0 \in C(f)$, $A \in \mathbf{R}$. Then*

$$f'(x_0) = A \Leftrightarrow D(\mu, \lambda_1, x_0) = A.$$

Theorem 1.17 (Lebesgue)**.** *Let f be a monotone function on an interval I. Then we have*

(a) *$f'(x)$ exists a.e. in I,*

(b) *f' is measurable and $\left|\int_a^b f'\right| \le |f(b) - f(a)|$, whenever $a, b \in I, a < b$,*

(c) *$f' \in \mathcal{L}^1_{loc}(I)$.*

Proof. Without any loss of generality we may assume that f is nondecreasing. Let $a, b \in I$, $a < b$. We define

$$g(x) = \begin{cases} \lim_{t\to a+} f(t), & x \in (-\infty, a] \\ \lim_{t\to x+} f(t), & x \in (a, b), \\ f(b), & x \in [b, \infty). \end{cases}$$

The function g is nondecreasing, continuous from the right at each point of $\mathbf{R}$, and $\{x \in (a, b)\ f(x) \ne g(x)\}$ is countable. By Remark there exists a Radon measure ν on $\mathbf{R}$ such that

$$\forall c, d \in \mathbf{R}, c < d\colon \nu((c, d]) = g(d) - g(c).$$

We find Radon measures μ, σ such that $\nu = \sigma + \mu$, $\sigma \ll \lambda$, and $\mu \perp \lambda$.

Claim. We have $D(\mu, \lambda, x) = 0$ λ-a.e.

Proof of Claim. There exists a Borel set N such that $\lambda(N) = 0$ and $\mu(\mathbf{R} \setminus N) = 0$. Denote $D = \{x \in \mathbf{R} \setminus N;\ D(\mu, \lambda, x) > c\}$. Then we have $0 = \mu(D) \geq c\lambda(D)$. This implies $\lambda(D) = 0$, and, consequently, $\lambda(\{x \in \mathbf{R} \setminus N;\ D(\mu, \lambda, x) > 0\}) = 0$. This gives the claim. □

Lemma 1.16 gives $g'(x) = D(\nu, \lambda, x)$ λ-a.e. in $[a, b]$, since g is continuous at each point $[a, b]$ except a countable set. For every $x_0 \in (a, b) \cap C(f)$ we have $f'(x_0) = A \in \mathbf{R}$ if and only if $g'(x_0) = A \in \mathbf{R}$ (Lemma 1.15), since $f(t) = g(t)$ whenever $t \in C(f) \cap (a, b)$. This implies (a).

(b) We have

$$\begin{aligned} f(b) - f(a) &\geq g(b) - g(a) = \nu((a, b]) \geq \sigma((a, b]) \\ &= \int_a^b D(\sigma, \lambda, x)\, d\lambda(x) \overset{\text{Claim}}{=} \int_a^b D(\nu, \lambda, x)\, d\lambda(x). \end{aligned}$$

(c) This follows from (b). □

Theorem 1.18. *Let I be a nonempty interval and $f \in BV(I)$. Then $f'(x)$ exists finite a.e. in I.*

Theorem 1.19. *Let $f\colon [a, b] \to \mathbf{R}$, $a < b$. Then the following assertions are equivalent.*

(i) $f \in AC([a, b])$.

(ii) *We have $\varphi \in \mathcal{L}^1([a, b])$ such that*

$$f(x) = f(a) + \int_a^x \varphi(t)\, dt, \qquad x \in [a, b].$$

(iii) *$f'(x)$ exists a.e. in $[a, b]$, $f' \in \mathcal{L}^1([a, b])$ and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt, \qquad x \in [a, b].$$

The end of the lecture no. 8, 26. 11. 2024

Theorem 1.20 (per partes for Lebesgue integral)**.** *Let* $f, g \in AC([a, b])$*. Then we have*

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 1.21. *Let* g *be a nonnegative function on* $[a, b]$ *with* $g \in \mathcal{L}^1([a, b])$*. Let* f *be a continuous function on* $[a, b]$*. Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b fg = f(\xi) \int_a^b g.$$

Theorem 1.22. *Let* $f \in \mathcal{L}^1([a, b])$ *and* g *be a monotone function on* $[a, b]$*. Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b fg = g(a) \int_a^\xi f + g(b) \int_\xi^b f.$$

1.6 Rademacher theorem

Definition. Let $M \subset \mathbf{R}^n$. We say that $f\colon M \to \mathbf{R}$ is **Lipschitz (on M)**, if there exists $K > 0$ such that

$$\forall x, y \in M\colon |f(x) - f(y)| \le K||x - y||.$$

Remark. If f is Lipschitz on M, then f is continuous on M.

