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CLASSES OF DUALLY SLENDER MODULES

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The natural categorial notion of a compact object, for which the covariant functor Hom commutes with direct sums, was first time studied in the context of module theory in 60's. Hyman Bass gave a non-categorial characterization of the notion (see Lemma 1.1 of the present paper) in the book [3] and basic properties of such a module were published by Rentschler in [10]. The notion has been studied under various terms (module of type Σ, small, Σ-compact, U-compact module), we use the term dually slender following the terminology of [6]. The study of dually slender modules has been motivated by progress of research in various branches of algebra. Probably the most frequent motivation (and the closest to the author of the present paper) comes from the context of representable equivalences of module categories ([4], [5], [13], [14] etc.). Dually slender modules has appeared also in the structure theory of graded rings [9] or almost free modules [12]. The structure theory of dually slender modules was developed also in [6],[18], [15].

The present paper has an expository and survey character, however it contains several new results (Lemma 2.1, Proposition 2.7, Proposition 3.5) which generalize and simplify older concepts. The first section, which introduces the central notions and their basic properties, is followed by an exposition of functorial properties of classes of dually slender modules and their consequences. The last part is devoted to a description of the structure of classes of dually slender modules over particular types of rings. The results cited in the paper are mostly published in [18], [11], [15], [16] and [17].

Throughout the paper a *ring* R means an associative ring with unit, and a *module* means a right R-module. We will use the letter R for a ring in all claims. The minimal cardinality of a set of generators of an R-module M is denoted by $gen_R(M)$. A ring R is (von Neumann) *regular* provided that each $x \in R$ has a pseudo-inverse element, i.e. there is $y \in R$ satisfying $xyx = x$. A regular ring R is *abelian regular* if all idempotents of R are central. We refer for non-explained terminology to [1].

1. Dually slender modules and rings conditions

A module M is said to be a *dually slender* module provided the natural $\mathbb{Z}$-monomorphism $\Psi : \bigoplus_i \operatorname{Hom}_R(M, A_i) \hookrightarrow \operatorname{Hom}_R(M, \bigoplus_i A_i)$ is surjective for every system of modules $\{A_i\}$. As it is shown in [3] and in [10, Section 1],

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dually slender modules can be described in natural way by language of systems of submodules:

Lemma 1.1. *The following conditions are equivalent for an arbitrary module M:*

(1) *M is dually slender,*
(2) *if $M = \bigcup_{i<\omega} M_n$ for an increasing chain of submodules $M_n \subseteq M_{n+1} \subseteq M$, then there exists n such that $M = M_n$,*
(3) *if $M = \sum_{i<\omega} M_n$ for a system of submodules $M_n \subseteq M$, $n < \omega$, then there exists n such that $M = \sum_{i<\omega} M_n$.*

Proof. (1)⇒(2) Let $M_n \subseteq M_{n+1}$ be an increasing chain of submodules such that $M = \bigcup_{i<\omega} M_n$ and $M \neq M_n$ for each n. We define a homomorphism $\varphi : M \to \bigoplus_{n<\omega} M/M_n$ by the condition $\varphi(m) = (\pi_n(m)|\ n < \omega)$ where π_n are the natural projections of M onto M/M_n. Since $\varphi(M) \nsubseteq \bigoplus_{n<j} M/M_n$ for every $j < \omega$, φ cannot be expressed as a finite sum of homomorphisms from $Hom(M, M/M_n)$, i.e. the natural monomorphism Ψ is not onto $Hom(M, \bigoplus_{n<\omega} M/M_n)$ and so M is not dually slender.

(2)⇒(1) If M is not dually slender, there exists a homomorphism $\varphi \in Hom(M, \bigoplus_{\alpha<\kappa} A_\alpha)$ such that $\pi_\alpha\varphi \neq 0$ for infinitely many α (where π_α is the natural projection $\bigoplus A_\alpha \to A_\alpha$), w.l.o.g. suppose $\kappa = \omega$ and $\pi_\alpha\varphi \neq 0$ for each α. Put $M_n = \{m \in M|\ \pi_\alpha\varphi(m) = 0\ \forall\alpha \geq n\}$. Then $M = \bigcup_{i<\omega} M_n$, $M \neq M_n$ and $(M_n|\ n < \omega)$ forms an increasing chain of submodules.

