ABELIAN GROUPS WITH CHAIN CONDITIONS UP TO ISOMORPHISM

PATRICK W. KEEF, M. TAMER KOŞAN, JAN ŽEMLIČKA

Abstract. For a module G over a ring R, the concepts of iso-noetherian and iso-artinian are studied. Particularly it is shown that iso-noetherian modules over perfect rings are noetherian and the rest of paper is devoted to the the case where $R = \mathbb{Z}$, so that G is an abelian group. If G is such a group with torsion T and $A = G/T$, it is shown that G has either property if and only if it splits as $T \oplus A$ where both T and A have the corresponding property. The torsion groups satisfying either property are completely characterized, and when A is a Butler group, a complete description of when it is either iso-noetherian or iso-artinian is given.

Introduction

Let R be an associative ring with a unit element and M_R is a unitary right R-module. Recall that a module M_R is called *iso-noetherian* (resp., *iso-artinian*), if M satisfies iso-acc (resp., iso-dcc) on submodules, i.e., for every ascending (resp., descending) chain $N_1 \subseteq N_2 \subseteq \cdots$ (resp., $N_1 \supseteq N_2 \supseteq \cdots$) of submodules of M, there exists $k \geq 1$ such that N_k is isomorphic to N_i for every $i \geq k$. A ring R is called *right iso-noetherian* (resp., *right iso-artinian*), if the right module R_R is iso-noetherian (resp., iso-artinian). Iso-noetherian and iso-artinian modules, which present central notions of this paper, were recently introduced in the works [2, 3]. While the paper [2] proves basic structural properties of the notions and states several sufficient conditions for a module to be iso-noetherian ore iso-artinian, the following works [3, 4, 1] use the properties as new tools for refining the structural description of other finiteness conditions. The paper [8] then examines the analogies between noetherian and iso-noetherian rings.

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Our work continues the paper [2] by further developing a structural theory for both studying classes of modules. Namely, Section 2 summarizes our results on iso-noetherian modules over general rings. Clearly, any noetherian (resp., artinian) module is iso-noetherian (resp., iso-artinian), and we prove the converse is true for modules over right perfect rings (Theorem 2.6). It is also shown that if M is an iso-noetherian module and F is a finitely generated submodule of M, then each finitely generated submodule of M/F is noetherian (Proposition 2.2), and several cardinality bounds for iso-noetherian modules are presented (Proposition 2.3).

The rest of the paper is devoted to structural description of iso-noetherian Abelian groups (Section 3) and iso-artinian ones (Section 4), note that throughout the text, the term group will mean an additively written Abelian group. Recall that the group G is said to be *torsion-splitting* if $G = T \oplus A$, where T is the torsion part of G. In describing either iso-noetherian groups or iso-artinian groups, it is shown that G has the property if and only if it is torsion-splitting and both T and A share the corresponding property (see, Theorems 3.1 and 4.1). Determining if T has one of these two properties is straightforward. It is easily seen that T is iso-noetherian iff it is noetherian (i.e., finitely generated). With a little more effort, it is shown that T is iso-artinian iff, for almost all primes p, $T_p = 0$, and whenever $T_p \neq 0$, then pT_p is artinian (i.e., finitely co-generated).

Characterizing when a torsion-free group satisfies one of these two definitions is more complicated, and we do not solve the problem completely. If A is a torsion-free group that is iso-noetherian, then it is easily seen that it must have finite rank. On the other hand, a free group, even one of infinite rank, is clearly iso-artinian. The most tractable class of torsion-free groups are the so-called *completely decomposables.* They are those that decompose into summands of rank 1 and can also be described as those that are *balanced-projective.* An important class of groups which generalizes the completely decomposable groups is known as the *Butler groups* (see, for example, [7, Chapter 14]).

We completely describe the torsion-free Butler groups that are either iso-noetherian (see, Theorem 3.12) or iso-artinian (see, Theorem 4.10). Since the theory of Butler groups varies considerably between the finite-rank case and the infinite-rank case, our approaches to these characterizations also differ considerably. On the other hand, it is perhaps surprising that the iso-noetherian case, where A has finite rank, is more complicated that the iso-artinian case, where A may have infinite rank.

Our notation and notions are all standard and may be found in the books [5, 6, 7]. We write $N \leq M$ if N_R is a submodule of M_R. As usual, ω is the first infinite cardinal or ordinal and $\mathbb{N} = \{1, 2, 3, \dots\}$. The symbol G will typically be reserved for an arbitrary group, with $T \leq G$ its torsion; and if p is a prime, then $T_p \leq T$ will denote its p-torsion. The expression *rank of G* will refer to its torsion-free rank. The symbol $\mathbb{Z}(n)$ ($n \in \mathbb{N}$) will denote the cyclic group of order n and $\mathbb{Z}(p^\infty)$ will be the infinite cocyclic group (i.e., Prüfer group) corresponding to the prime p. Of course, for a prime p, the *p-rank* of G will be the dimension of the *p-socle*, $G[p]$, as a vector space over $\mathbb{Z}(p)$.

Any other specific concepts will be explained as needed in the sequel.

1. Preliminaries

Firstly, we review some standard ideas in torsion-free groups of finite rank. If A and B are groups, which in our applications will be torsion, we write $A \sim B$ if there are finite subgroups, $F_A \leq A$ and $F_B \leq B$ such that $A/F_A \cong B/F_B$. This is clearly an equivalence relation on the class of all groups. If M is a finite rank torsion-free group and X, Y are both free subgroups of M with M/X and M/Y torsion, then $M/X \sim M/Y$ (this follows from the fact that $(X + Y)/X \leq M/X$ and $(X + Y)/Y \leq M/Y$ are both finite with corresponding factors isomorphic to $M/(X + Y)$).

If M is a torsion-free group of finite rank, let $c(M)$ be the equivalence class of M/X under $\sim$, which is usually known as the *Richman type* of M. If M has torsion-free rank k, then the p-torsion of such an M/X will have p-rank at most k, and if M/X has j copies of $\mathbb{Z}(p^\infty)$ in its divisible subgroup, then so does M/Y whenever Y is another free subgroup with M/Y torsion. Clearly, if $L \leq M$, then we can think of $c(L)$ as being contained in $c(M)$ (if $X \leq M$ is free with M/X torsion, then $X \cap L$ is free and $L/(X \cap L)$ embeds in M/X and so is torsion).

Recall the following well known fact.

Fact 1. If M_1 and M_2 are torsion-free groups of finite rank with $M_1 \leq M_2$ and $M_1 \cong M_2$, then M_2/M_1 is finite (see, for example, [7, Chapter 12, Proposition 1.10]).

Actually, Fact 1 follows from the preceding paragraph:

If $X \leq M_1$ is free with M_1/X torsion, then $c(M_1) = c(M_2)$ easily implies the conclusion.

Proposition 1.1. *Suppose G is a torsion-free group of finite rank. If G is iso-noetherian or iso-artinian, then for one (and hence for all) free subgroups $F \leq G$ such that G/F is torsion, the p-torsion of G/F is 0 for almost all primes p.*

Proof. We establish this when G is iso-noetherian; the other argument being very similar. Let F be some free subgroup with G/F torsion. Let

$\mathcal{Q}$ be the set of primes such that $(G/F)[p] \neq 0$; by way of contradiction, assume that $\mathcal{Q}$ is infinite. Write $\mathcal{Q}$ as an ascending union $\mathcal{Q}_0 \subseteq \mathcal{Q}_1 \subseteq \mathcal{Q}_2 \subseteq \cdots$ such that for all k, $\mathcal{Q}_{k+1} \backslash \mathcal{Q}_k$ is infinite. For each $k < \omega$, let M_k be the subgroup of G containing F such that M_k/F is the direct sum of the p-torsion subgroups of G/F for all $p \in \mathcal{Q}_k$. Since each $\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k$ is infinite, it follows that each M_{k+1}/M_k is infinite, so that M_{k+1} is not isomorphic to M_k, contradicting that G is iso-noetherian. □

Proposition 1.2. *Suppose G is a torsion-free group of rank 1. Then G is iso-noetherian or iso-artinian iff the condition of Proposition 1.1 holds. That is, if $\overline{\tau} = (\tau_p)$ is the characteristic of some $x \in G$, then $\tau_p = 0$ for almost all primes p.*

Proof. Necessity follows directly from Proposition 1.1.

Suppose $x \in G$ has a characteristic $\overline{\tau} = (\tau_p)$, with almost all $\tau_p = 0$. Let $M_0 \leq M_1 \leq \cdots \leq G$; we may clearly assume that $x \in M_0$. For each k, let $\overline{\tau}^k = (\tau_p^k)$ be the characteristic of x as an element of M_k. Let $\mathcal{Q}$ be the set of primes p such that $\tau_p \neq 0$. Then $\mathcal{Q}$ is finite. Let $\mathcal{Q}' \leq \mathcal{Q}$ be those primes p such that $\tau_p^k = \infty$ for some k. Since $\mathcal{Q}'$ is finite, there is some N such that $\tau_p^N = \infty$ for all $p \in \mathcal{Q}'$. It follows that M_{k+1}/M_k is finite for all $k \geq N$. But, since these group are torsion-free of rank 1, we can come to the conclusion that $M_{k+1} \cong M_k$ for $k \geq N$. □

We consider now the case of torsion-free groups of finite rank. Note that if $A \sim B$, then A is artinian iff B is artinian. So, if G is a torsion-free group of finite rank, it makes sense to say "$c(G)$ is artinian" which essentially says that $c(G)$ uses only finitely many primes, i.e. it satisfies the condition of Proposition 1.1.

So we can ask

"when is a torsion-free group of finite rank, whose Richman-type is artinian, iso-noetherian or iso-artinian?"

We pause for some useful ideas that we will encounter.

Lemma 1.3. *Suppose G is a finite rank torsion-free group and M is a pure subgroup of G. If $F \leq G$ is a free subgroup such that G/F is torsion and $\pi : G \to G/M$ is the canonical epimorphism, then there is a short-exact sequence*

$$0 \to M/[M \cap F] \to G/F \to \pi(G)/\pi(F) \to 0.$$

Since $\pi(F)$ will be a free subgroup of $\pi(G) = G/M$, we can interpret this result as saying that there is a short exact sequence

$$0 \to c(M) \to c(G) \to c(G/M) \to 0.$$

In particular, $c(G)$ is artinian iff $c(M)$ and $c(G/M)$ are artinian.

The following is pretty clear.

Lemma 1.4. *Suppose $K \leq M$ are torsion-free groups of the same finite rank. Then $c(M) = c(K)$ if and only if M/K is finite if and only if M/K is bounded.*

So, the above result says that if $c(M) = c(K)$, then for some $n \in \mathbb{N}$ we have $mM \leq K \leq M$, i.e. M and N are *quasi-equal*, written $K \approx M$. Recall that two torsion-free groups of finite rank are said to be *quasi-isomorphic* if they are isomorphic to a pair of quasi-equal groups.

We next observe that whenever G is a torsion-free group of finite rank such that $c(G)$ is artinian, then G is both "quasi-iso-noetherian" and "quasi-iso-artinian."

Proposition 1.5. *Let G be a torsion-free group of finite rank such that $c(G)$ is artinian.*

(a) If $M_0 \leq M_1 \leq M_2 \leq \cdots \leq G$, then, for some N, $M_n \approx M_{n+1}$ for all $n \geq N$.

(b) If $G \geq M_0 \geq M_1 \geq M_2 \geq \cdots$, then, for some N, $M_n \approx M_{n+1}$ for all $n \geq N$.