Theorem 1.23. *Let $G \subset \mathbf{R}^n$ be open nonempty and $f\colon G \to \mathbf{R}$ be Lipschitz on G. Then f is differentiable a.e. on G.*

Lemma 1.24. *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be continuous and $i \in \{1, \dots, n\}$. Then the set*

$$D_i = \Big\{x \in \mathbf{R}^n;\ \tfrac{\partial f}{\partial x_i}(x) \text{ exists}\Big\}$$

is Borel.

Proof. We have

$$\begin{aligned}
&\frac{\partial f}{\partial x_i}(x) \text{ exists} \\
&\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall t_1, t_2 \in (-\delta, \delta) \setminus \{0\}\colon \Big|\tfrac{f(x+t_1 e_i)-f(x)}{t_1} - \tfrac{f(x+t_2 e_i)-f(x)}{t_2}\Big| < \varepsilon \\
&\Leftrightarrow \forall \varepsilon \in \mathbf{Q}^+\ \exists \delta \in \mathbf{Q}^+\ \forall t_1, t_2 \in \big((-\delta, \delta) \cap \mathbf{Q}\big) \setminus \{0\}\colon \Big|\tfrac{f(x+t_1 e_i)-f(x)}{t_1} - \tfrac{f(x+t_2 e_i)-f(x)}{t_2}\Big| < \varepsilon.
\end{aligned}$$

For $\varepsilon > 0$ and nonzero t_1, t_2 denote

$$D(\varepsilon, t_1, t_2) = \Big\{x \in \mathbf{R}^n;\ \Big|\tfrac{f(x+t_1 e_i)-f(x)}{t_1} - \tfrac{f(x+t_2 e_i)-f(x)}{t_2}\Big| < \varepsilon\Big\}.$$

The set $D(\varepsilon, t_1, t_2)$ is open since f is continuous. We have

$$D_i = \bigcap_{\varepsilon \in \mathbf{Q}^+} \bigcup_{\delta \in \mathbf{Q}^+} \bigcap_{\substack{t_1 \in (-\delta,\delta)\cap\mathbf{Q} \\ t_1 \neq 0}} \bigcap_{\substack{t_2 \in (-\delta,\delta)\cap\mathbf{Q} \\ t_2 \neq 0}} D(\varepsilon, t_1, t_2),$$

therefore D_i is Borel. $\square$

The end of the lecture no. 9, 3. 12. 2024

Lemma 1.25. *Let $\beta > 0$, $A \neq \emptyset$, $f_\alpha, \alpha \in A$, be β-Lipschitz function on $\mathbf{R}^n$ and $x \in \mathbf{R}^n$ be such that $\sup_{\alpha\in A} f_\alpha(x)$ is finite. Then the function $z \mapsto \sup_{\alpha\in A} f_\alpha(z)$ is β-Lipschitz on $\mathbf{R}^n$.*

Proof. Let $u, v \in \mathbf{R}^n$. Then $|f_\gamma(u) - f_\gamma(x)| \leq \beta||u - x||$ for any $\gamma \in A$, therefore

$$f_\gamma(u) \leq f_\gamma(x) + \beta||u - x|| \leq \sup_{\alpha\in A} f_\alpha(x) + \beta||u - x||.$$

This implies

$$\sup_{\gamma\in A} f_\gamma(u) \leq \sup_{\alpha\in A} f_\alpha(x) + \beta||u - x||,$$

thus $\sup_{\gamma\in A} f_\gamma(u) \in \mathbf{R}$. Further we have

$$f_\gamma(u) \leq f_\gamma(v) + \beta||u - v|| \leq \sup_{\alpha\in A} f_\alpha(v) + \beta||u - v|| \quad \text{for every } \gamma \in A.$$

We get

$$\sup_{\gamma\in A} f_\gamma(u) \leq \sup_{\alpha\in A} f_\alpha(v) + \beta||u - v||.$$