(2) $\Longleftrightarrow$ (3) Obvious, when we put $N_n = \sum_{i<n} M_i$. □

The condition (2) shows immediately that every finitely generated module is dually slender. Moreover, it is clear from (3) that there is no infinitely countably generated dually slender module. An another easy consequence of Lemma 1.1 is an observation that for every infinite cardinal κ of uncountable cofinality a union of strictly increasing chain of the length κ consisting of dually slender submodules produces a dually slender module as well. Such an example of a dually slender module is every κ-generated uniserial module, which is a union of κ-many cyclic submodules. This construction motivates the following definition of particular classes of dually slender modules. For an arbitrary cardinal number λ we say that a module M is *λ-reducing* if for every submodule $N \subseteq M$ such that $gen(N) \leq \lambda$ there exists a finitely generated submodule F for which $N \subseteq F \subseteq M$.

For an arbitrary ring R we will denote by $\mathcal{DS}(R)$, $\mathcal{R}_\kappa(R)$, $\mathcal{FG}(R)$ and $\mathcal{FP}(R)$ respectively the classes of all dually slender, κ-reducing, finitely generated and finitely presented right R-modules. Now, it is easy to see that the following inclusions of classes holds true:

Lemma 1.2. *Let $\lambda < \kappa$ be infinite cardinal numbers. Then $\mathcal{FP}(R) \subseteq \mathcal{FG}(R) \subseteq \mathcal{R}_\kappa(R) \subseteq \mathcal{R}_\lambda(R) \subseteq \mathcal{DS}(R)$.*

We have remarked that the inclusion $\mathcal{FG}(R) \subseteq \mathcal{R}_\kappa(R)$ is strict over suitable rings (for example in case of valuation domains with κ^+-generated ideals).

Moreover, it is proved in [14, Theorem 2.8] that a ring power R^ω contains a dually slender right ideal which is not ω-reducing.

Recall that classes $\mathcal{R}_\kappa(R)$ and $\mathcal{DS}(R)$ have similar class properties as the class $\mathcal{FG}(R)$:

Proposition 1.3. *Let κ be an infinite cardinal. The classes $\mathcal{R}_\kappa(R)$ and $\mathcal{DS}(R)$ are closed under taking homomorphic images, extensions and finite sums.*

Proof. We can see immediately from Lemma 1.1 (from the definition of κ-reducing modules) that factors and finite direct sums of dually slender (κ-reducing) modules are dually slender (κ-reducing) as well. Fix $A \subseteq M$.

Suppose $A, M/A \in \mathcal{DS}(R)$ and $M = \bigcup_{i<\omega} M_n$ for an increasing chain of submodules $(M_n \subseteq \mid n < \omega)$. Then there exists n_0 such that $M_{n_0} + A = M$ and n_1 such that $A \subseteq M_{n_1}$ by Lemma 1.1 (2). Hence $M = M_n + A = M_n$ for $n = max(n_0, n_1)$ which proves M is dually slender.

Suppose $A, M/A \in \mathcal{R}_\kappa(R)$ and take a set $\{m_\alpha \mid \alpha < \kappa\} \subset M$. By the definition, there exists a finitely generated submodule $F \subseteq M$ such that $\{m_\alpha + A \mid \alpha < \kappa\} \subset F + A$. Since $F + A \in \mathcal{R}_\kappa(R)$, there exists a finitely generated submodule $G \subseteq F + A \subseteq M$ containing $\{m_\alpha \mid \alpha < \kappa\}$ which finishes the proof. □

It is not hard to characterize rings over which dually slender modules are precisely finitely presented ones.

