4 Quotient field

Let $\mathbf{R}$ be a domain, and $\mathbf{M} = \mathbf{R} \setminus \{0\}$ and $\sim$ a relation on $\mathbf{R} \times \mathbf{M}$ given by $(a,b) \sim (c,d) \Leftrightarrow ad = bc$.

4.1. Prove that the relation $\sim$ is an equivalence.

We define the *fraction* $\frac{a}{b}$ to be the equivalence class $[(a,b)]_\sim = \{(c,d) \colon (c,d) \sim (a,b)\}$. The *quotient field* $\mathbb{Q}$ consists of the set $\mathbb{Q}$ of all fractions, together with the operations and constants:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{ad}, \quad -\frac{a}{b} = \frac{-a}{b}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}, \quad 0 = \frac{0}{1}, \quad 1 = \frac{1}{1}.$$

4.2. Prove that the operations on $\mathbf{R} \times \mathbf{M}$ are correctly defined.

Solutions: It is enough to show for each $\frac{a}{b} = \frac{\tilde{a}}{\tilde{b}}$, $\frac{c}{d} = \frac{\tilde{c}}{\tilde{d}}$ *that* $\frac{a}{b} \pm \frac{c}{d} = \frac{\tilde{a}}{\tilde{b}} \pm \frac{\tilde{c}}{\tilde{d}}$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{\tilde{a}}{\tilde{b}} \cdot \frac{\tilde{c}}{\tilde{d}}$.

4.3. Prove for each $a, c, e \in \mathbf{R}$ and $b, d, f \in \mathbf{M}$

(a) $\frac{ad}{bd} = \frac{a}{b}$,

(b) associativity of addition: $\frac{a}{b} + (\frac{c}{d} + \frac{e}{f}) = (\frac{a}{b} + \frac{c}{d}) + \frac{e}{f}$,

(c) commutativity of addition: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$,

(d) additive identity: $\frac{a}{b} + \frac{0}{1} = \frac{a}{b}$,

(e) additive inverse: $\frac{a}{b} + \frac{-a}{b} = 0$,

(f) associativity and commutativity of multiplication $\cdot$,

(g) unity element: $\frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b}$,

(h) distributivity: $\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{ac}{bd} + \frac{ae}{bf}$,

(i) $\frac{b}{d}$ is the multiplicative inverse of $\frac{d}{b}$.

4.4. Prove that the quotient field $(\mathbb{Q}, +, -, \cdot, 0)$ is a field and $(\mathbf{R}, +, -, \cdot, 0)$ is its subdomain where we identify elements $r \in \mathbf{R}$ with $\frac{r}{1}$.

4.5. Describe the quotient fields of the domains

(a) integers $\mathbb{Z}$,

(b) real polynomials $\mathbb{R}[x]$

(c)* arbitrary field F.

Solutions: (a) $\mathbb{Q}$, *(b) rational functions, (c)* F.

4.6. Let $\mathbb{Z}[i] = \{a + ib \colon a, b \in \mathbb{Z}\}$ (the Gaussian integers). Prove that

(a) $\mathbb{Z}[i]$ forms a subring of the field of complex numbers $\mathbb{C}$,

(b) $(\mathbb{Z}[i], +, -, \cdot, 0)$ is a domain,

(c)* $\mathbb{Q}[i] = \{a + ib \colon a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$ is a quotient field of $\mathbb{Z}[i]$.

4.7. (a)* Show that the empty set cannot be a ring.

(b)* Discuss the one-element ring. Is it a domain? Is it a field?