UNJ-RINGS

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Abstract. In analogy to the elementwise definitions of UU and UJ-rings, a ring R is called UNJ if $1 + N(R) + J(R) = U(R)$. After presenting several characterizations and properties, we consider the UNJ property within many well-studied classes of rings. In particular, we examine Dedekind finite rings, 2-primal rings, (semi)regular rings, π-regular rings and rings has the identity $x^2 = x$. Finally, we close the paper with group rings.

1. Introduction

All rings are associative with unity and all modules are unitary right modules. For a ring R, the Jacobson radical and the set of nilpotent elements and the set of invertible elements of R are denoted by $J(R)$ and $N(R)$ and $U(R)$, respectively. The symbols $M_n(R)$ and $T_n(R)$ stand for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R, respectively. $R[x]$ stands for the polynomial ring over R. For an endomorphism σ of a ring R, let $R[x; \sigma]$ denote the ring of left skew polynomials over R. Hence, elements of $R[x; \sigma]$ are polynomials in x with coefficients in R written on the left, subject to the relation $xr = \sigma(r)x$ for all $r \in R$. The group ring of a group G over a ring R is denoted by RG. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}_n$ be the ring of $\mathbb{Z}$ modulo n. We also use $\mathbb{N}$ to denote the set of natural numbers.

We recall two popular facts on the ring theory: $1 + J(R) \subseteq U(R)$ and $1 + N(R) \subseteq U(R)$. A ring R which satisfies the equalities $1 + J(R) = U(R)$ and $1 + N(R) = U(R)$ is said to be a UJ-ring ([14] and [21]) or a UU-ring ([6]), respectively. Let us remark that for any elements $n \in N(R)$ and $j \in J(R)$ we

2010 *Mathematics Subject Classification.* 16N20; 16D60; 16U60; 16W10.

Key words and phrases. UU-ring, UJ-ring, UNJ-ring, unit, Jacobson radical, (semi)regular ring, clean ring, group ring.

February 2, 2019.

M. Tamer Koşan dedicates this study to memorize of Sarp Can who was past away in 19.05.2018.

have $1+n+j \in U(R)$. They offer us the natural definition: a ring R is a UNJ-ring if $1 + N(R) + J(R) = U(R)$.

The outline of this paper is as follows. We begin by giving several examples of both UNJ and UU (or UJ)-rings. For example, we prove that 2-primal UNJ-rings are UJ. Furthermore, if R is a semilocal ring then R is UJ iff R is UNJ iff there exists $n \in \mathbb{N}$ such that $R/J(R) \cong \mathbb{Z}_2^n$. We then develop some preliminary results which we utilize throughout the paper. In this section, we also study the UNJ property within the well-studied classes of Dedekind finite rings, 2-primal rings, (semi)regular rings, π-regular rings. It is shown that every UNJ-ring is Dedekind finite, and if R is a UNJ-ring then R is semiregular iff R is an exchange ring iff R is a clean ring. Finally, we close this section with $*$-rings (or rings with the involution). For a $*$-ring R, we prove that R is a $*$-regular UNJ-ring iff R is a π-$*$-regular, reduced and UNJ-ring iff R has the identity $x^2 = x = x^*$. In the last section, we study the group rings. Let G be a locally finite group and R be a ring. Put $H = G \cap 1 + J(RG)$. Then the group RG is UNJ if and only if G/H is a 2-group, $J(R) \subseteq J(R(G/H))$ and R is a UNJ-ring.

2. UNJ rings

It is easy to see that $1+n+j \in U(R)$ for any elements $n \in N(R)$ and $j \in J(R)$. Hence R is said to be a UNJ-ring if we have $1 + N(R) + J(R) = U(R)$.

We begin with several examples of both UNJ and UU (or UJ)-rings to initiate the reader and to motivate our study.

Examples 2.1. (1) Any UU or UJ-ring is a UNJ-ring. Let R be the $\mathbb{F}_2$-algebra generated by x, y and with the relation $x^2 = 0$. Then R is a UNJ-ring which is not a UJ-ring.

(2) A field is UNJ if and only if it is the field $\mathbb{F}_2$.

(3) If R is a UU-ring, then $S = R[[x]]$ is a UNJ-ring (since the ideal (x) generated by x in S is contained in $J(S)$).