Theorem 1.4. *$\mathcal{DS}(R) = \mathcal{FP}(R)$ iff R is right noetherian.*

Proof. It is well known that $\mathcal{FG}(R) = \mathcal{FP}(R)$ iff R is right noetherian and it is proved in [10, 7^o] or [4, Proposition 1.9] that $\mathcal{DS}(R) = \mathcal{FG}(R)$ for every right noetherian ring R. □

The ring-theoretical characterization of the class condition $\mathcal{DS}(R) = \mathcal{FG}(R)$ is still an open problem and it leads to the following definition. We say that a ring R is *right steady*, provided $\mathcal{DS}(R) = \mathcal{FG}(R)$ (left steadiness is defined by the same condition for left modules). Beside noetherian rings large classes of rings are known to be steady, for instance perfect rings [5, Corollary 1.6], semiartinian rings with countable socle lengths [6, Theorem 2.2], countable commutative rings [10, 11^0] or abelian regular rings with countably generated ideals [18, Corollary 7]. A characterization of steadiness is available only for a few particular interesting classes of rings, as we will show in the sequel.

Remind that a ring is called right semiartinian if every non-zero cyclic module contains simple submodule (more about semiartinian rings see [8]). In [6] they are shown large classes of examples both steady and non-steady abelian regular semiartinian rings. The papers [11] and [17] are devoted to characterize steadiness of abelian regular semiartinian rings [11, Theorem 3.4] and regular semiartinian rings with primitive factors artinian [17, Theorem 3.5] as follows:

Theorem 1.5. *Let R be a regular semiartinian ring with primitive factors artinian. Then the following conditions are equivalent:*

(1) *R is right steady;*
(2) *R is left steady;*
(3) *There exists no infinitely generated dually slender right ideal of any factor-ring of R.*
(4) *There exists no infinitely generated dually slender left ideal of any factor-ring of R.*

A criterion of steadiness of valuation rings is given in [18, Theorem 13] and more general case of chain rings (i.e. rings with linearly ordered both lattices of right and left ideals) is characterized in [16, Theorem 2.4]:

Theorem 1.6. *For a chain ring R the following conditions are equivalent:*

(1) *R is right steady.*
(2) *There exists no ω_1-generated uniserial right module.*
(3) *$R/rad(R)$ contains no uncountable strictly decreasing chain of ideals, R contains no uncountably generated right ideal and for every ideal I and for every prime ideal $P \subseteq I$ there exists an ideal K such that $P \subset K \subset I$.*

However countable valuation rings are steady, it is presented an example of a countable chain ring which is not right steady in [16, Example 1.9].

2. Functorial properties

In this section we generalize several nice properties of the tensor functor applied on the classes $\mathcal{DS}(R)$ and $\mathcal{R}_\kappa(R)$ which are introduced for particular cases in [11, section 4] and [16, section 3].

Lemma 2.1. *Let R and S be rings, κ be an infinite cardinal and let X be an R-S-bimodule. Denote by T_X the tensor functor $-\otimes_R X : Mod\text{-}R \to Mod\text{-}S$.*

(1) *If $X_S \in \mathcal{DS}(S)$, then $T_X(\mathcal{DS}(R)) \subseteq \mathcal{DS}(S)$,*
(2) *if $X_S \in \mathcal{R}_\kappa(S)$, then $T_X(\mathcal{R}_\kappa(R)) \subseteq \mathcal{R}_\kappa(S)$,*
(3) *if there exists a pure embedding of ${}_RR$ into ${}_RX$ as left R-modules, then $T_X(Mod\text{-}R \setminus \mathcal{FG}(R)) \subseteq Mod\text{-}S \setminus \mathcal{FG}(S)$.*

Proof. Let $A \subseteq M$ be R-modules with the inclusion monomorphism i. Throughout the proof $A \otimes X$ will mean $Im(i \otimes_R X)$ in $M \otimes_R X$.

(1) Suppose $M \in \mathcal{DS}(R)$. Let $M \otimes_R X = \bigcup_{n<\omega} N_n$ for an increasing chain of S-submodules N_i, $i < \omega$. Define R-submodules $P_n = \{m \in M \mid m \otimes X \in N_n\}$ for each $n < \omega$. Note that $m \otimes X$ is dually slender since it is a factor of dually slender S-module X for every $m \in M$. Hence for each $m \in M$ there exists n such that $m \otimes X \subseteq N_n$ which proves that $M = \bigcup_{n<\omega} P_n$. Clearly, the sequence $(P_n \mid n < \omega)$ forms an increasing chain of R-submodules of M and $P_n \otimes X \subseteq N_n$, for each $n < \omega$. As M is a dually slender R-module, there exists

n such that $P_n = M$, hence $P_n \otimes X = N_n = M \otimes X$. Thus $M \otimes X$ is a dually slender S-module.