Proof. We establish (a), the proof of (b) being analogous. Clearly, since the ranks are increasing and bounded by the rank of G, they must eventually be constant. Similarly, since the number of infinite co-cyclic summand of $c(G)$ is finite, and the number of such summands of $c(M_n)$ is increasing, it also must be eventually constant. Therefore, since only finitely many primes are used, the result follows from Lemma 1.4. $\square$

We will use repeatedly the following construction: Let $\hat{\mathbb{Z}}_p$ be the p-adic integers. Then $\mathbb{Z} \leq \hat{\mathbb{Z}}_p$ and $\mathbb{Q} \leq \mathbb{Q}\hat{\mathbb{Z}}_p$, the latter being the field of quotients of $\hat{\mathbb{Z}}_p$. Let $\alpha \in \hat{\mathbb{Z}}_p$ be chosen so that 1 and α are linearly independent over $\mathbb{Q}$. For all $n < \omega$, choose $\sigma_n \in \mathbb{Z}$ such that $\sigma_n - \alpha \in p^n\hat{\mathbb{Z}}_p$. Let

$$B_\alpha := \mathbb{Z} + \langle (1/p^n)(\sigma_n - \alpha) : n < \omega \rangle \leq \hat{\mathbb{Z}}_p.$$

We can conclude that B_α has rank 2. Let

$$F_\alpha := \langle 1, \alpha \rangle \leq \hat{\mathbb{Z}}_p.$$

Clearly, F_α satisfies $B_\alpha/F_\alpha \cong \mathbb{Z}(p^\infty)$. We can think of B_α as the p-purification of F_α in $\hat{\mathbb{Z}}_p$. Now, since $F_\alpha \cong \mathbb{Z}^2$ (with $1 \mapsto \mathbf{e}_1, \alpha \mapsto \mathbf{e}_2$) extends to an embedding

$$B_\alpha \to \mathbb{Z}[1/p]^2$$

as a p-pure subgroup, we will often identify B_α with this image.

The next (well-known) result is a key property of this construction (cf., [7, Lemma 12.4.6]).

Lemma 1.6. *If p is a prime number and α is a unit in $\hat{\mathbb{Z}}_p$ that is a transcendental over $\mathbb{Q} \leq \mathbb{Q}\hat{\mathbb{Z}}_p$, then every endomorphism $B_\alpha \to B_\alpha$ will be multiplication by some integer.*

Proof. Let $\phi : B_\alpha \to B_\alpha$ be some endomorphism. Since $\mathbb{Z}$ is p-pure and dense in the p-adic topology, we obtain that ϕ extends to an endomorphism of $\hat{\mathbb{Z}}_p$; in particular, ϕ must be multiplication by some $\beta \in \hat{\mathbb{Z}}_p$. Since $\beta = \beta \cdot 1 = \phi(1) \in B_\alpha$ for some m, we must have $\beta = (1/p^m)(u + v\alpha)$, where $u, v \in \mathbb{Z}$. If we can show $v = 0$, then it follows that $\beta = (1/p^m)u \in \mathbb{Z}[1/p] \cap \hat{\mathbb{Z}}_p = \mathbb{Z}$, as required.

If $v \neq 0$, then

$$\frac{1}{p^m}(u\alpha + v\alpha^2) = \beta\alpha = \phi(\alpha) \in B_\alpha \leq \mathbb{Q} \oplus \mathbb{Q}\alpha$$

implies that

$$\alpha^2 = \frac{1}{v}(p^m\phi(\alpha) - u\alpha) \in \mathbb{Q} \oplus \mathbb{Q}\alpha$$

but this contradicts that α is transcendental over $\mathbb{Q}$. So ϕ must be multiplication by an integer. □

Observe in Lemma 1.6 that all that was actually necessary was that α was not a root of a quadratic equation over $\mathbb{Q}$.

Lemma 1.7. *Suppose G and H are quasi-equal finite rank torsion-free groups. If E_G, the endomorphism ring of G, is isomorphic to a subring of $\mathbb{Q}$, then E_H, the endomorphism of H, is isomorphic to the same subring.*

Proof. Let $n \in \mathbb{N}$ with $nH \leq nG \leq H \leq G$. clearly E_G and E_H are torsion-free and finite rank as abelian groups. In addition, there are natural inclusions $nE_G \leq E_H$ and $nE_H \leq E_G$, so that $E_G \approx E_H$. Since E_G is a subring of $\mathbb{Q}$, so is E_H. But, clearly, quasi-equal subrings of $\mathbb{Q}$ are, in fact, equal as desired. □

Recall that if p is a prime, then -1 is a quadratic residue modulo p iff $p = 4k + 1$ for some $k \in \mathbb{N}$; i.e. $p \equiv 1, 5 \pmod 8$.

Recall also that 2 is a quadratic residue modulo p iff $p \equiv \pm 1 \pmod 8$.

Lemma 1.8. *If $p_1, p_2, \dots, p_j$ and $q_1, q_2, \dots, q_k$ are distinct primes, then there is a prime s of the form $4k + 1$ such that each p_i is a quadratic residue modulo s but each q_i is not a quadratic residue modulo s.*

Proof. There is no loss of generality in assuming that 2 is one of these primes. We consider 2 cases:

Case 1. $p_1 = 2$: Let a_i be a quadratic residue modulo p_i for $i = 2, \dots, j$ and let b_i fail to be a quadratic residue modulo q_i for $i = 1, \dots, k$. By the Chinese Remainder Theorem, we can find an integer s such that

$$\begin{aligned} s &\equiv 1 \pmod{8} \\ s &\equiv a_i \pmod{p_i} \\ s &\equiv b_i \pmod{q_i}. \end{aligned}$$

Since s is relatively prime to all the p's and q's, by Dirichlet's Theorem, we may assume s is a prime. Clearly, s is of the form $4k+1$. By Gauss's Law of Quadratic Reciprocity, for $i = 2, \dots, j$, we have

$$\left(\frac{p_i}{s}\right)\left(\frac{s}{p_i}\right) = (-1)^{\frac{p_i-1}{2}\frac{s-1}{2}} = 1.$$

Therefore,

$$\left(\frac{p_i}{s}\right) = \left(\frac{s}{p_i}\right) = \left(\frac{a_i}{p_i}\right) = 1.$$

On the other hand, for $i = 1, \dots, k$, a similar computation with p_i replaced by q_i implies that

$$\left(\frac{q_i}{s}\right) = \left(\frac{s}{q_i}\right) = \left(\frac{b_i}{q_i}\right) = -1$$

giving the result.

Case 2. $q_1 = 2$: An analogous computation pertains with the first congruence replaced by "$s \equiv 5 \pmod{8}$". □

Finally, we will frequently use the easily verified fact that an arbitrary submodule of either an iso-noetherian module or an iso-artinian module retains that property.

2. ISO-NOETHERIAN MODULES

As we remarked in the introduction, through this section, all rings are associative with unity and all modules are unitary right modules.

We continue with an elementary observation:

Lemma 2.1. *If a module M contains a chain of finitely generated submodules $F_0 \subset F_1 \subset F_1 \subset \dots$ such that F_{i+1}/F_i is not noetherian for each i, then M is not iso-noetherian.*

Proof. Since F_{i+1}/F_i is not noetherian, there exists an infinitely generated submodule C_i satisfying $F_i \subset C_i \subset F_{i+1}$. As $F_i \not\cong C_i$ for each i, M is not iso-noetherian. □

Recall that N is an *essential submodule* of a module M if N has non-zero intersection with each non-zero $L \leq M$.

Proposition 2.2. *Let M be an iso-noetherian module. Then, there exist a finitely generated submodule F and a chain of submodules $(M_i \mid i < \omega)$ such that each finitely generated submodule containing F is isomorphic to F and*

1. *each finitely generated submodule of M/F is noetherian,*
2. *$M_0 = F$, $M_i \subseteq M_{i+1}$ for each i and $M = \bigcup_i M_i$,*
3. *M_{i+1}/M_i is essential in M/M_i and it is a direct sum of noetherian cyclic submodules.*

Proof. We remark that if M is finitely generated, then it is enough to put $F = M_i := M$ for all i. Let M be infinitely generated and $\mathcal{N}$ denote a set of all finitely generated submodules of M.

(1) By [2, Lemma 2.1], there exists $N \in \mathcal{N}$ such that every finitely generated submodule containing N is isomorphic to N. If for each $G \in \mathcal{N}$ containing N there exists $H \in \mathcal{N}$ containing G such that H/G is not noetherian, then we can construct a chain of finitely generated submodules $(F_i \mid i < \omega)$ such that F_{i+1}/F_i is not noetherian, which contradicts to the hypothesis that M is iso-noetherian by Lemma 2.1. Thus there exists a finitely generated module F containing N such that $G \cong N \cong F$ and G/F is noetherian for each finitely generated module G containing F.

(2), (3) Now we put $M_0 := F$ and assume that M_i is defined. Since all cyclic submodules of M/F are noetherian, all cyclic submodules of the factor M/M_i are noetherian as well, hence there exist a maximal sum of cyclic submodules $\bigoplus_{\alpha \in A_i} x_\alpha R$ which is essential in the module M/M_i by the maximality. It remains to show that $M = \bigcup_i M_i$. Assume that $x \in M \setminus \bigcup_i M_i$. Since the module $xR + M_0/M_0$ is noetherian, there exists k such that $xR \cap M_i = xR \cap M_k$ for each $i \geq k$. Now, it is easy to obtain that $(xR + M_k/M_k) \cap (M_{k+1}/M_k) = \{0 + M_k\}$, a contradiction as M_{k+1}/M_k is essential in M/M_k. □

A *semi-local ring* is a ring for which $R/J(R)$ is a semisimple ring, where $J(R)$ (shortly, J) is the Jacobson radical of R.

Proposition 2.3. *Let M be an iso-noetherian module over a ring R.*

1. *If R is a semi-local ring, then $Gen(M) \leq \omega$,*
2. *if R is commutative and κ is an infinite cardinal grater than the cardinality of the set of all maximal ideals, then $Gen(M) \leq \kappa$.*

Proof. Let us remark that there exist a finitely generated submodule F of M and a chain of submodules $(M_i \mid i < \omega)$ such that

$\bigoplus_{\alpha\in A_i} X_{i\alpha} \cong M/M_i$ for each i and non-zero noetherian cyclic modules $X_{i\alpha}$ by Lemma 2.2.
(1) We show that M_i is finitely generated by induction on i. Clearly, $M_0 = F$ is finitely generated by the hypothesis, so suppose that M_i is finitely generated. Then

$$\frac{M_{i+1}}{M_{i+1}J + M_i} \cong \frac{M_{i+1}/M_i}{(M_{i+1}/M_i)J} \cong \bigoplus_{\alpha\in A_i} \frac{X_{i\alpha}}{X_{i\alpha}J},$$

where $X_{i\alpha}/X_{i\alpha}J \neq 0$ as $X_{i\alpha}$ is non-zero finitely generated for each $\alpha \in A_i$. Assume that A_i is infinite, which implies that $\frac{M_{i+1}}{M_{i+1}J+M_i}$ is infinitely generated semisimple module. Then we can chose a finitely generated submodule G of M_{i+1} for which $M_i \subseteq G \subseteq M_{i+1}$ and

$$\dim\frac{G}{GJ} \geq \dim\frac{G + M_{i+1}J}{M_{i+1}J} \geq \dim\frac{G + M_{i+1}J}{M_{i+1}J + M_i} > \dim F/FJ,$$

where dim denotes the number of members of a semisimple decomposition. Since it contradicts to the fact that $G \cong F$, $M = \bigcup_i M_i$ for finitely generated modules M_i which implies $Gen(M) \leq \omega$.
(2) Similarly as in (1), we prove that $Gen(M_i) \leq \kappa$ by induction on. It is easy to see that if $Gen(M_i) \leq \kappa$, then it is enough to show that $Gen(M_{i+1}/M_i) \leq \kappa$. Note that, for each $\alpha \in A_i$, there exists a maximal ideal I such that $X_{i\alpha}/X_{i\alpha}I \neq 0$. Hence we may assume that $\operatorname{card}(A_i) > \kappa$. Then there exists a maximal ideal I such that cardinality of the set $A = \{\alpha \in A_i \mid X_{i\alpha}/X_{i\alpha}I \neq 0\}$ is greater than κ. Since $M_{i+1}/M_i/(M_{i+1}/M_i)I \cong \bigoplus_{\alpha\in A} \frac{X_{i\alpha}}{X_{i\alpha}I}$, there exists a finitely generated submodule G of M_{i+1} such that $M_i \subseteq G \subseteq M_{i+1}$ and $\dim G/GI > \dim F/FJ$ which contradicts to the fact that $G \cong F$ again. Finally, we easily say that $Gen(M) = \sum_i Gen(M_i) \leq \kappa$. $\square$

Since any submodule of an iso-noetherian module is iso-noetherian, we get the following immediate consequence of Proposition 2.3(2).