(4) For a UU-ring R and $n \geq 1$, the ring of upper triangular matrix $T_n(R)$ is a UNJ-ring (since the strictly upper matrices (only 0's on the diagonal) form a nilpotent ideal and hence are contained in the Jacobson radical of R).

(5) The polynomial ring $\mathbb{F}_2[x]$ is a UNJ-ring (since its a unit trivial ring and both its Jacobson radical and its nilpotent elements set are equal to $\{0\}$).

(6) If R is a local ring with the residue field $\mathbb{F}_2$, then R is UNJ, but it is not UU unless its Jacobson radical is nil.

The following observations characterize UNJ-rings in terms of $N(R)$ and $U(R)$ for a ring R.

Proposition 2.2. *The following conditions are equivalent for a ring R:*

(1) *R is a UNJ-ring,*
(2) $U(R/J(R)) = \{1 + n + J(R) : n \in N(R)\}$,
(3) $1 + U(R) = N(R) + J(R)$.

Proof. This is straightforward. □

Proposition 2.3. *The following conditions are equivalent for a ring R such that $N(R)$ is closed under the addition.*

(1) *R is UNJ.*
(2) $U(R) + U(R) \subseteq N(R) + J(R)$.

Proof. (2) ⇒ (1) This is obvious.
(1) ⇒ (2) Let $u, u' \in U(R)$. We write $u = 1 + n + j$ and $u' = 1 + n' + j'$ for some $n, n' \in N$ and $j, j' \in J$. Then $u - u' = n - n' + j - j' \in N + J$ by our assumption which implies $U(R) + U(R) \subseteq N(R) + J(R)$. □

First, make an elementary but useful observation on invertible elements.

Lemma 2.4. *Let R be a ring, $I \subseteq J(R)$ an ideal and $u \in R$. Then $u \in U(R)$ if and only if $u + I \in U(R/I)$.*

In the following proposition, we collect a few basic properties of UNJ-rings. Recall that a ring R is called 2-primal if its prime radical contains $N(R)$.

Proposition 2.5. *For a ring R, the following hold.*

(1) *If R is a UNJ-ring, then $R/J(R)$ is a UU-ring.*
(2) *If $R/J(R)$ is a UU-ring, then, for any $u \in U(R)$, there exists $r \in \mathbb{N}$ such that $u^r \in 1 + N(R) + J(R)$.*
(3) *If R is a UNJ-ring then, for any ideal I of R contained in $J(R)$, R/I is also a UNJ-ring.*
(4) *Let I be a nil ideal in R such that $N(R)$ is closed under addition. If R/I is a UNJ-ring then R is a UNJ-ring.*
(5) *If R is a UNJ-ring, then a power of 2 belongs to $J(R)$.*
(6) *If R is a 2-primal UNJ-ring then R is a UJ-ring.*

Proof. (1) If R is a UNJ-ring, then

$$\begin{aligned} U(R/J(R)) &= \{u + J(R) \mid u \in U(R)\} \\ &= \{1 + n + J(R) \mid n \in N(R)\} \end{aligned}$$

by Lemma 2.4. Hence the quotient ring $R/J(R)$ is clearly a UU-ring.
(2) Suppose that R/J is a UU-ring and $u \in U(R)$. Then $u + J(R) \in U(R/J(R))$ and so there exists $n \in N(R/J(R))$ such that $u - 1 - n \in J(R)$. Now we can find $l \in \mathbb{N}$ such that both $2^l, n^l \in J(R)$. This leads to $(1+n)^s \in 1 + J(r)$ for some $s \in \mathbb{N}$. Now we obtain $u^s \in 1 + J(R)$, as required.
(3) This is clear since $J(R/I) = J(R)/I$.
(4) Let $u \in U(R)$ and I be a nil ideal of R. Then $u + I \in U(R/I)$ and $u + I = 1 + n + j + I$ with $n + I$ a nilpotent in R/I and $j \in J(R)$ by the hypothesis. Since I is a nil ideal, we obtain n is also a nilpotent in R and so there exists $n' \in I$. Hence it is a nilpotent in R, with $u = 1 + n + n' + j$. Finally, $n + n' \in N(R)$. We deduce that R is UNJ.
(5) Since $-1 + J(R) \in U(R/J(R))$ and $R/J(R)$ is a UU-ring by (1), we have $2 + J(R) = n + J(R)$ for some $n + J(R) \in N(R/J(R))$. This leads to the fact that a power of 2 belongs to $J(R)$.
(6) If R is a 2-primal ring, then we have that $N(R)$ is the prime radical of R and hence $N(R) \subseteq J(R)$. This implies that $U(R) = 1 + N(R) + J(R) = 1 + J(R)$. Hence R is a UJ-ring. □

Remark 2.6. The converse implication of Proposition 2.5(4) is not true in general. For example, the ring $\mathbb{Z}/2\mathbb{Z}$ is UNJ. But, $\mathbb{Z}$ is not a UNJ-ring.