Thus we have

$$\sup_{\alpha\in A} f_\alpha(u) - \sup_{\alpha\in A} f_\alpha(v) \leq \beta||u - v||.$$

Interchanging the roles of u and v we obtain

$$\sup_{\alpha\in A} f_\alpha(v) - \sup_{\alpha\in A} f_\alpha(u) \leq \beta||u - v||,$$

which proves β-Lipschitzness. □

Lemma 1.26. *Let $\beta > 0$, $E \subset \mathbf{R}^n$ be nonempty and $f\colon E \to \mathbf{R}$ be β-Lipschitz. Then there exists β-Lipschitz function $\tilde{f}\colon \mathbf{R}^n \to \mathbf{R}$ with $\tilde{f}|_E = f$.*

Proof. The function $f_x\colon y \mapsto f(x) - \beta \cdot ||y - x||$ is β-Lipschitz for every $x \in E$ since

$$|f_x(u) - f_x(v)| = \big|\beta \cdot ||u - x|| - \beta \cdot ||v - x||\big| \leq \beta||u - v||$$

for every $u, v \in \mathbf{R}^n$. For every $y \in E$ we have $\sup_{x\in E} f_x(y) \leq f(y)$. Using Lemma 1.25 we get the mapping defined by

$$\tilde{f}(y) = \sup_{x\in E}(f(x) - \beta||y - x||)$$

is β-Lipschitz on $\mathbf{R}^n$. For $z \in E$ we have $\tilde{f}(z) \geq f_z(z) = f(z)$. Moreover $f_x(z) = f(x) - \beta||z - x|| \leq f(z)$, which gives $\tilde{f}(z) \leq f(z)$. Thus we prove $\tilde{f}(z) = f(z)$. □

Proof of Theorem 1.23. By Lemma 1.26 we may suppose that f is Lipschitz with the constant β on $\mathbf{R}^n$, i.e.,

$$\forall x, y \in \mathbf{R}^n\colon |f(x) - f(y)| \leq \beta||x - y||.$$

We show that f is differentiable a.e. This gives also the statement of the theorem. Let $E \subset \mathbf{R}^n$ be a set of those points where at least one partial derivative does not exist. The set $\mathbf{R}^n \setminus D_i$ is by Lemma 1.24 measurable. We use Fubini theorem and Rademacher theorem for $n = 1$ (see Remark) to get $\lambda_n(\mathbf{R}^n \setminus D_i) = 0$. Then we have $\lambda_n(E) = 0$, since $E = \bigcup_{i=1}^n (\mathbf{R}^n \setminus D_i)$.

For $p, q \in \mathbf{Q}^n$, $m \in \mathbf{N}$, denote

$$S(p,q,m) = \Big\{x \in \mathbf{R}^n;\ \forall i \in \{1,\dots,n\}\ \forall t \in (-1/m, 1/m) \setminus \{0\} \colon p_i \le \tfrac{f(x+te_i)-f(x)}{t} \le q_i\Big\}.$$

It is easy to verify that the set $S(p,q,m)$ is Borel. Let $\tilde{S}(p,q,m)$ be the set of all points of $S(p,q,m)$, where $S(p,q,m)$ has density 1. Then Theorem 1.12 gives

$$\lambda_n\big(S(p,q,m) \setminus \tilde{S}(p,q,m)\big) = 0.$$

The set

$$N = \bigcup \{S(p,q,m) \setminus \tilde{S}(p,q,m);\ p,q \in \mathbf{Q}^n, m \in \mathbf{N}\}$$

is of measure zero.

We show that f is differentiable at each point $x \in \mathbf{R}^n \setminus (E \cup N)$. Take $x \in \mathbf{R}^n \setminus (E \cup N)$ and $\varepsilon \in (0,1)$. Choose $p, q \in \mathbf{Q}^n$ such that

$$q_i - \varepsilon < p_i < \frac{\partial f}{\partial x_i}(x) < q_i, \quad i = 1,\dots,n.$$

Then there is $m \in \mathbf{N}$ such that $x \in S(p,q,m)$. Since $x \notin N$, the point x is a point of density of the set $S(p,q,m)$. Denote $S = S(p,q,m)$.