Corollary 2.4. *Let M be an iso-noetherian module over a commutative ring R and λ be the cardinality of the set of all maximal ideals. Then $Gen(N) \leq \max(\omega, \lambda^+)$ for every submodule N.*

The following consequence of Proposition 2.3(1) presents the core argument of the proof that iso-noetherian modules over perfect rings are exactly noetherian ones.

Corollary 2.5. *If M is an iso-noetherian module over a semilocal ring, then, for every submodule N, there exist a finitely generated submodule $F \subseteq N$ such that $N/F = (N/F)J$.*

Proof. Since N is iso-noetherian, $Gen(N) \leq \omega$ by Proposition 2.3(1), hence there exists a chain of finitely generated submodules $(F_i \mid i < \omega)$ such that $F_0 \cong F_i$ for each i and $N = \bigcup_i F_i$. As $N/NJ = \bigcup_i F_i + NJ/NJ$ we get

$$\begin{aligned} \dim N/NJ &\leq \sup_i \dim(F_i + NJ/NJ) \\ &\leq \sup_i \dim(F_i/F_iJ) = \dim(F_0/F_0J) < \omega. \end{aligned}$$

Thus the module N/NJ is finitely generated, which means there exists a finitely generated submodule F satisfying $F + NJ = N$. Now $N/F = (N/F)J$, as desired. □

A submodule N of a module M is *superfluous*, in case for any submodule L of M, $L + N = M$ implies $L = M$.

Theorem 2.6. *Let M be a module over a right perfect ring R. Then M is iso-noetherian iff it is noetherian.*

Proof. If N is an arbitrary submodule of an iso-noetherian module M, then there exist a finitely generated submodule $F \subseteq N$ such that $N/F = (N/F)J$ by Corollary 2.5. As R is right perfect, $N/F = (N/F)J$ is superfluous in N/F, and hence $N/F = 0$. Now $N = F$ is finitely generated, as desired.

The converse is clear. □

Since valuation rings are semiperfect, the following example shows that Theorem 2.6 does not hold for semiperfect rings:

Example 2.7. Let R be a discrete valuation domain of Krull dimension > 2 with a prime ideal P such that R/P is a discrete valuation domain of Krull dimension 2. Then

(1) R is not iso-noetherian by [2, Theorem 5.6],
(2) R/P is an iso-noetherian ring by [2, Proposition 5.7], thus it is iso-noetherian as a module over R,
(3) R/P is a non-noetherian domain.

3. ISO-NOETHERIAN ABELIAN GROUPS

Theorem 3.1. *Suppose G is a group with torsion T. The following statements are equivalent:*

(1) *G is iso-noetherian;*
(2) *$G = T \oplus A$, where T is noetherian (i.e., finite), and A is iso-noetherian and hence of finite (torsion-free) rank.*

Proof. (1) ⇒ (2). Suppose G is iso-noetherian. It immediately follows that $T \leq G$ is also iso-noetherian. Since a countably infinite group is the ascending union of finite subgroups of strictly ascending orders, it

follows that T must be finite, i.e. noetherian, so there is a splitting as indicated. Note that $A \leq G$ must also be iso-noetherian. Since a torsion-free group of countably infinite rank is the ascending union of subgroups A_k of rank k (for $k < \omega$), we obtain that A must have finite rank.
(2) $\Rightarrow$ (1). Suppose that T is finite and A is iso-noetherian. If $M_0 \leq M_1 \leq M_2 \leq \cdots \leq G$, then $M_N \cap T = M_{N+1} \cap T = \cdots$ for some N. Replacing M_i by M_{N+i}, we may assume all of the M_i have the same (finite) torsion subgroup, which we denote by $\hat{T}$. Consider the canonical projection $\pi : G \to A$. Clearly, $M_i \cong \hat{T} \oplus \pi(M_i)$ for all $i \geq 0$ and $\pi(M_0) \leq \pi(M_1) \leq \cdots \leq A \leq G$. Therefore, since A is iso-noetherian, there is a K such that $\pi(M_K) \cong \pi(M_{K+1}) \cong \cdots$. Therefore, $M_K \cong M_{K+1} \cong \cdots$, as required. □

Remark 3.2. By Theorem 3.1, a description of the iso-noetherian abelian groups reduces to the case of torsion-free groups of finite rank. Recall that a torsion-free group is *completely decomposable* if it is isomorphic to a direct sum of groups of rank 1. More generally, a torsion-free group G of finite rank is said to be a *Butler group* if one of two equivalent conditions holds: Either G is the epimorphic image of a finite rank completely decomposable group, or G embeds as a pure subgroup of a finite rank completely decomposable group (see, for example, [7, Theorem 14.1.4]). Our objective in this section is to completely describe when a group G such that G/T is a Butler group is iso-noetherian. By Theorem 3.1, this reduces to the case where G is torsion free of finite rank.

The following shows that we may ignore free summands (of finite rank).

Theorem 3.3. *Suppose A is a free group of finite rank and G is any group. The following statements are equivalent:*

1. *G is iso-noetherian;*
2. *$G \oplus A$ is iso-noetherian.*

Proof. (2) $\Rightarrow$ (1). As any subgroup of an iso-noetherian group retains that property, if $G \oplus A$ is iso-noetherian, then so is G.
(1) $\Rightarrow$ (2). Suppose that G is iso-noetherian. Basically, the obvious proof works: Let $M_0 \leq M_1 \leq \cdots \leq G \oplus A$ be an ascending chain of subgroups. Let $\pi : G \oplus A \to A$ be the usual projection and $B_k := M_k \cap G$. Then $\pi(M_k) \leq A$, which implies that it is free. Therefore, for each k, $M_k \cong \pi(M_k) \oplus B_k$. Since A is noetherian, for some N_1, we have $\pi(M_{N_1}) = \pi(M_k)$ for all $k \geq N_1$. Also, since G is iso-noetherian,

for some N_2, $B_{N_2} \cong B_k$ for all $k \geq N_2$. Setting N as the max of N_1 and N_2 shows that $M_N \cong M_k$ for all $k \geq N$, as required. □

As usual, if $n \in \mathbb{N}$ (where usually n is a prime), $\mathbb{Z}[1/n]$ will denote the fractions whose denominator is a power of n. Recall $c(G)$ denotes the Richman type of a group G.

Proposition 3.4. *If* $\{p_1, \dots, p_k\}$ *is a collection of distinct prime numbers, then the group* $G := \mathbb{Z}[1/p_1] \oplus \cdots \oplus \mathbb{Z}[1/p_k]$ *is iso-noetherian.*

Proof. Let

$$\begin{aligned} Z_i &:= \mathbb{Z}[1/p_i] \ \text{ for } i = 1, \dots, k, \\ F_i &:= \mathbb{Z} \leq Z_i, \\ F &:= F_1 \oplus \cdots \oplus F_k. \end{aligned}$$

Note that $c(G) = G/F \cong \oplus_{i \leq k} \mathbb{Z}(p_i^\infty)$. Let $M_0 \leq M_1 \leq \cdots \leq G$; leaving off a finite number of terms, we may assume all have the same rank. Since F is noetherian, leaving off a finite number of terms, we may assume that $M_n \cap F = M_0 \cap F$ for all n. Let $I \subseteq \{1, \dots, k\}$ be all i such that, for some $n < \omega$, $M_n/[M_0 \cap F]$ has an unbounded p_i-torsion component, i.e. a summand which is isomorphic to $\mathbb{Z}(p_i^\infty)$. After possibly relabelling, we may assume that $I = \{1, 2, \dots, k'\}$, where $k' \leq k$. Since there are only a finite number of such primes, after leaving off a finite number of terms, there is no loss of generality in assuming that, for all $n < \omega$,

$$M_n/[M_n \cap F] = M_n/[M_0 \cap F] \cong \left(\bigoplus_{i \leq k'} \mathbb{Z}(p_i^\infty) \right) \oplus J_n,$$

where J_n is finite.

For each $1 \leq i \leq k'$, let $W_i := \bigoplus_{j \neq i} Z_j$ and $\pi_i : G \to W_i$ be the usual projection. For each $n < \omega$ there is a short exact sequence

$$0 \to (Z_i \cap M_n)/(F_i \cap M_n) \to M_n/(F \cap M_n) \to \pi(M_n)/\pi(F \cap M_n) \to 0.$$

The middle term of this sequence has a summand which is isomorphic to $\mathbb{Z}(p_i^\infty)$. And, since $c(\pi(M_n))$ can be viewed as a subgroup of $c(W_i)$, which has no such summand, we can conclude that the left term of this sequence also has a summand which is isomorphic to $\mathbb{Z}(p_i^\infty)$. Since the left group is $c(Z_i \cap M_n)$ and $c(Z_i) \cong \mathbb{Z}(p_i^\infty)$, we can conclude that $Z_i \cap M_n$ has finite index in Z_i. And since this is true whenever $1 \leq i \leq k'$, we can conclude that if $G' = \bigoplus_{1 \leq i \leq k'} Z_i$, then $G' \cap M_n$ has finite index in G'. It follows that if we ignore some M_n at the beginning of the sequence, then we may assume that $G' \cap M_n = G' \cap M_0$ for all

$n < \omega$. Let $G'' := \bigoplus_{k<i\leq k} Z_i$ and $\pi'' : G \to G''$ be the usual projection. Consider the exact sequence

$$0 \to [M_n \cap G']/[M_n \cap F'] \to M_n/[M_n \cap F] \to \pi''(M_n)/\pi''(M_n \cap F) \to 0.$$

The middle term is $c(M_n)$ and the left term is $c(M_n \cap G')$; however up to finite summands, both of these are isomorphic to $c(G') \cong \oplus_{i\leq k'} \mathbb{Z}(p_i^\infty)$. It follows that the right-hand term must be finite, but since it is $c(\pi''(M_n))$, we can conclude that $\pi''(M_n)$ must be free.

Now, we can complete the proof. Note that all of the $\pi''(M_n)$ will have the same rank (namely $\mathrm{rank}(M_0) - k'$) and

$$\begin{aligned} M_n &\cong (M_n \cap G') \oplus \pi''(M_n) \\ &= (M_0 \cap G') \oplus \pi''(M_n) \\ &\cong (M_0 \cap G') \oplus \pi''(M_0) \cong M_0 \end{aligned}$$

for all n. □

Proposition 3.5. *If p is a prime number, then the group $G := \mathbb{Z}[1/p]^2$ is iso-noetherian.*

Proof. Let $F := \mathbb{Z}^2 \leq G$ and $M_1 \leq \cdots \leq G$ be a chain of subgroups. We need to show that eventually all of the M_n are isomorphic.