A ring R is reduced if R has no nonzero nilpotents.

Proposition 2.7. *The following are equivalent for a ring R:*

1. *R is a UJ-ring.*
2. *R is a UNJ-ring and $R/J(R)$ is reduced.*

Proof. (1) $\Rightarrow$ (2) This follows from [14, Proposition 2.3].
(2) $\Rightarrow$ (1) Let $u \in U(R)$. Then $u - 1 - j \in N(R)$ for some $j \in J(R)$, and so $u - 1 + J(R)$ is a nilpotent in $R/J(R)$. By (2), we get $u - 1 \in J(R)$. □

For an endomorphism σ of R, R is called σ-compatible if, for any $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$, and in this case σ is clearly injective. An automorphism σ of R is said to be of locally finite order if for every $a \in R$ there exists an integer $n(a)$ with $\sigma^n(a) = a$.

Theorem 2.8. *Let R be a 2-primal ring and σ a locally finite order automorphism of R such that R is σ-compatible. The following are equivalent:*

1. *$R[x, \sigma]$ is a UNJ-ring.*
2. *$J(R) = N(R)$ and $U(R) = 1 + J(R)$ (R is a UJ-ring).*

Proof. (1) $\Rightarrow$ (2) Assume that $R[x,\sigma]$ is a UNJ-ring. By [4, Theorem 3.1],

$$1 + J(R) \subseteq 1 + J(R[x,\sigma]) + N(R[x,\sigma]) \subseteq 1 + IR[x,\sigma] + N(R[x,\sigma])$$

for some nil ideal I of R. It follows that $J(R) \subseteq IR[x,\sigma] + N(R[x,\sigma])$. Once can check that $J(R) \subseteq I + N(R)$. Since R is a 2-primal ring, $J(R) \subseteq N(R)$. Hence $J(R) = N(R)$.

For the rest, take an arbitrary unit u of R. Since $R[x,\sigma]$ is a UNJ-ring, we get $u = 1 + a_0 + b_0$ for some nilpotent elements a_0, b_0 of R. Then $u \in 1 + N(R) = 1 + J(R)$.

(2) $\Rightarrow$ (1) Since R is a 2-primal ring, we get by (2) that $J(R) = Nil_*(R) = P(R)$, where $P(R)$ is the prime radical and $Nil_*(R)$ is the lower nilradical of R. Then $R/J(R)$ is a reduced ring. Since $\sigma(J(R)) \subseteq J(R)$, the map $\bar{\sigma} : R/J(R) \to R/J(R)$ defined by $\bar{\sigma}(\bar{x}) = \sigma(a)$ is an endomorphism of $R/J(R)$. We next show that $R/J(R)$ is $\bar{\sigma}$-compatible. To this, we must show that $ab \in P(R) \Leftrightarrow a\sigma(b) \in P(R)$ for any $a, b \in R$. But this equivalence " $\Leftrightarrow$ " has been established in the proof of Claims 1 and 2 of [1, Theorem 3.6].

Now since $R/J(R)$ is a reduced ring and it is $\bar{\sigma}$-compatible, we get

$$U(\frac{R}{J(R)}[x,\bar{\sigma}]) = U(\frac{R}{J(R)}),$$

which is equal to $\{1\}$ by [7, Corollary 2.12] and (2). Now

$$\frac{R[x,\sigma]}{J(R)[x,\sigma]} \cong \frac{R}{J(R)}[x,\bar{\sigma}],$$

which shows that $\dfrac{R[x,\sigma]}{J(R)[x,\sigma]}$ is a UNJ-ring. On the other hand, since R is a 2-primal ring that is σ-compatible, and $Nil_*(R[x,\sigma]) = Nil_*(R)[t,\sigma]$ by [7, Lemma 2.2], we have $J(R)[x,\sigma] = Nil_*(R[x,\sigma])$ is nil. By [10, Theorem 2.4], we obtain that $R[x,\sigma]$ is a UNJ-ring. $\square$

Corollary 2.9. . *A 2-primal ring $R[x]$ is a UNJ ring if and only if $J(R) = N(R)$ and $U(R) = 1 + J(R)$.*

The following result gives a description of UNJ-rings and UJ-ring that are semilocal. UU-rings and UJ-rings were handled by Danchev and Lam in [10, Theorem 2.8] and Koşan, Leroy and Matczuk in [14, Proposition 1.4], respectively.

