8 Divisibility of polynomials

8.1. Find irreducible decompositions in the domains $\mathbb{R}[x]$, $\mathbb{C}[x]$, $\mathbb{Z}[x]$ of polynomials

(a) $6x - 6$,

(b) $2x^2 + 2$.

Solutions: (a) irreducible in $\mathbb{R}[x]$, $\mathbb{C}[x]$, $2 \cdot 3 \cdot (x - 1)$ *in* $\mathbb{Z}[x]$, *(b) irreducible in* $\mathbb{R}[x]$, $(2x + 2i) \cdot (x - i)$ *in* $\mathbb{C}[x]$, $2 \cdot (x^2 + 1)$ *in* $\mathbb{Z}[x]$.

8.2. Calculate $\gcd(f, g)$ in $\mathbb{Z}[x]$ if

(a) $f = 6x^3 - 6$, $g = 8x^2 - 8$,

(b) $f = 6x^2 + 3x - 3$, $g = 6x^2 + 6x$.

Solutions: (a) $2(x - 1)$ *(b)* $3(x + 1)$.

8.3. Let $\mathcal{R}$ be a UFD, $\mathcal{Q}$ its quotient field, $f = \sum_{i=0}^{n} a_i x^i \in R[x]$, and $\frac{r}{s} \in Q$ be a root of f such that $r, s \in R$ are coprime. Prove that

(a) $r \mid a_0 s^n$ and so $r \mid a_0$,

(b) $s \mid a_n r^n$ and so $s \mid a_n$.

The previous exercise proved Proposition 6.9 from the lecture notes so we can apply it to search rational roots of polynomials:

8.4. Find all rational roots of the given polynomials in $\mathbb{Z}[x]$:

(a) $3x^5 - 2x^2 + x + 1$,

(b) $x^3 - 7x^2 + 11x + 3$,

(c) $2x^3 - x^2 + 3$.

Solutions: (a) no rational root, (b) 3, *(c)* -1.

8.5. Let $\mathbf{R}$ be a UFD, $f = \sum_{i=0}^{n} a_i x^i$ be a primitive polynomial in $\mathbf{R}[x]$, and suppose that there exists an irreducible element $p \in R$, such that $p \mid a_0, p \mid a_1, \ldots, p \mid a_{n-1}$, and $p^2 \nmid a_0$. Prove that f is irreducible in $\mathbf{R}[x]$.

Hint: $f = gh$ *where* $g = \sum_{i=0}^{k} g_i x^i$, $h = \sum_{i=0}^{l} h_i x^i \in R[x] \setminus \{R\}$ *show (a)* $p \mid g_0$ *or* $p \mid h_0$ *(b)* $p \mid g_i$ *by induction for* $i > 0$ *if* $p \mid g_0$.

The previous assertion is called Eisenstein's criterion (Theorem 6.10 in the lecture notes).

8.6. Using either 8.3 or 8.5 prove that the following polynomials are irreducible:

(a) $x^3 + x^2 + x + 3$ in $\mathbb{Z}[x]$,

(b) $4x^3 - 15x^2 + 60x + 180$ in $\mathbb{Z}[x]$,

(c) $\frac{10}{17}x^8 + 5x^6 + \frac{9}{2}x^5 - 12x^4 + \frac{4}{3}x - 6$ in $\mathbb{Q}[x]$.

Hint: (a) show that the polynomial has no rational root, (b) apply the Eisenstein's criterion for the prime 5, *(c) take associate polynomial in* $\mathbb{Z}[x]$ *which is primitive and then apply the Eisenstein's criterion for the prime* 17.