If all of the M_n have rank 1, then so does $H := \cup_{n<\omega} M_n$. Now any rank 1 subgroup of G is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}[1/p]$, both of which are iso-noetherian. Therefore, H is iso-noetherian, so that the M_n's are eventually all isomorphic. So, disregarding a finite number of initial terms, we may assume that all of M_n have rank 2.

Since F is noetherian, there is an N such that $M_n \cap F = M_N \cap F$ for all $n \geq N$. Ignoring the first N terms, we may assume $N = 1$. Since each M_n has rank 2, we have that $M_n \cap F = M_1 \cap F \cong \mathbb{Z}^2$. Therefore, since $M_n/(M_n \cap F)$ embeds in $G/F \cong \mathbb{Z}(p^\infty)^2$, we obtain that each $M_n/(M_n \cap F) \cong M_n/(M_1 \cap F)$ will be either

$$\text{finite}, \ldots (1)$$

$$\cong \mathbb{Z}(p^\infty)^2, \ldots (2)$$

or

$$\cong \mathbb{Z}(p^\infty) \oplus C_n \ldots (3)$$

where $C_n \cong \mathbb{Z}(p^{j_n})$ for some non-negative integer j_n.

First, if (1) happens for all $n \in \mathbb{N}$, then there is a $k \in \mathbb{N}$ such that $M_n \cong p^k M_n \leq (M_1 \cap F) \leq F$, which implies that each $M_n \cong \mathbb{Z}^2$. So we may suppose (1) fails for some N; so for each $n \geq N$, $M_n/(M_1 \cap F)$ is also infinite. Eliminating the first N terms, we may assume this quotient is always infinite.

Next, if (2) happens for some N, i.e.

$$M_N/(M_N \cap F) \cong M_N/(M_1 \cap F) \leq G/F \cong \mathbb{Z}(p^\infty)^2,$$

then, for all $n \geq N$, we obtain that

$$\mathbb{Z}(p^\infty)^2 \cong M_N/(M_1 \cap F) \leq M_n/(M_1 \cap F) = M_n/(M_n \cap F) \leq \mathbb{Z}(p^\infty)^2,$$

i.e. $M_n = M_N$, as desired.

Finally, we assume that, for all n,

$$M_n/(M_1 \cap F) \cong \mathbb{Z}(p^\infty) \oplus C_n,$$

where $C_n \cong \mathbb{Z}(p^{j_n})$ for some non-negative j_n. Note that, for all n, $M_1/(M_1 \cap F) \leq M_n/(M_1 \cap F)$. Hence, if $M_1 \cap F \leq B \leq M_1 \leq G$ and $B/(F_1 \cap M_1)$ is a maximal divisible subgroup of $M_1/(M_1 \cap F)$, then $B/(F_1 \cap M_1)$ is also a maximal divisible subgroup of $M_n/(M_1 \cap F)$ for all n. Clearly,

$$M_n/(M_1 \cap F) \cong B/(M_1 \cap F) \oplus C_n.$$

In addition, if $n \leq m$, then we have $j_n \leq j_m$, and $M_n = M_m$ if and only if $j_n = j_m$. Now, if there is an N such that $j_n = j_N$ for all $n \geq N$, we have $M_n = M_N$ for all $n \geq N$, as desired. Therefore, we may assume that the j_n increase without the bound. Let $H := \cup M_n$. Note that,

$$H \cap F = M_1 \cap F \cong \mathbb{Z}^2$$

and the map

$$H/(M_1 \cap F) \to G/F \cong \mathbb{Z}(p^\infty)^2$$

is an isomorphism. Hence

$$H \leq \mathbb{Z}[1/p](M_1 \cap F)$$

and

$$H/(M_1 \cap F) \leq \{\mathbb{Z}[1/p](M_1 \cap F)\}/(M_1 \cap F)$$

are both isomorphic to $\mathbb{Z}(p^\infty)^2$, which implies that

$$H = \mathbb{Z}[1/p](M_1 \cap F) \cong \mathbb{Z}[1/p]^2.$$

Hence, there is no loss of generality in replacing G by H and F by $F \cap M_1$.

Finally, we show that $B = p^{j_n} M_n$ for all n, which will complete the proof. Clearly,

$$M_n/B \cong (M_n/F)/(B/F) \cong (\mathbb{Z}(p^\infty) \oplus C_j)/\mathbb{Z}(p^\infty) \cong C_j \cong \mathbb{Z}(p^{j_n}),$$

which implies that $p^{j_n} M_n \leq B$. On the other hand, since

$$p^{-j_n} F/F = (G/F)[p^{j_n}] = (B/F)[p^{j_n}] \oplus C_n = (M_n/F)[p^n],$$

we obtain that $p^{-j_n}F \leq M_n$ which implies $F \leq p^{j_n}M_n$. The divisibility of B/F implies that

$$B = p^{j_n}B + F \leq p^{j_n}M_n + p^{j_n}M_n = p^{j_n}M_n$$

which gives that $p^{j_n}M_n = B$, as stated. ☐

We produced two examples of iso-noetherian groups in Propositions 3.4 and 3.5. Now, we produce some examples that fail to have that property.

Recall that $\hat{\mathbb{Z}}_p \cong \mathrm{End}(\mathbb{Z}_{p^\infty})$ and $\mathbb{M}_2(\hat{\mathbb{Z}}_p) \cong \mathrm{End}(\mathbb{Z}_{p^\infty}^2)$, where $\mathbb{M}_n(R)$ denotes the full matrix ring of $n \times n$ matrices over a ring R.

Proposition 3.6. *If p and q are distinct prime numbers, then the group $G := \mathbb{Z}[1/pq] \oplus \mathbb{Z}[1/p]$ is not iso-noetherian.*

Proof. Suppose that $\alpha \in \hat{\mathbb{Z}}_p$ is a p-adic integer that is a unit of $\hat{\mathbb{Z}}_p$ and transcendental over $\mathbb{Q}$. Let $B := B_\alpha \leq \mathbb{Z}[1/p]^2$ and $F := F_\alpha = \mathbb{Z}^2$. By Lemma 1.6, every endomorphism of B is multiplication by an integer.

As usual, we set $\mathbf{e}_1 = (1, 0) \in G$ and we let $M_n := B + \langle (1/q^n)\mathbf{e}_1 \rangle \leq G$ for $n < \omega$. Clearly, $B \approx M_n$. By Lemma 1.7, every endomorphism of M_n is multiplication by an integer. Hence, it will suffice to show that for all n that M_n is not isomorphic to M_{n+1}.

To get a contradiction, suppose $\phi : M_{n+1} \to M_n$ is such an isomorphism. Then ϕ is multiplication by some $y \in \mathbb{Z}$. Now, it is easy to see that $(\mathbb{Z}[1/pq] \oplus 0) \cap M_n = \langle (1/q^n)\mathbf{e}_1 \rangle$ is cyclic, and since $\phi(\langle 1/q^{n+1}\mathbf{e}_1 \rangle) = \langle 1/q^n \mathbf{e}_1 \rangle$, we can conclude that $y = \pm q$. Note that the localization $\mathbb{Z}_{(q)}M_{n+1} \leq \mathbb{Z}_{(q)}\langle 1/q^{n+1}, \alpha \rangle \leq \mathbb{Q}^2$ will be isomorphic to the free $\mathbb{Z}_{(q)}$-module $\mathbb{Z}_{(q)}^2$, which implies that $M_{n+1}/yM_{n+1} \cong \mathbb{Z}(q)^2$. Hence, $M_{n+1}/\phi(M_{n+1}) \cong M_{n+1}/M_n \cong \mathbb{Z}(q)$, and this contradiction completes the proof. ☐

Proposition 3.7. *If p is a prime number, then the group $G := \mathbb{Z}[1/p]^3$ is not iso-noetherian.*

Proof. We completely follow the construction in Lemma 1.6. Let α and β be algebraically independent transcendental units of $\hat{\mathbb{Z}}_p$. In $\hat{\mathbb{Z}}_p^2$, let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $F := \langle \mathbf{e}_1, \mathbf{e}_2, (\alpha, \beta) \rangle \cong \mathbb{Z}^3$ and H be the p-pure closure of F in $\hat{\mathbb{Z}}_p^2$. Clearly, $H \leq G$.

Claim: Any endomorphism $\phi : H \to H$ is multiplication by some integer $x \in \mathbb{Z}$.

Note that ϕ extends to a $\hat{\mathbb{Z}}_p$-module homomorphism $\hat{\mathbb{Z}}_p^2 \to \hat{\mathbb{Z}}_p^2$. Since $\phi(\mathrm{e}_1), \phi(\mathrm{e}_2), \phi((\alpha, \beta)) \in H$, there is a $k \in \mathbb{N}$ such that

$$p^k\phi(\mathrm{e}_1), p^k\phi(\mathrm{e}_2), p^k\phi((\alpha, \beta)) \in F = \langle \mathbf{e}_1, \mathbf{e}_2, (\alpha, \beta) \rangle,$$

where the matrix M, for $p^k\phi$, is of the form

$$\begin{bmatrix} c_{11}+a\alpha & c_{12}+a\beta \\ c_{21}+b\alpha & c_{22}+b\beta \end{bmatrix},$$

where a, b and the c's are integers. Since $(\alpha,\beta)M = p^k\phi(\alpha,\beta) \in F$, we obtain that

$$(c_{11}\alpha + a\alpha^2 + c_{21}\beta + b\alpha\beta, c_{12}\alpha + a\alpha\beta + c_{22}\beta + b\beta^2) \in F.$$

It easily follows that $a = b = c_{12} = c_{21} = 0$ and $c_{11} = c_{22}$. Therefore, $p^k\phi$ is multiplication by this c_{11}, which implies that ϕ is multiplication by the rational number $p^{-k}c_{11} \in \mathbb{Q}$. Since $\phi(H) \le H$ and $\hat{\mathbb{Z}}_p^2$ has no p-divisible subgroups, we obtain that $p^{-k}c_{11}$ must be an integer.