Theorem 2.10. *The following conditions are equivalent for a semilocal ring R:*

(1) *R is a UNJ-ring,*
(2) *R is a UJ-ring,*

(3) *there exists* $n \in \mathbb{N}$ *such that* $R/J(R) \cong \mathbb{Z}_2^n$.

Proof. (1)⇒(3). Note that $R/J(R)$ is a UU-ring by Proposition 2.5(1). Hence $R/J(R) \cong \mathbb{Z}_2^n$ for some n by [10, Theorem 2.8].
(2)⇔(3) is proved in [14, Proposition 1.4].
(2)⇒(1) This is obvious. □

Example 2.11. By Theorem 2.10, any semilocal domain, say R, satisfying $R/J(R) \cong \mathbb{Z}_2^n$ for some n (for example, the localization of integers in the ideal $2\mathbb{Z}$) is a UNJ-ring and a UJ-ring. On the other hand, if $J(R) \neq 0$, it is not a UU-ring by [10, Theorem 2.8].

The following technical lemma generalizes [10, Theorem 2.7].

Lemma 2.12. *Let* R *be a ring,* $g, h \in R$ *be non-zero orthogonal idempotents and* $a \in gRh$, $b \in hRg$ *such that* $ab = g$ *and* $ba = h$. *If* $f = g + h$, *then* f *is an idempotent and both the rings* fRf *and* R *are not UU-rings.*

Proof. Since $gh = 0 = hg$ and $a = gah$, $b = hbg$, we can easily compute that $ag = 0 = ha$, $gb = 0 = bh$, and so $a^2 = aha = 0$ and $b^2 = bgb = 0$. Obviously, $f = g + h$ is a non-zero idempotent and $g, h, a, b \in fRf$. Put $u = g + a + b$ and $v = f - u = h - (a + b)$. Then

$$u^2 = (g + a + b)^2 = g + ga + ab + bg + ba = f + g + a + b = f + u,$$

hence $uv = u(f - u) = u - u^2 = -g$ and symmetrically $vu = u - u^2 = -g$, which proves that $u, v \in U(fRf)$. As $N(fRf) \cap U(fRf) = \emptyset$, we obtain that $f - u = v \notin N(fRf)$. Thus fRf is not a UU-ring. By [10, Theorem 2.6(2)], R is not a UU-ring. □

The following results show that UNJ-rings and UU-rings are Dedekind finite. UJ-rings were handled by Koşan, Leroy and Matczuk in [14, Proposition 1.3(6)].

Theorem 2.13. *Every UU-ring is Dedekind finite.*

Proof. Assume that a ring R is not Dedekind finite. Hence there exist $x, y \in R$ such that $xy = 1$ and $yx \neq 1$. Let $e = yx$ and observe that e is a nonzero idempotent and it holds $xe = x$ and $ey = y$. Put $a = (1 - e)xe$, $b = ey(1 - e)$, $g = ab$, and $h = ba$. We will verify that these elements satisfy the hypothesis of Lemma 2.12. First note that

$$g = ab = (1 - e)xey(1 - e) = (1 - e)(xy)(xy)(1 - e) = (1 - e)^2 = (1 - e)$$

is a nonzero idempotent. Moreover, $h^2 = b(ab)a = b(1-e)a = h$ and $ahb = 1-e \neq 0$, so h is a nonzero idempotent as well. Since $a^2 = 0 = b^2$, we obtain that $gh = 0 = hg$, i.e. f and g forms an orthogonal pair of nonzero idempotents. Clearly, $a = ga = aba = ah$ and $b = bg = bab = hb$. Hence $a \in gRh$ and $b \in hRg$. Now R is not a UU-ring by Lemma 2.12. □

Theorem 2.14. *Every UNJ-ring is Dedekind finite.*

Proof. Suppose that R is not Dedekind finite and let $x, y \in R$ such that $xy = 1$ and $e = yx \neq 1$. Then $0 \neq e \notin J(R)$ and $1 - e \notin J(R)$. Hence $R/J(R)$ is not Dedekind finite. As $R/J(R)$ is not UU-ring by Theorem 2.13, R is not a UNJ-ring by Proposition 2.5(1). □

Recall that a ring R is said to be regular in the sense of von Neumann if for every $a \in R$, there is an $x \in R$ such that $axa = a$, and R is said to be π-regular if for each $a \in R$, $a^n \in a^n R a^n$ for some positive integer n.