(2) Suppose $M \in \mathcal{R}_\kappa(R)$ and take an arbitrary set $\{m_\alpha \otimes x_\alpha | \ \alpha < \kappa\} \subseteq M \otimes X$ (since κ is infinite we may w.l.o.g. consider the set of generators of κ-generated module is formed by simple tensors). As $M \in \mathcal{R}_\kappa(R)$, there exists $n_1, \dots, n_k \in M$ such that $m_\alpha = \sum_{i=1}^k n_i r_{\alpha i}$ for $r_{\alpha i} \in R$ and $\alpha < \kappa$. Moreover, there exists a finitely generated S-submodule $Y \subseteq X_S$ such that $\{r_{\alpha i} x_\alpha | \ i = 1, \dots, k, \ \alpha < \kappa\} \subseteq Y$. Now, $m_\alpha \otimes x_\alpha = \sum_{i=1}^k n_i \otimes r_{\alpha i} x_\alpha \in \sum_{i=1}^k n_i \otimes Y$, for each $\alpha < \kappa$, where $\sum_{i=1}^k n_i \otimes Y$ is a finitely generated submodule of the S-module $M \otimes X$.

(3) If M is infinitely generated, there exists an infinite cardinal λ and a strictly increasing chain of submodules $(M_\alpha \mid \alpha < \lambda)$ such that $M = \bigcup_{\alpha<\lambda} M_\alpha$ and $M/M_\alpha \neq 0$. Note that $M \otimes X = M_\alpha \otimes X$ iff $M/M_\alpha \otimes X = 0$. Since there is a pure monomorphism ${}_R R \hookrightarrow_R X$ we have $M/M_\alpha \cong M/M_\alpha \otimes R \hookrightarrow M/M_\alpha \otimes X$. Hence $M/M_\alpha \neq 0$ iff $M/M_\alpha \otimes X \neq 0$. Finally, observe that $M \otimes X = \bigcup_{\alpha<\lambda} M_\alpha \otimes X$ where $M \otimes X \neq M_\alpha \otimes X$, i.e. $M \otimes X$ is infinitely generated. $\square$

Corollary 2.2. *If S is a ring and R is a subring of S, then $T_S(\mathcal{DS}(R)) \subseteq \mathcal{DS}(S)$ for the functor $T_S = - \otimes_R S$*

Applying Lemma 2.1 we are ready to describe the correspondence between classes of dually slender modules over a ring and over its pure extension [11, Lemma 4.1]:

Corollary 2.3. *Let S be a ring and let R be its subring such that R is a pure left R-submodule of S. Assume that S is right steady. Then R is right steady.*

Note that every extension of a regular ring is pure, therefore right steadiness of an extension of a regular ring R implies R is right steady as well [11, Corollary 4.2].

If a ring R contains an idempotent e, we can define the natural ring structure on eRe. The following claim [16, Proposition 3.3] shows that an occurrence of an idempotent $e \in R$ allows to reduce the question about steadiness of R to rings eRe and $(1-e)R(1-e)$:

Proposition 2.4. *Let $e \in R$ be an idempotent. Then R is right steady if and only if both the rings eRe and $(1-e)R(1-e)$ are right steady.*

Proof. Let $M \in \mathcal{DS}(eRe) \setminus \mathcal{FG}(eRe)$. Remark that eR is eRe-R-bimodule which is finitely generated as a right R-module. Moreover $eR \cong eRe \oplus eR(1-e)$ as left eRe-modules, hence eRe is a left pure submodule of eR. Applying Lemma 2.1 (1) and (3) for $X = eR$ we get that $M \otimes eR$ is an infinitely generated dually slender R-module.

