On structure of group modules

M. T. Koşan and J. Žemlička

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- Define for all $\sum_{g \in G} m_g g$, $\sum_{g \in G} n_g g \in MG$ and $\sum_{g \in G} r_g g \in RG$ operations on MG:

$$\sum_{g \in G} m_g g + \sum_{g \in G} n_g g = \sum_{g \in G} (m_g + n_g) g,$$

$$(\sum_{g \in G} m_g g) \cdot (\sum_{g \in G} r_g g) = \sum_{g \in G} (\sum_{hh'=g} m_h r_{h'}) g.$$

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- MG has structure of a module over RG and R.

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  (M is called *regular* if for each $m \in M$ there is $f \in \mathrm{Hom}(M, R)$ such that $m = mf(m)$.)
- MG is an injective RG-module iff M is a injective and G is finite

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The functor $- \otimes_{RH} RG : RH - \mathbf{Mod} \to RG - \mathbf{Mod}$ is exact, preserves direct limits, and $A \otimes_{RH} RG \neq 0$ for each nonzero RH-module A.

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- **Theorem**

  MG is a flat RG-module iff M is a flat R-module.

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If $\mathrm{Soc}(MG_{RG}) \neq 0$, *then* $\mathrm{Soc}(M_R) \neq 0$.

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Let G *be a finite group with order invertible in* $End_R(M)$*. Then* MG_{RG} *is semiartinian iff* M_R *is semiartinian.*

Lemma

Let $M \neq 0$. If either

1. *G is an infinite cyclic group, or*
2. *G contains an infinite strictly increasing chain of finite subgroups,*

then MG_{RG} is not artinian.

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Let M be a nonzero artinian module.

- If $G = \mathbb{Z}_{p^\infty}$ is a Prüfer p-group for a prime p, then G is a periodic artinian group and MG_{RG} is non-artinian.

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- If $G = \mathbb{Z}_{p^\infty}$ is a Prüfer p-group for a prime p, then G is a periodic artinian group and MG_{RG} is non-artinian.
- If G is an infinite locally finite group, then MG_{RG} is non-artinian.
- If G contains an infinite cyclic subgroup, then MG_{RG} is non-artinian.

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Lemma

1. *If M is artinian (noetherian) and G is finite, then MG_{RG} is artinian (noetherian).*
2. *If MG_{RG} is artinian then M_R is artinian and G is periodic.*

Lemma

Let S be a simple R-module, G be a group and $T = End(S_R)$. Then T is a skew-field and

1. *if SG is an artinian RG-module, then TG is a right artinian ring,*
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2. *if SG is a noetherian RG-module, then TG is a right noetherian ring.*

Theorem

Let $M \neq 0$. Then MG_{GR} is artinian iff M_R is artinian and G is finite.

Theorem

Let $M \neq 0$. If MG_{GR} is noetherian, then both M_R and G are noetherian.