Research statement
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Summary

My research interest lies in a mathematical analysis of partial differential equations, especially those arising in fluid mechanics. I primarily study the long-time dynamics of their weak solutions by means of global attractors. Such a question closely relates to investigating the uniqueness of a weak solution and some kind of additional higher regularity in time or space. Recently, I have also been interested in linearization techniques and the stability of steady solutions. In the future, I would like to carry on with deeper problems related to long-time behaviour in fluid dynamics or expand my experience to the stability of different systems.

Past research

I started to focus on differential equations during my master's studies. More elaborately, I studied the existence results for various ordinary or partial differential equations using tools from fixed-point theory. Already at that point, I was interested in models with more complex non-linearities such as the so-called maximal monotone graphs, i.e. monotone graphs which may also contain vertical components.

Moving on to my doctoral studies, I was curious not so much about the existence theory any more, but rather about the possible qualitative results concerning solutions to evolutionary partial differential equations. To be specific, the model of an incompressible non-Newtonian fluid in the two- or three-dimensional domain Ω was considered. The set was initially a standard bounded Lipschitz, or more regular, domain, but the results were later also extended to cover certain unbounded domains with simple geometry.

The main novelty of my work was investigating a broad class of the so-called *dynamic slip boundary condition*; a fairly new problem in continuum dynamics. To be precise, the additional evolutionary problem is prescribed on the boundary of Ω:

$$\begin{aligned} \beta\partial_t \boldsymbol{u} + [\alpha\boldsymbol{s} + \mathbb{S}(\mathrm{D}\boldsymbol{u})\vec{n}]_\tau &= \boldsymbol{h}, \\ \boldsymbol{u}\cdot\vec{n} &= 0. \end{aligned}$$

Here, α and β are given positive parameters, $\boldsymbol{h}$ is a given external force, $\vec{n}$ is the outward unit normal vector to the boundary of Ω, $[\cdot]_\tau$ denotes the tangential component, $\mathbb{S}$ is the viscous part of the Cauchy stress, and $\boldsymbol{s}$ is an additional boundary non-linearity. Let me mention that $\boldsymbol{s}$ is either a direct analogue to $\mathbb{S}$, i.e. a function of the velocity field $\boldsymbol{u}$ that satisfies certain monotonicity and growth conditions; or even a more general object such as the following maximal monotone graph:

$$|\boldsymbol{s}| \le 1 \Leftrightarrow [\boldsymbol{u}]_\tau = 0 \text{ and } |\boldsymbol{s}| > 1 \Leftrightarrow \boldsymbol{s} = \frac{[\boldsymbol{u}]_\tau}{|[\boldsymbol{u}]_\tau|} + [\boldsymbol{u}]_\tau.$$

It should be pointed out that the model of the Navier–Stokes-like system subjected to the dynamic slip boundary condition is physically meaningful in the context of molten polymers. What happens is that the boundary condition is not enforced immediately but after a certain time delay; this represents some kind of polymer memory. Although I am not a physicist, I appreciate that we study equations and models that have physical and real-life relevance, potentially even in more applied sciences.

The fundamental existence result for the usual weak solutions was established shortly before I started my doctoral studies. Naturally, the following general question arises:

How does the time derivative on the boundary affect the dynamics of the whole problem?

It should be noted that for $\beta = 0$, the dynamic boundary condition degenerates into a whole spectrum of more typical conditions depending on the value of α. For $\alpha = 0$, one obtains the perfect slip boundary condition, while $\alpha \to +\infty$ formally gives rise to the Dirichlet condition. Thus, if we find any quantitative result, we hope it to be consistent with the known results for these standard boundary conditions, i.e. some 'reasonable' dependence on our parameters α and β is expected.

Research on the long-time behaviour of solutions to differential equations is founded on the study of *global attractors*. Being a fundamental concept of dynamical systems, a global attractor $\mathcal{A}$ captures all the states toward which initial conditions evolve under the action of the solution operator. Let me now outline the key results to the problem above.

The first theorem states that $\mathcal{A}$ exists whenever there is a weak solution satisfying the *energy equality*. In particular, it covers three-dimensional domains and general non-linearities $\mathbb{S}$ and $\boldsymbol{s}$. The proof is based on the so-called method of *short trajectories*. Its main feature and strength is that weak solutions are not required to be unique on the whole time interval; one just needs to ensure that no solution can branch out after a certain short time. To guarantee this property, a higher time-integrability of $\boldsymbol{u}$ is proven.