For $n < \omega$, let $M_n = H + \langle p^{-n}\mathbf{e}_1\rangle \le \mathbb{Z}[1/p]F = G$. If G were iso-noetherian, then for all n sufficiently large, we would have an isomorphism $\phi : M_{n+1} \to M_n$. Since $M_{n+1} \approx H$, ϕ must be multiplication by some integer y. But since $\phi(\langle p^{-(n+1)}\mathbf{e}_1\rangle) = \langle p^{-n}\mathbf{e}_1\rangle$, we could conclude that $y = \pm p$. However, it is readily checked that $M_{n+1}/pM_{n+1} \cong \mathbb{Z}(p)^2$, whereas $M_{n+1}/M_n \cong \mathbb{Z}(p)$. Therefore, $\phi(M_{n+1}) \ne M_n$ and this contradiction shows that G cannot be iso-noetherian. □

Proposition 3.8. *Suppose p, q and s are distinct prime numbers. If $G := \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/sq] \le \mathbb{Q}^2$, then the group G is not iso-noetherian.*

Proof. By Lemma 1.8, there exists a prime number t such that -1, p and q are quadratic residues (mod t), but s is not. Let $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1) \in G$. For $n < \omega$, let

$$M_n = t\mathbb{Z}[1/p]\mathbf{e}_1 + (t/s^n)\mathbb{Z}[1/q]\mathbf{e}_2 + \langle \mathbf{e}_1 + \mathbf{e}_2\rangle.$$

Claim: For all n, $M_n \ncong M_{n+1}$. Assume not. Let $\phi : M_{n+1} \to M_n$ be an isomorphism. Clearly, ϕ restricts to an isomorphism

$$q^\infty M_{n+1} = (t/s^{n+1})\mathbb{Z}[1/q]\mathbf{e}_2 \to q^\infty M_n = (t/s^n)\mathbb{Z}[1/q]\mathbf{e}_2,$$

which means that $\phi(t\mathbf{e}_2) = bq^kst\mathbf{e}_2$ where $k \in \mathbb{Z}$ and $b = -1$ or 1. Similarly, $\phi(t\mathbf{e}_1) = atp^j\mathbf{e}_1$, where $j \in \mathbb{Z}$ and $a = -1$ or 1. Hence

$$ap^j\mathbf{e}_1 + bq^ks\mathbf{e}_2 = \phi(\mathbf{e}_1 + \mathbf{e}_2) \in M_n,$$

which implies that

$$ap^j \equiv bq^ks \pmod{t}.$$

This, however, contradicts that a, b, p and q are quadratic residues modulo t, but s fails to have this property. □

Proposition 3.9. *Suppose p and q are distinct prime numbers. Then the group $G := \mathbb{Z}[1/p]^2 \oplus \mathbb{Z}[1/q]$ is not iso-noetherian.*

Proof. Let s be a prime number which is of the form $4k+1$ such that q is a quadratic residue modulo s, but p is not. Let $B := B_\alpha \leq \mathbb{Z}[1/p]^2$ such that B is a p-pure hull of $\langle 1, \alpha\rangle$ in $\hat{\mathbb{Z}}_p$, where α is a unit transcendental over $\mathbb{Q}$. If $Z \leq B$ is a pure subgroup of rank 1, then we can conclude that $Z \cong \mathbb{Z}$ since $B \leq \hat{\mathbb{Z}}_p$ has no p-divisible subgroups, which means that $B/Z \cong \mathbb{Z}[1/p]$. In particular, this means that any homomorphism $B \to \mathbb{Z}[1/q]$ is necessarily 0.

For each $n \in \mathbb{N}$, let

$$M_n := \langle \mathbf{e}_1 + \mathbf{e}_3\rangle + s\{(p^{-n}B) \oplus \mathbb{Z}[1/q]\}.$$

Assume that $\phi : M_n \to M_{n-1}$ is an isomorphism. Consider the natural projection $M_{n-1} \subseteq G \to \mathbb{Z}[1/q]$. Clearly, the composite

$$\phi(s(p^{-n}B)) \subseteq \phi(M_n) = M_{n-1} \to \mathbb{Z}[1/q]$$

is 0, so that $\phi(sp^{-n}B) \subseteq sp^{-(n-1)}B$. Since an endomorphism of B is multiplication by some integer, it follows that ϕ is multiplication by $\pm p$ on $s(p^{-n}B))$. We also have

$$\phi(s\mathbb{Z}[1/q]\mathbf{e}_3) = \phi(q^\infty M_n) = q^\infty M_{n-1} = s\mathbb{Z}[1/q]\mathbf{e}_3,$$

and it follows, on $s\mathbb{Z}[1/q]\mathbf{e}_3$, that ϕ is a multiplication by $\pm q^j$ for some $j \in \mathbb{Z}$. Hence

$$\phi(\mathbf{e}_1 + \mathbf{e}_3) = \pm p\mathbf{e}_1 \pm q^j\mathbf{e}_3 \in M_{n-1}$$

must satisfy

$$\pm p \equiv \pm q^j \pmod{s},$$

which cannot happen, since the right side is a quadratic residue modulo s, but the left side is not. ☐

Remark 3.10. If $G \approx H$, then G and H are each isomorphic to a subgroup of the other. Since a subgroup of an iso-noetherian group inherits that property, G is iso-noetherian iff H shares that property.

Remark 3.11. Recall that if G is a torsion-free group and $\mathbf{t}$ is a type, then $G^*(\mathbf{t})$ is the subgroup generated by all $x \in G$ such that $\tau(x) > \mathbf{t}$ and $G^*(\mathbf{t})_*$ is the purification of $G^*(\mathbf{t})$ (so that $G^*(\mathbf{t})_*/G^*(\mathbf{t})$ is the torsion subgroup of $G/G^*(\mathbf{t})$).

We have come to the main result of this section.

Theorem 3.12. *Suppose G is a torsion-free Butler group of finite rank. Then G is iso-noetherian exactly in the following three cases:*

(a) $G \cong \mathbb{Z}[1/(p_1 \cdots p_k)] \oplus A$, where $p_1, \ldots, p_k$ are distinct prime members and A is free.

(b) $G \cong \mathbb{Z}[1/p]^2 \oplus A$, where p is a prime number and A is free.

(c) G is quasi-isomorphic to $G' \cong \mathbb{Z}[1/p_1] \oplus \mathbb{Z}[1/p_2] \oplus \cdots \oplus \mathbb{Z}[1/p_k] \oplus A$, where $p_1, \ldots, p_k$ are distinct prime numbers and A is free.

Proof. By Theorem 3.3, we can ignore the free summands A. Regarding sufficiency, (a) follows from Proposition 1.2, (b) follows from Proposition 3.5 and (c) follows from Proposition 3.4 since groups which are quasi-isomorphic to iso-noetherian groups share that property.

For the necessity, suppose G is an iso-noetherian Butler group of finite rank. Let $\overline{\mathbf{0}} = \tau(\mathbb{Z})$. Obviously, $G = G(\mathbf{t})$ and there exists a decomposition $G \cong G^*(\overline{\mathbf{0}})_* \oplus A$ by [7, Theorem 14.1.7], where A is free and $G^*(\overline{\mathbf{0}})_*/G^*(\overline{\mathbf{0}})$ is finite. There is clearly no loss of generality in assuming that $G = G^*(\overline{\mathbf{0}})_*$, which means that there is a maximal linearly independent set $x_1, \dots, x_k$ such that $\tau(x_i) > \overline{\mathbf{0}}$ for each $1 \leq i \leq k$, where k is the rank of G. Hence, we have $G^*(\overline{\mathbf{0}}) \approx G$.

Suppose first that $X := \langle x \rangle_*$, for some non-zero $x \in G$, is divisible by the primes p and q. It follows from Propositions 3.6 and 3.8 that, for all $y \in G$, $\tau(y) > \overline{\mathbf{0}}$ implies that $y \in X$. Therefore, $G = G^*(\overline{\mathbf{0}})_* = X$ has rank $k = 1$, and we are in the case (a).

Now, suppose that $\tau(x) > \overline{\mathbf{0}}$ for every non-zero $x \in G$. It is easy to see that $X := \langle x \rangle_* \cong \mathbb{Z}[1/p]$ for some prime number p. Hence there are linearly independent elements x, y of G satisfying $X \cong \mathbb{Z}[1/p] \cong \langle y \rangle_* =: Y$, which implies that , $k = 2$ and $G^*(\mathbf{0}) = G$ is p-divisible by Propositions 3.7 and 3.9. Therefore, if $\mathbf{t} = \tau(\mathbb{Z}[1/p])$, then G is a $\mathbf{t}$-homogeneous Butler group. So G is completely decomposable (see [7, Corollary 14.1.5]) and we are in the case (b).

So we may assume, for all primes p, that $p^\infty G$ has rank at most 1. If $x_1, \dots x_k \in G^*(\mathbf{0})$ are linearly independent elements such that $\langle x_i \rangle_* = p_i^\infty G \cong \mathbb{Z}[1/p_i]$ (for $1 \leq i \leq k$), then we claim that

$$G^*(\mathbf{0}) = p_1^\infty G \oplus p_2^\infty G \oplus \cdots \oplus p_k^\infty G.$$

Note that, since $G = G^*(\mathbf{0})_* \approx G^*(\mathbf{0})$, verifying this claim will complete the proof.

The inclusion $\geq$ being obvious, and so we show that this containment cannot be strict. Denote the right side of this equation by X. Since X and G have the same rank, we can conclude that G/X is a torsion group. If $X \neq G^*(\mathbf{0})$, then for some prime $q \neq p_i$ $(1 \leq i \leq k)$, we have $q^\infty G \neq 0$. There is clearly a $y \in X \cap q^\infty G$ such that $q^{-1}y \in q^\infty G \setminus X$. For $j < \omega$, let $y_j := q^{-j}y$ and

$$Y_j := X + \langle y_j \rangle.$$

To show G is not iso-noetherian, it suffices to show that $Y_j \ncong Y_{j+1}$ for all $j < \omega$. So, we assume that $\phi : Y_j \to Y_{j+1}$ is such an isomorphism. For $i = 1, \dots, k$, we obtain that

$$p_i^\infty Y_j = p_i^\infty G = p_i^\infty Y_{j+1},$$

which implies $\phi(X) = X$. Hence

$$\mathbb{Z}(q^j) \cong Y_j/X \cong \phi(Y_j)/\phi(X) = Y_{j+1}/X \cong \mathbb{Z}(p^{j+1}),$$

which is our desired contradiction. □

Corollary 3.13. *Let G be a Butler group of finite rank. Then G is iso-noetherian iff it is quasi-isomorphic to an iso-noetherian completely decomposable group.*

Let us continue to produce examples of non-iso-noetherian groups.

We will fix a prime number p in the rest of the section. First, we formulate easy linear-algebraic observation.

Lemma 3.14. *Let $A_1, A_2, A_3 \le \mathbb{Z}_{p^\infty}^2$ be a triple of distinct subgroups which are isomorphic to $\mathbb{Z}_{p^\infty}$ and $\psi \in \operatorname{End}(\mathbb{Z}_{p^\infty}^2)$ be an epimorphism. If $\psi(A_i) \subseteq A_i$ for all $i = 1, 2, 3$, then there exists $\lambda \in \hat{\mathbb{Z}}_p$ satisfying $\psi = \lambda \operatorname{id}$, where id is the identity map.*

Proof. Observe that, for each $i = 1, 2, 3$, there exists $\lambda_i \in \hat{\mathbb{Z}}_p$ such that $\psi_i(a) = \lambda_i(a)$ for every $a \in A_i$ and vectors $(\alpha_i, \beta_i) \in \hat{\mathbb{Z}}_p^2$ for which

$$A_i = \{(\alpha_i(a), \beta_i(a)) \mid a \in \hat{\mathbb{Z}}_p\}.$$

Moreover, the epimorphism can be represented by a non-singular matrix $U \in M_2(\hat{\mathbb{Z}}_p)$, i.e. $\psi(v) = Uv$ for every column vector $v \in \hat{\mathbb{Z}}_p^2$. If $i \ne j$, then $A_i \ne A_j$. Thus, the vectors (α_i, β_i) and (α_j, β_j) are linearly independent eigenvectors of U corresponding to eigenvalues λ_i and λ_j. Now, it is easy to obtain that $\lambda = \lambda_1 = \lambda_2 = \lambda_3$, $U = \lambda I_2$ and so $\psi = \lambda \operatorname{id}$. □

Let M be a torsion-free group of a finite rank and $F \le M$ be a finitely generated subgroup of the same rank. Denote by $d_p(M)$ the rank of p-component of the divisible part of M/F.