Let $M \in \mathcal{DS}(R) \setminus \mathcal{FG}(R)$. Then either M/MeR or $M/M(1-e)R$ is infinitely generated (otherwise $M \in \mathcal{FG}(R)$). We may suppose w.l.o.g. that $M = M/M(1-e)R$ is infinitely generated, so $MeR = M$. Suppose $Me = \bigcup_{i<\omega} N_i e$

where $(N_ie|\ i<\omega)$ is an increasing chain of eRe-submodules. Then $M = MeR = \bigcup_{i<\omega} N_ieR$. Since M is dually slender, there exists i such that $M = N_ieR$. Hence $Me = N_ieRe = N_ie$, i.e. $Me \in \mathcal{DS}(eRe)$. Assume Me is a finitely generated eRe-module. Then $Me = \sum_{i\leq n} m_ieRe$. Since $M = MeR$, we get $M = \sum_{i\leq n} m_ieReR = \sum_{i\leq n} m_ieR$, hence M would be a finitely generated R-module. Thus Me is infinitely generated. □

An inductional extension of Proposition 2.4 yields:

Corollary 2.5. *Let $\{e_i|\ 1\leq i\leq n\}$ be an orthogonal set of idempotents satisfying $\sum_{i\leq n} e_i = 1$. Then R is right steady if and only if e_iRe_i is right steady for every $i\leq n$.*

Recall that a ring R is serial, provided there exists a (complete) set of orthogonal idempotents $\{e_i,\ i\leq n\}$ such that $\sum_{i\leq n} e_i = 1$ and e_iR, Re_i respectively are right, left uniserial modules for all $i\leq n$. Combining Theorem 1.6 and Corollary 2.5 we can formulate the claim [16, Theorem 3.4]:

Theorem 2.6. *The following conditions are equivalent for a serial ring R with a complete set of orthogonal idempotents $\{e_i,\ i\leq n\}$:*

(1) *R is right steady,*
(2) *e_iRe_i is right steady for every $i\leq n$,*
(3) *there exists no ω_1-generated uniserial R-module.*

It is well known that for Morita equivalent rings R and S there exists κ-generated dually slender R-module iff there exists κ-generated dually slender S-module, hence R is right steady iff S is right steady. The following observation helps us to produce examples of (non-)steady rings in an another way:

Proposition 2.7. *Let R be a ring and let $\{e_i|\ 1\leq i\leq n\}$ be an orthogonal set of idempotents satisfying $\sum_{i\leq n} e_i = 1$. Suppose that S is a subring of R such that $e_iRe_i\subseteq S$ for each $i\leq n$. Then R is right steady if and only if S is right steady.*

Proof. Note $e_iRe_i = e_iSe_i$ for every $i\leq n$. Applying Corollary 2.5 for both the rings R and S we see that R is right steady iff $e_iRe_i = e_iSe_i$ is right steady for every $i\leq n$ iff S is right steady. □

Using Proposition 2.7 we can easily see that any subring of the full matrix ring over R of degree n containing all diagonal matrices is right steady iff R is right steady.

3. Largeness and density of $\mathcal{DS}(R)$

This section is devoted to studying the question how large can be the class $\mathcal{DS}(R)$. Obviously, representative class of dually slender modules over a right steady ring is only a set, more precisely, $\mathcal{DS}(R)$ contains at most

$max(2^{card(R)}, \omega)$ non-isomorphic dually slender (i.e. finitely generated) modules. We will show that even when we restrict our considerations to regular non-steady rings, the representative class of dually slender modules can be quite small set as well as a proper class.

First, denote by $\mathcal{I}(R)$ the class of all injective modules over R and recall two elementary lemmas [14, Theorem 1.6]:

Lemma 3.1. *Let κ be an infinite cardinal. If there exists an embedding $R_R^{(\kappa)} \to R_R$, then $\mathcal{I}(R) \subseteq \mathcal{R}_\kappa(R)$ $(\subseteq \mathcal{DS}(R))$.*

Proof. Fix $E \in \mathcal{I}$. If N is a κ-generated submodule of E, there exists an epimorphism $f : R_R^{(\kappa)} \to N$ which may be extended to a homomorphism $g : R \to E$ such that $N \subseteq g(R)$. As $g(R)$ is a cyclic module we have $E \in \mathcal{R}_\kappa(R)$. $\square$

Lemma 3.2. *If there exists an embedding $R_R^2 \to R_R$, then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$ $(\subseteq \mathcal{DS}(R))$.*