The next result is about the *fractal dimension* of $\mathcal{A}$, denoted by $\dim \mathcal{A}$. The method of short trajectories also gives additional information $\dim \mathcal{A} < +\infty$ provided that $\boldsymbol{s}$ is truly a monotone *function* of $\boldsymbol{u}$, i.e. its graph is without any 'vertical jumps'.

The previous result shows only finiteness of the dimension; it lacks any meaningful quantitative character. However, we can find a much better estimate if we move into the two-dimensional setting, as the method of *Lyapunov exponents* can be applied there. For a bounded smooth domain Ω, one can show:

$$\dim \mathcal{A} \leq \frac{\kappa}{\min^{3/2}\{1, \alpha \operatorname{diam} \Omega\}} \Big(1 + \frac{\beta}{\operatorname{diam} \Omega}\Big) G,$$

where κ is some universal constant and G is a non-dimensional Grashof number. For the channel domain $\{(x, y) \in \mathbb{R} \times \mathbb{R}; 0 < y < L\}$, we have a similar upper bound:

$$\dim \mathcal{A} \leq \kappa \Big(\frac{32}{\pi^2} + \frac{8\beta}{L(1 + 8\alpha L)}\Big)^2 G^2.$$

These are fairly satisfactory and explicit estimates. Note that for $\alpha \to +\infty$, they degenerate into the best (up to the value of κ) known upper bounds in the Dirichlet setting. In addition, the estimate in the channel is also uniform for $\alpha \to 0_+$; this is essentially due to the character of the constant in the Poincaré inequality.

The main technical assumption to apply Lyapunov exponents is that the (unique) weak solution is actually strong, i.e. it has $L^\infty_{\mathrm{loc}}(0, T; W^{2,2}(\Omega))$-regularity. To show this, one uses the bootstrap argument together with the corresponding regularity for the stationary Stokes problem, which was necessary to investigate first. In comparison with the usual boundary conditions, the time derivative on the boundary forces us to require much stronger assumptions on the non-linearities $\mathbb{S}$ and $\boldsymbol{s}$, and also the right-hand side.

The by-product of the strong regularity of the stationary Stokes problem is also an L^p-theory, the existence of an analytic semigroup, and the determination of the domain of the Stokes operator. In our context, in the Hilbert setting, the domain is equal to fractional space $W^{3/2,2}_{\mathrm{div},\vec{n}}(\Omega)$ instead of the usual $W^{2,2}_{\mathrm{div},0}(\Omega)$ for the Dirichlet boundary condition.

Current and future research

The previous results are also true when considering the Navier–Stokes system in the two-dimensional domains. However, we do not have anything in the three-dimensional setting, as there is a weak solution that satisfies only the energy *inequality* instead of equality. An attempt to address the long-time behaviour of such a problem is by studying the *linearized system* around the steady state. First, the analogue of the well-known theorem from the theory of ordinary differential equations can be established: If the spectrum of the linearized operator lies in the appropriate half-plane, then the norm of a solution to the non-linear problem exponentially decays toward zero. Second, it is expected to apply the result in a simple geometry, such as two parallel plates or concentric cylinders, for some special steady states, such as the Couette or Taylor–Couette flows. This result will hopefully lead to at least some stability comparison between the dynamic, Dirichlet, and Navier boundary conditions.

Next, in view of the discussion above, there are several open questions that I would like to potentially address in the future. Naming the most interesting of them:

(i) Can we show that $\dim \mathcal{A} < +\infty$ if $\boldsymbol{s}$ is a maximal monotone graph as above?

(ii) Can we show that $\dim \mathcal{A} < +\infty$ for Ω being the half-space in $\mathbb{R}^2$?

(iii) Can we obtain at least some *lower* bounds on the attractor dimension?

(iv) How to obtain strong solutions for $\mathbb{S}$ with non-quadratic growth?

Finally, I would welcome the opportunity to broaden my expertise by studying the long-time behaviour and dynamics of other complex systems arising in mathematical physics. Let me mention the well-known Navier–Stokes–Cahn–Hilliard equations that describe fluid mixtures or models of currents and waves. An intriguing direction could also be the study of electrically conducting fluids, i.e. magnetohydrodynamics. This is useful to describe Earth's magnetic field or the solar wind and also in engineering for problems related to nuclear fusion. Besides all these macroscopic models, where we perceive fluid as a continuum, we could also investigate a mesoscopic model, i.e. kinetic equations, where we work with a system of N particles. This concept is interesting, as some effects are not captured by fluid models since they are kinetic by nature. As an example of such a phenomenon in plasma, recall the so-called Landau damping, which prevents an instability from developing.