Proposition 3.15. *Let G be a torsion-free iso-noetherian group and F be a finitely generated subgroup such that $G/F \cong \mathbb{Z}_{p^\infty}^2$. Then*

1. *there exist indecomposable subgroups $A_1, A_2 \le G$ such that $d_p(A_1) = d_p(A_2) = 1$ and $d_p(A_1 + A_2) = 2$.*
2. *if A_1 and A_2 are subgroups as in (1) and $A = A_1 + A_2$, then there exists an automorphism $\alpha \in \operatorname{Aut}(A)$ and a sequence of finitely generated subgroups $F_0 \le F_1 \le \dots$ such that $A = \bigcup_i F_i$, $F_{i+1}/F_i \cong \mathbb{Z}_p^2$ and*
    - (a) *$\tilde{\alpha} = p\mu \operatorname{id}$ for algebraic $\mu \in \hat{\mathbb{Z}}_p^*$, where $\tilde{\alpha} \in \operatorname{End}(A/F_0)$ is the endomorphisms induced by α,*
    - (b) *$\alpha(C) = C$ for each $C \le A$ which is indecomposable and pure, and $d_p(C) = 1$,*

(c) $A_1 \cap A_2 \leq F_0$, $\alpha(F_i) = F_{i-1}$ *for all* $i > 0$, *and* $\alpha(A_1) = A_1$, $\alpha(A_2) = A_2$, $\alpha(A_1 \cap A_2) = A_1 \cap A_2$.

Proof. (1) Let D be a finitely generated direct summand of G with a maximal possible rank, i.e. there exists $A \leq G$ such that $G = D \oplus A$, A has no non-zero finitely generated direct summand and $d_p(A) = 2$. Let $n = \operatorname{rank}(A)$. Then there are subgroups $D_1, D_2, \dots D_n$ of A of rank $n-1$ such that $\bigcap_{i=1}^{n} D_i = 0$. Let us denote by $H_1, \dots H_n \leq A$ the pure closures of $D_1, D_2, \dots D_n$ in A. Then $\bigcap_{i=1}^{n} H_i = 0$ and, for each i, A/H_i is an infinitely generated torsion-free group of the rank 1, since H_i is pure in A and A has no non-zero finitely generated direct summand. Thus $d_p(H_i) = 1$ and there is a decomposition $H_i = A_i \oplus F_i$, where A_i is pure and indecomposable and F_i is finitely generated for each i. Furthermore, if $A/(A_i + A_j)$ is infinitely generated for $i \neq j$, then $A_i + A_j/A_j \cong A_i/(A_i \cap A_j)$ is finitely generated torsion-free, which implies that $A_i = A_i \cap A_j$. Using the symmetric argument, we obtain that $A_j = A_i \cap A_j = A_i$. Since $\bigcap_{i=1}^{n} H_i \leq \bigcap_{i=1}^{n} H_i = 0$, there are $i \neq j$ such that $A/(A_i + A_j)$ is finitely generated, we may suppose that $i = 1$ and $j = 2$.

(2) Let $A = A_1 + A_2$ for groups satisfying the previous condition and fix a finitely generated module F such that $A/F \cong \mathbb{Z}_{p^\infty}^2$. Consider the natural projection $\pi : A \to A/F \cong \mathbb{Z}_{p^\infty}^2$ and define subgroups

$$B_\rho = \{(a, \rho(a)) \in \mathbb{Z}_{p^\infty}^2 \mid a \in \mathbb{Z}_{p^\infty}\}$$

for each $\rho \in \hat{\mathbb{Z}}_p$ and sets of subgroups

$$\overline{\mathcal{A}} = \{B_\rho \mid \rho \in \hat{\mathbb{Z}}_p\},$$

$$\mathcal{A} = \{\pi^{-1}(B) \leq A \mid B \in \overline{\mathcal{A}}\}.$$

Let $F_n := \pi^{-1}(\mathbb{Z}_{p^n}^2)$ for every $n \geq 0$. Then, for each pair of distinct p-adic numbers $\rho, \mu \in \hat{\mathbb{Z}}_p$, we have $B_\mu \cong B_\rho \cong \mathbb{Z}_{p^\infty}$, $B_\rho + B_\mu = \mathbb{Z}_{p^\infty}^2$ and $B_\rho \cap B_\mu = \{(a, \rho(a)) \mid [\rho - \mu](a) = 0\}$ is finite. It is easy to see that $B/F \cong \mathbb{Z}_{p^\infty}$, $B + C = A$ and $B \cap C$ is finitely generated for each distinct $B, C \in \mathcal{A}$. Now, we can define an increasing sequence $(B + F_n \mid n < \omega)$ of subgroups for every $B \in \mathcal{A}$. Clearly,

$$(B + F_{n+1})/(B + F_n) \cong (\overline{B} + \mathbb{Z}_{p^{n+1}}^2)/(\overline{B} + \mathbb{Z}_{p^n}^2) \cong \mathbb{Z}_p$$

and

$$\bigcup_n (B + F_n) = A.$$

Since A is iso-noetherian, there exists n_B for any $B \in \mathcal{A}$ such that $(B + F_{n_B}) \cong (B + F_i)$ for all $i \geq n_B$. Note that $\mathcal{A}$ is an uncountable set, and hence there exists n for which the set

$$\mathcal{B} = \{B \in \mathcal{A} \mid (B + F_n) \cong (B + F_i) \forall i \geq n\}$$

is again uncountable, i.e., without loss of generality, we may suppose that $n = 0$.

Now, we can chose an isomorphism $\varphi_B : B + F \to B + F_1$ for every $B \in \mathcal{B}$. Let us observe that it can be extended to an automorphism $\overline{\varphi}_B \in \mathrm{Aut}(E(A))$ of the injective envelope $E(A) = E(F) \cong \mathbb{Q}^{\operatorname{rank} F}$. Since $\mathrm{Aut}(E(A))$ is countable, there exists an uncountable set $\mathcal{C} \subseteq \mathcal{B}$ and an automorphism $\overline{\varphi} \in \mathrm{Aut}(E(A))$ such that $\overline{\varphi} = \overline{\varphi}_B$ for each $B \in \mathcal{C}$. Then for every pair of distinct $B, C \in \mathcal{C}$, we obtain that

$$\overline{\varphi}(A) = \overline{\varphi}(B + C) = \overline{\varphi}_B(B) + \overline{\varphi}_C(C) = \varphi_B(B) + \varphi_C(C) \subseteq A,$$

$$A = B + C \subseteq \varphi_B(B) + \varphi_C(C) = \overline{\varphi}(B + C) = \overline{\varphi}(A),$$

which shows that the restriction of $\overline{\varphi}$ on the group A forms an automorphism on A. Let us denote it by φ and put $D := \bigcap_{B \in \mathcal{C}} B$. It is easy to see that $F \subseteq D$, D is finitely generated and

$$D = \bigcap_{B \in \mathcal{C}} B \subseteq \bigcap_{B \in \mathcal{C}} \varphi_B(B) = \varphi(\bigcap_{B \in \mathcal{C}} B) = \varphi_B(D).$$

Consider the natural projection $\tilde{\pi} : A \to A/D \cong \mathbb{Z}_{p^\infty}^2$. Let $B \in \mathcal{C}$ and $D_n := \tilde{\pi}^{-1}(\mathbb{Z}_{p^n}^2)$ for every n . Since we have

$$B/D \cong \mathbb{Z}_{p^\infty}, \quad D \subseteq \varphi^n(B) \subseteq \varphi^{n+1}(B), \quad \varphi^{n+1}(B)/\varphi^n(B) \cong \mathbb{Z}_p$$

for each $n \geq 0$, we get that $\varphi^n(B)/B$ is a subgroup of $\mathbb{Z}_{p^\infty}$. Hence $\varphi^n(B)/B \cong \mathbb{Z}_{p^n}$. As B/D is a divisible subgroup, it is a direct summand in $\varphi^n(B)/D$. Then $\varphi^n(B)/D \cong B \oplus \mathbb{Z}_{p^n}$, which implies that $\varphi^n(B) = B + D_n$. This means that we can suppose, without loss of generality, that $F = D$ and $F_n = D_n$.

(a) Let $\alpha := \varphi^{-1} \in \mathrm{Aut}(A)$. Then,

$$\alpha(B) \subseteq B, \quad F + pB = B, \quad pB \subseteq \alpha(B) \text{ and } B/\alpha(B) \cong \mathbb{Z}_p$$

for each $B \in \mathcal{C}$. Furthermore, $\bigcup_n \alpha(F_n) = \alpha(\bigcup_n F_n) = \alpha(A) = A$, and hence there exists n for which $F \subseteq \alpha(F_n)$. Since $\alpha(F) \subseteq F$, the map $\overline{\alpha}(a + F) = a + \alpha(F)$ is an epimorphism of the group $A/F \cong \mathbb{Z}_{p^\infty}^2$ satisfying $\overline{\alpha}(C/F) \subseteq C/F$ for arbitrary $C \in \mathcal{C}$. By Lemma 3.14, there exists $\lambda \in \hat{\mathbb{Z}}_p$ such that $\overline{\alpha} = \lambda\,\mathrm{id}$. Since $B/F \cong \mathbb{Z}_{p^\infty}$ and

$$\begin{aligned}(B/F) \oplus \mathbb{Z}_{p^n} \cong (B + F_n)/F \quad &= \alpha(B + F_{n+1})/F \\ &= \overline{\alpha}((B/F) \oplus \mathbb{Z}_{p^{n+1}}) \\ &= (B/F) \oplus \lambda(\mathbb{Z}_{p^{n+1}}),\end{aligned}$$

we get $\mathbb{Z}_{p^n} \cong \lambda(\mathbb{Z}_{p^{n+1}})$, and hence there exists an invertible $\mu \in \hat{\mathbb{Z}}_p^*$ satisfying $\lambda = p\mu$. Now,

$$\begin{aligned} \alpha(F_n)/F = \overline{\alpha}(F_n/F) = \lambda(\mathbb{Z}_{p^n}^2) &= p\mu(\mathbb{Z}_{p^n}^2) \\ &= p\mathbb{Z}_{p^n}^2 = \mathbb{Z}_{p^{n-1}}^2 = F_{n-1}/F \subsetneq F_n/F, \end{aligned}$$

which implies that $\alpha(F_n) \subsetneq F_n$ and

$$F_n/\alpha(F_n) \cong \frac{F_n/F}{\alpha(F_n)/F} \cong \mathbb{Z}_{p^n}^2/\mathbb{Z}_{p^{n-1}}^2 \cong \mathbb{Z}_p^2.$$

Since α may be extended to automorphism of a vector space $E(A)$ over the field $\mathbb{Q}$, we obtain that it is a root of its characteristic polynomial, and hence λ is a root of the same polynomial with rational coefficients. Thus λ and $\mu = \frac{\lambda}{p}$ are algebraic, which finishes the proof of (a).

(b) Suppose that C is an indecomposable pure subgroup of A with $d_p(C) = 1$. Note that there exists n such that $C + F_n/F_n \cong \mathbb{Z}_{p^\infty}$, and hence we may suppose that $F_n = F$. Then

$$\alpha(C)+F/F = \overline{\alpha}(C+F/F) = p\mu(C+F/F) = p(C+F/F) = C+F/F.$$

Thus $\alpha(C) + F = C + F$, which implies that

$$\alpha(C)/(\alpha(C) \cap C) \cong (\alpha(C) + C)/C \leq (F + C)/C$$

and

$$C/\alpha(C) \cap C \cong C + \alpha(C)/\alpha(C) \leq F + \alpha(C)/\alpha(C).$$

As $\alpha(C)$ and C are pure subgroups, $\alpha(C) \cap C$ is a pure subgroup as well, which implies that $\alpha(C)/(\alpha(C)\cap C)$ and $C/(\alpha(C)\cap C)$ are finitely generated torsion-free and so free groups. Indecomposability of $\alpha(C)$ and C give that $C = \alpha(C) \cap C = \alpha(C)$.

