Variants of absolute direct summand property

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ADS-modules

Type-ADS modules

Essentially ADS modules

Rings

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Example

(1) If every idempotent of R is central (in particular if R is commutative or reduced), then R_R is ADS.
(2) Every cyclic module over commutative ring is ADS.

A module A is *B-injective* if every homomorphism $C \to A$ for for every submodule $C \leq B$ can be extendede to a homomorphism $B \to A$.

Theorem (Alahmadi,Jain Leroy, 2012)

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Theorem (Alahmadi,Jain Leroy, 2012)

Let R be an simple ring. If R_R is ADS, then either R_R is indecomposable or R is a right self-injective regular ring.

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- $M = A \oplus B$ is a *type decomposition*, if A and B are type submodules of M.
- An R-module M is *type-ADS* if for every type decomposition $M = A \oplus B$ and every arbitrary type complement C of A, we have $M = A \oplus C$.

Example

(1) Every ADS module is type-ADS.

(2) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then R_R is type-ADS, however it is not ADS.

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Corollary

A type direct summand of a type-ADS module is type-ADS.

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(2) $\alpha(M) \leq M$ for all idempotents $\alpha \in End(E(M))$ such that $(1-\alpha)(E(M)) \cap M$ is a type direct summand of M.

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(3) For every decomposition $E(M) = E_1 \oplus E_2$ where $E_1 \cap M$ is a type direct summand of M, $M = (E_1 \cap M) \oplus (E_2 \cap M)$.

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A submodule X of M is called *fully invariant* if for every $f \in End(M)$, $f(X) \leq X$.

Lemma

Let $M = \oplus_{i \leq n} M_i$. If each M_i is type-ADS fully invariant submodule of M and M_i is $\oplus_{j \neq i} M_j$-injective for all i, then M is type-ADS.

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Example

Let $M_1 = \mathbb{Z}$ and $M_2 = \mathbb{Z}_2$ be $\mathbb{Z}$-modules. Then M_1 and M_2 are indecomposable, hence type-ADS, but $M = M_1 \oplus M_2$ is not type-ADS.

M is called an essentially ADS-module if $M = S \oplus T'$ for each decomposition $M = S \oplus T$ and each complement T' of S with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$

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Theorem (Koșan, Quynh, Ž. 2019)

Let M be an R-module.

1. *If $E(A) \ncong E(B)$ for each decomposition $M = A \oplus B$, then M is e-ADS.*
2. *If M is an e-ADS module with a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, then $A \cong B$ and the modules A and B are automorphism invariant.*

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Example

(1) Every ADS module is e-ADS.
(2) Let T be a non-divisible torsion abelian group and $M := \mathbb{Z} \oplus T$. Since $E(A) \ncong E(B)$ for every $M = A \oplus B$, M is an e-ADS abelian group and it M is not ADS, since T is not $\mathbb{Z}$-injective.
(3) Let $M := \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ for some prime p. Then M is e-ADS and $Z_p \oplus \mathbb{Z}_{p^2}$ is not e-ADS.

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(2) For every decomposition $M = S \oplus T$, if T' is a complement of S in M and T is a complement of T' in M, then $M = S \oplus T'$.

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Lemma

Let M be an e-ADS module. If M has a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, then A is e-ADS.

A module M is *trivial e-ADS* if it has no a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$.

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A module M is *trivial e-ADS* if it has no a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$.

Lemma

M is trivial e-ADS if and only if for every decomposition $M = A \oplus B$ no complement of A is a complement of B.

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(2) Either $eQ \ncong (1-e)Q$ for any idempotent $e \in R$ or $R \cong M_2(S)$ for a suitable right automorphism invariant ring S,

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Let R be a right non-singular ring and Q be its the maximal right ring of quotients. Then the following is equivalent:

(1) R is right e-ADS,

(2) Either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong M_2(S)$ for a suitable right automorphism invariant ring S,

(3) Either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong T \times M_2(S)$ for a suitable self-injective ring T and a normal right automorphism invariant ring S.