Proof. Fix elements $a, b \in R$ such that $R_R^2 \cong aR \oplus bR$. It is easy to see that $b^n aR \cong R$ and $b^n aR \cap \sum_{i<n} b^i aR = 0$, hence $R^{(\omega)} \cong \bigoplus_{n<\omega} b^n aR$ is a right ideal of R. $\square$

Remark that the hypothesis of Lemma 3.1 is satisfied for example by the endomorphism ring $End(V)$ for any κ-dimensional vector space V. Moreover, any non-commutative domain which does not satisfy right Ore condition (for instance polynomials in two non-commuting variables $\mathbb{Z}[x, y]$) satisfies the hypothesis of Lemma 3.2. Hence over such rings we have a proper class of non-isomorphic dually slender modules.

In [5, Lemma 1.10] it is proved the following analogical claim:

Proposition 3.3. *Let R be a simple ring containing an infinite orthogonal set of idempotents. Then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$ $(\subseteq \mathcal{DS}(R)$.*

Corollary 3.4. *Let R be a non-artinian simple von Neumann regular ring. Then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$ $(\subseteq \mathcal{DS}(R)$.*

In case the regular ring is “close enough” to commutativity (such as abelian regular rings are), we are able to bound the cardinality of an arbitrary dually slender module, however the ring need not be right steady.

Denote by $\mathcal{G}_\kappa(R)$ the class of all R-modules which are at most κ-generated for any infinite cardinal κ.

Proposition 3.5. *Let R be a regular ring with primitive factors artinian and put $\kappa = max(2^{card(R)}, \omega)$. Then $\mathcal{DS}(R) \subseteq \mathcal{G}_\kappa(R)$.*

Proof. The claim is trivial for finite rings, suppose R is infinite. Let $M \in \mathcal{DS}(R)$ and denote by $\{I_\alpha |\ \alpha < \lambda\}$ the set of all maximal two-sided ideals. It is well known (see e.g. [7, 6.19]) that $\bigcap_{\alpha<\lambda} MI_\alpha = 0$, hence M embeds into $\prod_{\alpha<\lambda} M/MI_\alpha$. Since the ring R/I_α is right artinian and so steady and

$M/MI_\alpha \in \mathcal{DS}(R/I_\alpha) = \mathcal{FG}(R/I_\alpha)$, we can take a finitely generated R-module F_α such that $F_\alpha + MI_\alpha = M$, $\alpha < \lambda$. Then $(M/\sum_{\alpha<\lambda} F_\alpha)I_\beta = 0$ for each $\beta < \lambda$, hence $M = \sum_{\alpha<\lambda} F_\alpha$. Now, it remains to bound $card(F) \leq card(R)$ and the number of maximal two-sided $\lambda \leq 2^{card(R)}$. Thus $gen(M) \leq card(M) \leq \kappa$. $\square$

The paper [15] contains a generalization of the previous observation which makes clear how "dense" (with respect to the subclasses of κ-generated modules $\mathcal{G}_\kappa(R)$) is the class $\mathcal{DS}(R)$ in Mod-R. We will sum up claims [15, 1.1 – 1.4] in the following formulation:

Theorem 3.6. *Let R be a ring and put $\kappa = card(R)^+$. Denote by Simp the representative set of all simple modules. Then R is not right steady iff $\prod_{S\in Simp} S^\kappa \oplus \bigoplus_{S\in Simp} E(S)$ contains an infinitely generated dually slender submodule.*

Corollary 3.7. *Put $\kappa = 2^{2^{card(R)}}$. If R is not a right steady ring, then $\mathcal{DS}(R) \cap \mathcal{G}_\kappa \nsubseteq \mathcal{FG}(R)$.*

For commutative regular rings it is proved in [15, Theorem 2.7] the following easier form of a module-theoretical criterion of existence of an infinitely generated dually slender module:

Theorem 3.8. *Let R be a commutative regular ring. Then R is steady if and only if the R-module $R^* = Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ contains no infinitely generated dually slender submodule.*

We conclude the paper with a short list of open problems:

- Provide a ring-theoretical criterion of steadiness (at least for commutative rings).
- Does the condition $\mathcal{DS}(R) = \mathcal{R}_\kappa(R)$ imply R is right steady?
- Exists a ring R over which $\mathcal{DS}(R)$ is closed under products?

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