(c) Note that $\alpha(A_i) = A_i$ for $i = 1, 2$ by (b), which implies that$\alpha(A_1 \cap A_2) = A_1 \cap A_2$. □

Lemma 3.16. *Let* $A_1, A_2 \leq A$ *be as in the Proposition 3.15 and suppose that* $A_1 \cap A_2 = 0$*, i.e.* $A = A_1 \oplus A_2$*. If* $C \leq A$ *is pure indecomposable and* $d_p(C) = 1$*, then*

$$\operatorname{rank}(C) = \operatorname{rank}(A_2) + \operatorname{rank}(A_1 \cap C) = \operatorname{rank}(A_1) + \operatorname{rank}(A_2 \cap C).$$

Proof. Since C is indecomposable and $\tilde{C} = C + A_1/A_1 \cong C/(A_1 \cap C)$ is a non-zero subgroup of the torsion-free group $\tilde{A} = A/A_1 \cong A_2$, the group $\tilde{C}$ is infinitely generated, hence $d_p(\tilde{C}) = 1$ and $\tilde{A}/\tilde{C}$ is finitely generated. Let $\hat{C}$ be the pure closure of $\tilde{C}$ in $\tilde{A}$. Then $\tilde{A}/\hat{C}$ is a finitely generated torsion-free group, and hence $\tilde{A}/\hat{C} = 0$ as $\tilde{A} \cong A_2$ is indecomposable. This implies that $\operatorname{rank}(C) - \operatorname{rank}(A_1 \cap C) = \operatorname{rank}(\tilde{C}) = \operatorname{rank}(\tilde{A}) = \operatorname{rank}(A_2)$ and the second equality we get if A_1 and A_2 are swapped. □

Proposition 3.17. *Let* A_1 *is an indecomposable torsion-free group of finite rank such that* $\operatorname{rank}(A_1) > 1$ *and* $A_1/G \cong \mathbb{Z}_{p^\infty}$ *for a finitely generated subgroup* G*. Then* $A_1 \oplus \mathbb{Z}[1/p]$ *is not iso-noetherian.*

Proof. Put $A_2 = \mathbb{Z}[1/p]$, $A = A_1 \oplus A_2$, and $m := \operatorname{rank}(A_1) > 1$, and assume that A is an iso-noetherian indecomposable torsion-free group.

First, we find similarly as in the proof of Proposition 3.15 an indecomposable pure subgroup $C \leq A$ such that $C \cap A_2 = 0$, $d_p(C) = 1$ and $C \neq A_1$. Suppose that G be a finitely generated subgroup of A of rank m satisfying $G \cap A_1 \neq A_1$ and $G \cap A_2 = 0$ and denote by $\hat{G}$ the pure closure of G in A. Then $d_p(\hat{G}) \in \{1, 2\}$ and there exists a decomposition $\hat{G} = D \oplus H$ such that $d_p(\hat{G}) = d_p(D)$ and H is finitely generated. If $d_p(D) = 1$, then put $C := D$, otherwise, we can find a pure subgroup C of D with $d_p(C) = 1$

It follows from Lemma 3.16 that $\operatorname{rank}(C) \geq \operatorname{rank}(A_1) = m$, hence $\operatorname{rank}(C) \in \{m, m+1\}$. If $\operatorname{rank}(C) > m$, then $\operatorname{rank}(C) = m + 1 = \operatorname{rank}(A)$, which means that $C = A$ as C is a pure subgroup of A, a contradiction. Thus $\operatorname{rank}(C) = m$, which implies that $\operatorname{rank}(C \cap A_1) \in \{m - 1, m\}$. If $\operatorname{rank}(C \cap A_1) = m = \operatorname{rank}(A_1)$, we get that $C = A_1$, a contradiction. Therefore, $\operatorname{rank}(C \cap A_1) = m - 1$ and $\operatorname{rank}(A_1/(C \cap A_1)) = 1$.

Now, by Proposition 3.15, we get an automorphism $\alpha \in \operatorname{Aut}(A)$ and a sequence of finitely generated subgroups $F_0 \leq F_1 \leq \dots$ satisfying the conditions of Proposition 3.15. Since $\alpha(C \cap A_1) = C \cap A_1$, we can easily see that the map α induces an automorphism $\hat{\alpha}$ on

$$A/(C \cap A_1) \cong A_1/(C \cap A_1) \oplus A_2 \cong \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p].$$

Let us consider $\hat{\alpha} \in \operatorname{Aut}(\mathbb{Z}[1/p]^2)$ and put $B := B_\gamma$ for a transcendental $\gamma \in \hat{\mathbb{Z}}_p$, where $\tilde{F}_0 = F_0/C \cap A_1$ corresponds to $\mathbb{Z}^2$. Since $\hat{\alpha}(B) \leq B$, we can get an integer z such that $(\hat{\alpha} - z)(B) = 0$ by Lemma 1.6. It implies that the induced endomorphism $\tilde{\alpha} = \lambda\,\mathrm{id} \in \operatorname{Aut}(A/F_0)$ has an eigenvalue z, i.e. $\lambda = z$. Thus $(\alpha - z)(A_i) \leq C \cap A_1$ is finitely generated for both $i = 1, 2$, and so $(\alpha - z)(A_i) = 0$ as A_i is indecomposable. Hence $\alpha = z\,\mathrm{id}$ and so $F_n/zF_n = F_n/\alpha(F_n) \cong \mathbb{Z}_p^{\operatorname{rank} F_n}$ for each n. As $\operatorname{rank}(A) = \operatorname{rank}(F_n) > 2$, we have got a contradiction with the assertion of Proposition 3.15 that $F_n/\alpha(F_n) \cong \mathbb{Z}_p^2$. □

Since $\mathbb{Z}[1/p]^2$ contains an indecomposable subgroup B of rank 2 and with $d_p(B) = 1$, we obtain that $B \oplus \mathbb{Z}[1/p]$ is not iso-noetherian, which gives an alternative proof of Proposition 3.7.

4. ISO-ARTINIAN ABELIAN GROUPS

Parallel to Theorem 3.1, the following reduces the question of characterizing general iso-artinian groups to the case of torsion-free groups.

Theorem 4.1. *Suppose G is a group with torsion $T \leq G$ and p-torsion $T_p \leq T$. Then G is iso-artinian if and only if*

1. *$T_p = 0$ for all but finitely many prime numbers p.*
2. *$T_p \cong E_p \oplus B_p$ for all other prime numbers p, where E_p is elementary and B_p is finitely co-generated (i.e. artinian).*
3. *$G = T \oplus A$, where A is torsion-free and iso-artinian.*

Proof. First, suppose that $G = T_p$ is a p-group. If condition (2) does not hold, then it is easy to see that G would have a subgroup X which is isomorphic to $\mathbb{Z}(p^2)^{(\omega)}$. Then, we can easily construct a descending sequence of subgroups of X which is of the form $M_n \cong \mathbb{Z}(p)^n \oplus \mathbb{Z}(p^2)^{(\omega)}$ for all $n < \omega$. Since $M_n \not\cong M_{n+1}$ for all n, we have that G is not iso-artinian.

For the converse, we suppose that the condition (2) does hold and $M_0 \geq M_1 \geq M_2 \geq \cdots$ is a descending sequence of subgroups of G. Since $pG \geq pM_0 \geq pM_1 \geq pM_2 \geq \cdots$ and pG is artinian, this descending sequence is eventually constant. Disregarding a finite number of terms, we may assume that $pM_n = pM_0$ for all n. Now, for some n, let E_{n+1} be a pM_{n+1}-high subgroup of M_{n+1}. Since $M_{n+1} \leq M_n$ and $pM_n = pM_{n+1}$, the high subgroup E_{n+1} can be extended to a pM_n-high subgroup M_n. It follows that there are subgroups $B_n \leq M_n$ and $B_{n+1} \leq M_{n+1}$ such that $M_n = E_n \oplus B_n$ and $M_{n+1} = E_{n+1} \oplus B_{n+1}$. Note that $pB_{n+1} = pM_{n+1} = pM_n = pB_n$, and hence $B_{n+1} \cong B_n$. And since we have $r(E_0) \geq r(E_1) \geq r(E_2) \geq \cdots$, these p-ranks must eventually be constant, proving that the M_n will eventually all be isomorphic.

Secondly, we suppose that G is possibly mixed and iso-artinian and show that the items (1) and (3) hold. Let $M_n := \bigoplus_{p \geq n} T_p$. Clearly, $M_1 \geq M_2 \geq \cdots$, which implies that if all M_n are eventually isomorphic, then the T_p are eventually 0, i.e. the item (1) holds. Since any subgroup of an iso-artinian group is also iso-artinian, we can conclude that T is as in (1) and (2). Therefore, T is the direct sum of a divisible group and a bounded group, which implies that there is a splitting as in (3). Furthermore, since G is iso-artinian, so is A.

Finally, suppose G satisfies all of the items (1)-(3) and we show that G is iso-artinian. Consider the canonical projection $\pi : G \to A$. Since $T_p = 0$ for all but finitely many p, it easy to see that not only each T_p, but also T will be iso-artinian. Consider an arbitrary descending sequence $M_0 \geq M_1 \geq \cdots$. Since we have already shown that T is iso-artinian, the terms in $M_0 \cap T \geq M_1 \cap T \geq \cdots$ will eventually all be isomorphic. Since each $M_n \cap T$ will be iso-artinian, it will be a direct sum of a divisible and a bounded group. Therefore, we will have $M_n \cong (M_n \cap T) \oplus \pi(M_n)$. Since A is iso-artinian, we obtain that $\pi(M_n)$ will eventually all be isomorphic. Therefore, the M_n will also eventually be isomorphic. □

We have the following immediate consequence of Theorems 3.1 and 4.1.

Corollary 4.2. *If a torsion group T is iso-noetherian, then it is iso-artinian.*

The following example which is pretty clear shows that there are iso-artinian torsion-free groups that are not iso-noetherian, since they have infinite (torsion-free) rank.

Example 4.3. Free groups are iso-artinian.

More generally, we have the following result which parallels Proposition 3.3, and whose (straightforward) proof is analogous, and hence omitted.

Proposition 4.4. *Suppose A is any free group (of arbitrary rank), and G is any group. Then G is iso-artinian if and only if $G \oplus A$ is iso-artinian.*

Remark that Proposition 4.4 essentially allows us to ignore free summands (as in Proposition 3.3).

Proposition 4.5. *Suppose G is a torsion-free group of finite rank such that $c(G)$ is artinian. Then G is iso-artinian if and only if G does not have non-isomorphic subgroups M and M' that are quasi-isomorphic.*

Proof. Necessity is an immediate consequence of Proposition 1.5(b).

For the sufficiency, suppose, by way of contradiction, that subgroups M and M' are quasi-isomorphic but not isomorphic. Since there is no loss of generality in assuming $mM \leq M' \leq M$, we have

$$M \geq M' \geq mM \geq mM' \geq m^2M \geq m^2M' \geq \cdots$$

which, clearly, demonstrates that G is not iso-artinian. □

Note the contrast between the following and Corollary 4.2:

Corollary 4.6. *Suppose G is a torsion-free group of finite rank. If G is iso-artinian, then it is iso-noetherian.*

Proof. Suppose G is iso-artinian (note that $c(G)$ is artinian) but G is not iso-noetherian. Let $M_0 \leq M_1 \leq M_2 \leq \cdots$ be a sequence which demonstrates that G is not iso-noetherian. By Corollary 1.5, there is an N such that $M_n \approx M_{n+1}$ for all $n \geq N$. Since $M_n \ncong M_{n+1}$ for some $n \geq N$, by Proposition 4.5, we obtain that G fails to be iso-artinian. □

The following computation is key to our characterization of groups that are iso-artinian.