We find $\delta \in (0, 1/m)$ such that

$$\lambda_n\big(B(x,r) \setminus S\big) \le \left(\tfrac{\varepsilon}{2}\right)^n \lambda_n(B(x,r))$$

for every $r \in (0, 2\delta)$. Notice that the set $B(x, (1+\varepsilon)\tau) \setminus S$ does not contain a ball with radius $\varepsilon\tau$, whenever $\tau \in (0,\delta)$. Otherwise it would hold

$$c_n(\varepsilon\tau)^n \le (\varepsilon/2)^n c_n (1+\varepsilon)^n \tau^n,$$

a contradiction. (The symbol c_n denotes n-dimensional measure of the unit ball.)

Choose $y \in B(x,\delta)$, $y \ne x$. Denote

$$y^i = [y_1, y_2, \dots, y_i, x_{i+1}, \dots, x_n].$$

For every $i \in \{0,\dots,n\}$ define a ball $B_i = B(y^i, \varepsilon||y-x||)$. Using the preceding observation we have $B_i \cap S \ne \emptyset$. Find points $z^i \in S \cap B_i$, $i = 0,\dots,n-1$, and denote $w^i = z^{i-1} + (y_i - x_i)e_i$, $i = 1,\dots,n$.

Then we have

$$p_i \le \frac{f(w^i) - f(z^{i-1})}{y_i - x_i} \le q_i \quad \text{if } x_i \ne y_i,$$
$$p_i < \frac{\partial f}{\partial x_i}(x) < q_i,$$

therefore

$$\left|f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x)(y_i - x_i)\right| \leq (q_i - p_i)|y_i - x_i| \leq \varepsilon\|y - x\|.$$

Then we have

$$\begin{aligned}
&\left|f(y) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)(y_i - x_i)\right| \\
&\leq \sum_{i=1}^{n}\left|f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x)(y_i - x_i)\right| + \sum_{i=1}^{n}(|f(y^i) - f(w^i)| + |f(z^{i-1}) - f(y^{i-1})|) \\
&\leq n\varepsilon||y - x|| + 2n\beta\varepsilon||y - x|| = \varepsilon(n + 2n\beta)||y - x||,
\end{aligned}$$

thus the proof is finished. □

Remark. Let us mention the following two deep results of D. Preiss ([**?**]).

1. Let H be a Hilbert space and $f\colon H \to \mathbf{R}$ be Lipschitz. Then there exists $x \in H$, where f is *Fréchet differentiable*, i.e., there exists a continuous linear mapping $L\colon H \to \mathbf{R}$ such that
$$\lim_{h \to 0} \frac{|f(x + h) - f(x) - L(h)|}{||h||} = 0.$$

2. There exists a closed measure zero set $F \subset \mathbf{R}^2$ such that any Lipschitz function on $\mathbf{R}^2$ is differentiable at some point of F.

The end of the lecture no. 10, 10. 12. 2024

1.7 Lipschitz functions and $W^{1,\infty}$

Remark. We have

$$W^{1,\infty}(\Omega) = \big\{u \in L^\infty(\Omega);\ \partial_i u \in L^\infty(\Omega) \text{ (in the sense of distributions)}, i \in \{1, \dots, n\}\big\}.$$

Theorem 1.27. *Let $U \subset \mathbf{R}^n$ be open. Then $f\colon U \to \mathbf{R}$ is local Lipschitz on U if and only if $f \in W^{1,\infty}_{\text{loc}}(U)$.*

Without proof.

1.8 Maximal operator

Definition. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be measurable. For $x \in \mathbf{R}^n$ we define

$$Mf(x) = \sup_{B\in\mathcal{B}, x\in B} \frac{1}{\lambda_n(B)} \int_B |f|.$$

Theorem 1.28 (Hardy-Littlewood-Wiener)**.**

(a) *If $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, then Mf is finite a.e.*

(b) *There exists $c > 0$ such that for every $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$ we have*

$$\lambda_n(\{x \in \mathbf{R}^n;\ Mf(x) > \alpha\}) \le \frac{c}{\alpha}\|f\|_1.$$

(c) *Let $p \in (1, \infty]$. Then there exists A such that for every $f \in L^p(\mathbf{R}^n)$ we have $\|Mf\|_p \le A\|f\|_p$.*