Proposition 4.7. *If p and q are (not necessarily distinct) prime numbers, then the group $G = \mathbb{Z}[1/p]\mathbf{e}_1 \oplus \mathbb{Z}[1/q]\mathbf{e}_2 \leq \mathbb{Q}^2$ is not iso-artinian.*

Proof. Suppose first that $p \neq q$. Let s be a prime distinct from p and q. Consider $H := sG + \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$. Clearly, $G \approx H$. By Proposition 4.5, we need prove $G \ncong H$. Let $\phi : G \to H$ be an isomorphism. Note that $\mathbf{e}_1 \in p^\infty G$ which gives $\phi(\mathbf{e}_1) \in p^\infty H = \mathbb{Z}[1/p]s\mathbf{e}_1 \leq sG$. Similarly, $\phi(\mathbf{e}_2) \in q^\infty H = \mathbb{Z}[1/q]s\mathbf{e}_2 \leq sG$. Therefore, $\phi(\mathbf{e}_1 + \mathbf{e}_2) \in sG$, so that $H = \phi(G) \leq sG < H$, a contradiction.

Suppose now that $q = p$, and we use the letter p for both. As before, let $\alpha \in \hat{\mathbb{Z}}_p$ be a unit that is transcendental over $\mathbb{Q}$ and let $B := B_\alpha \leq \mathbb{Z}[1/p]^2 = G$ (in particular, any endomorphism of B is multiplication by some $k \in \mathbb{Z}$). Let $s \neq p$ be another prime and $H := sB + \langle \mathbf{e}_1 \rangle$. Clearly, $B \approx H$. Now, assuming that G is iso-artinian, there must be an isomorphism $\phi : B \to H$. Suppose ϕ is multiplication by some $k \in \mathbb{Z}$. Since

$$B \cap (\mathbb{Z}[1/p] \oplus 0) = \langle \mathbf{e}_1 \rangle = H \cap (\mathbb{Z}[1/p] \oplus 0),$$

we have $k = \pm 1$. Hence, this implies that $B = H$, which is clearly false. □

Corollary 4.8. *Suppose G is an iso-artinian torsion-free group of arbitrary rank. Then the followings hold:*

1. *If p and q are any prime numbers such that $p^\infty G \neq 0 \neq q^\infty G$, then $p^\infty G = q^\infty G$ has rank 1.*
2. *$p^\infty G = 0$ for all but finitely many prime numbers.*

Proof. Assume that (1) does not hold. Then G has a subgroup of rank 2 as in Proposition 4.7. Since this subgroup fails to be iso-artinian, so does G.

Since G is iso-artinian, so is the subgroup $\sum_{p \in \mathcal{P}} p^\infty G$, described in (1). This means that (2) must hold. □

We now give a complete description of the iso-artinian torsion-free groups of rank at most 2.

Proposition 4.9. *Suppose G is torsion-free of rank 2. Then G is iso-artinian if and only if $G \cong H \oplus \mathbb{Z}$, where H has rank 1 and $c(H)$ is artinian.*

Proof. Suppose G is iso-artinian (and torsion-free of rank 2). If, for some prime p, $c(G)$ has a summand of the form $\mathbb{Z}(p^\infty)^2$, then we obtain that $p^\infty G = G$ has rank 2, which gives that G is not iso-artinian by Corollary 4.8. Hence $c(G) \sim \mathbb{Z}({p_1}^\infty) \oplus \cdots \oplus \mathbb{Z}({p_k}^\infty)$ for some collection of distinct prime numbers $p_1, \ldots, p_k$.

If $p_i^\infty G \neq 0$ for each i, then G is of the form specified: Suppose, as in Corollary 4.8, $X \leq G$ is a pure subgroup of torsion-free rank 1 such that $X = p_i^\infty G$ for each i. It follows that $c(X) \sim c(G)$, so that $c(G/X)$ must be finite. Therefore, G/X must be free, so that G splits as indicated.

Hence, we may assume $p_i^\infty G = 0$ for some i. There is, clearly, a subgroup $G' \leq G$ of rank 2 such that $c(G') = \mathbb{Z}({p_i}^\infty)$. If we can show that G' fails to be iso-artinian, then G will also fail to be, as desired. So we replace G by G', and so we have only one prime number, which we label by p.

Let $0 \neq z \in G$ and let Z be the pure hull of $\langle z \rangle$ in G. Since $c(Z)$ can be viewed as a subgroup of $c(G)$ and $p^\infty G = 0$, we obtain that $Z/\langle z \rangle$ is finite, which means that Z must be cyclic. So, we can assume $Z = \langle z \rangle$ is cyclic and pure in G. It follows that G/Z is a rank 1 torsion-free group, and arguing as in Lemma 1.3, we must have $c(G/Z) \approx \mathbb{Z}(p^\infty)$. Now $G/Z \cong \mathbb{Z}[1/p]$, which means that Z is dense in G in the p-adic topology and G is Hausdorff in that topology since $p^\infty G = 0$. Therefore, the obvious isomorphism $\langle z \rangle \to \mathbb{Z}$ extends to an embedding $G \to \hat{\mathbb{Z}}_p$, where $\hat{\mathbb{Z}}_p$ is, again, the p-adic integers. So we can view G as a p-pure subgroup of $\hat{\mathbb{Z}}_p$ containing 1. On the other hand, any endomorphism $\phi : G \to G$ (uniquely) extends to an endomorphism $\hat{\phi} : \hat{\mathbb{Z}}_p \to \hat{\mathbb{Z}}_p$, which will necessarily be multiplication by $\alpha = \phi(1)$. As such, we can view the endomorphism ring of G as

$$E = \{\alpha \in \hat{\mathbb{Z}}_p : \alpha G \leq G\}.$$

Let q be a prime number which is distinct from p. Clearly, $G/qG \cong \mathbb{Z}(q)^2$. Let $\mu \in \hat{\mathbb{Z}}_p$ be any element of E. It is easy to see that μ determines an endomorphism $\overline{\mu} : G/qG \to G/qG$. Suppose $x \in G \setminus qG$ and $H = qG + \langle x \rangle$. Since G is iso-artinian and $G \approx H$, we obtain that $G \cong H$ by Proposition 4.5. If we compose this isomorphism with the inclusion $H \leq G$, we get an endomorphism of G which gives that there is an $\alpha \in H \leq \hat{\mathbb{Z}}_p$ such that $H = \alpha G$. Now, there is a $\beta \in G$ such that $x = \alpha\beta$. Therefore, $\mu\beta \in G$, and so we can conclude that $x\mu = (\alpha\beta)\mu = \alpha(\mu\beta) \in H$ by the commutativity of $\hat{\mathbb{Z}}_p$. Hence $\mu H \leq H$, which implies that $\overline{\mu} : G/qG \to G/qG$ maps every cyclic summand into itself. The only endomorphisms of $\mathbb{Z}(q)^2$ with this property are multiplications by some scalar from $\mathbb{Z}(q)$. However, an endomorphism of G, such as the above α, for which the image of $\overline{\alpha}(G/qG) = H/qG$ is a non-zero cyclic summand of this $G/qG \cong \mathbb{Z}(q)^2$, is certainly not multiplication by such a scalar. This contraction implies that G cannot be iso-artinian, as asserted.

The converse follows directly from earlier results. □

Generalizing the concept of a Butler group to the infinite-rank case, the torsion-free group G is said to be a *B_1-group* if

$$\mathrm{Bext}^1(G, T) = 0$$

for all torsion T (i.e., every balanced extension of T by G splits). There is a second such generalization of the definition of Butler groups, those said to be *B_2-groups.* Since every B_2-group is a B_1-group (see [7, Theorem 14.5.3]), we will not concern ourselves with this generalization. Since the balanced-projective torsion-free groups are precisely those that are completely decomposable, each such group is clearly a B_1 group.

This brings us to the main goal of this sections, giving a complete description of the torsion-free iso-artinian groups that are B_1:

Theorem 4.10. *Suppose That G is a torsion-free B_1-group. The following statements are equivalent.*

1. G is iso-artinian,
2. $G \cong H \oplus A$, where H has rank at most 1, $c(H)$ is artinian and A is free.

Proof. Again, only necessity needs to be considered, so assume G is iso-artinian. If

$$H := \sum_{p \in \mathcal{P}} p^\infty G,$$

then it follows from Corollary 4.8(a) that H is a pure subgroup of rank ≤ 1. If L is a pure subgroup of G containing H such that L/H has rank 1, then we have $L/H \cong \mathbb{Z}$ by Proposition 4.9. This implies that G/H is $\mathbb{Z}$-homogeneous.

Claim: H is balanced in G: Suppose X is a torsion-free group of rank 1. If $\phi : X \to G/H$ is any non-zero homomorphism, then we must have $X \cong \phi(X) \cong \mathbb{Z}$ (since G/H is $\mathbb{Z}$-homogeneous). But, as $\mathbb{Z}$ is (trivially) free, ϕ must factor through G, giving the claim.

Claim: H is a TEP subgroup of G (i.e., it has the torsion-extending property): Suppose T is a torsion group and $\phi : H \to T$ is a homomorphism. We must show that ϕ extends to $G \to T$. If K is the kernel of ϕ, then, since $H \leq G$ is iso-artinian, H/K will be artinian,

and hence pure-injective. So, the natural map $H \to H/K$ extends to $G \to H/K \leq T$ since H is pure in G, giving the result.
Claim: G/H is also a B_1-group: Let T be any torsion group. By our first claim, there is an exact sequence

$$\operatorname{Hom}(G,T) \to \operatorname{Hom}(H,T) \to \operatorname{Bext}^1(G/H,T) \to \operatorname{Bext}^1(G,T)$$

By hypothesis, $\operatorname{Bext}^1(G,T) = 0$ and the left map is surjective by our second claim. Hence $\operatorname{Bext}^1(G/H,T)$ is isomorphic to a subgroup of $\operatorname{Bext}^1(G,T) = 0$.

The result now follows easily. In fact, we have already observed that G/H is a $\mathbb{Z}$-homogeneous B_1-group. By [7, Corollary 14.8.3], G/H must be completely decomposable, which means that it is free. Therefore, the required splitting must occur, completing the proof. □

Corollary 4.11. *If G is an iso-artinian B_1 group, then it is completely decomposable.*

We mention a final conjecture. To what extent does Theorem 4.10 generalize to torsion-free groups that are not Butler?

Conjecture: Suppose G is any torsion-free group. Then G is iso-artinian if and only if $G \cong H \oplus A$ where H has rank at most 1, $c(H)$ is artinian and A is free.

Sufficiency being clear, this is really about necessity. Proposition 4.9 verifies it for the case where G has rank 2, which is evidence that it may be true whenever G has finite rank. It is also plausible that it holds when G has finite rank, but not when G has infinite rank.

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Department of Mathematics, Whitman College, Walla Walla, WA 99362, USA
E-mail address: `keef@whitman.edu`

Department of Mathematics, Faculty Sciences, Gazi University, Ankara, Turkey
E-mail address: `mtamerkosan@gazi.edu.tr, tkosan@gmail.com`

Department of Algebra, Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czechia
E-mail address: `zemlicka@karlin.mff.cuni.cz; jan.zemlicka@matfyz.cuni.cz`