Theorem 2.15. *The following are equivalent for a ring R:*

(1) *R is a regular UNJ-ring.*
(2) *R is a π-regular, reduced and UNJ-ring.*
(3) *R has the identity $x^2 = x$ (i.e., R is a Boolean ring).*

Proof. (1) ⇒ (2). Since R is regular, $J(R) = 0$ and every nonzero right ideal contains a nonzero idempotent. Assume R is not reduced. Then there exists a nonzero element $a \in R$ such that $a^2 = 0$. By [16, Theorem 2.1], there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some non-trivial ring T. If $u \in U(eRe)$, then $u + 1 - e \in U(R)$. Since R is a UNJ-ring, we have $u - e \in N(R)$, that is $(u-e)^k = 0$ for some positive integer k. It follows that $u - e \in N(eRe)$, and so $eRe \cong M_2(T)$ is a UNJ-ring. On the other hand, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(T)$ and $A - I_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \in U(M_2(T))$, a contradiction.

(2) ⇒ (3) Since reduced rings are abelian, R is strongly π-regular and $J(R) = N(R) = 0$ by [3, Lemma 5]. Let $x \in R$. By [19, Theorem 1], there exist $e^2 = e \in R$ and $u \in U(R)$ such that $x = e + u$ and $xe = ex \in N(R) = 0$. So we have

$$x = x - xe = x(1-e) = u(1-e) = (1-e)u = 1 - e,$$

since R is a UNJ-ring. Thus, $x^2 = x$.

(3) ⇒ (1). Clearly, R is regular. Let $u \in U(R)$. Then $u^2 = u$ which implies $u = 1$, and so R is a UNJ-ring. □

A ring R is semiregular ([17]) if $R/J(R)$ is regular and idempotents lift modulo $J(R)$, and R is exchange ([18]) if for each $a \in R$ there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$.

Theorem 2.16. *The following are equivalent for a ring R:*

(1) *R is a semiregular UNJ-ring.*
(2) *R is an exchange UNJ-ring.*
(3) *$R/J(R)$ is a Boolean ring.*

Proof. This follows from the proof of Theorem 2.15 and the fact that semiregular rings are exchange. □

R is called a clean ring if every element of R is a sum of an idempotent and a unit ([18]). By [12], R is a clean ring if and only if $R/J(R)$ is clean and idempotents lift modulo $J(R)$. Also, by [18], $R/J(R)$ is clean and so R is clean.

Corollary 2.17. *The following are equivalent for a UNJ-ring R:*

(1) *R is a semiregular ring.*
(2) *R is an exchange ring.*
(3) *R is a clean ring.*

A ring R is said to be a $*$-ring or ring with the involution if there exists a map $* : R \to R$ such that

$$(x + y)^* = x^* + y^*,$$
$$(xy)^* = y^* x^*$$

and

$$(x^*)^* = x$$

for all $x, y \in R$. A $*$-ring R is called $*$-regular if R is regular and the involution is proper (that is, $x^* = 0$ implies $x = 0$ for all $x \in R$) [5]. Equivalently, for each $r \in R$, $rR = pR$ for some projection $p \in R$ (that is, $p^2 = p = p^*$).

Theorem 2.18. *The following are equivalent for a $*$-ring R:*

(1) *R is a $*$-regular UNJ-ring.*
(2) *R is a π-$*$-regular, reduced and UNJ-ring.*
(3) *R has the identity $x^2 = x = x^*$.*

Proof. (1) $\Rightarrow$ (2) Remark that $*$-regular rings are π-$*$-regular and regular. Now (2) follows from Theorem 2.15.
(2) $\Rightarrow$ (3) From the proof of Theorem 2.15, we must show that every idempotent of R is a projection. Take e an idempotent of R. As R is π-$*$-regular, $eR = pR$

for some $p^2 = p^* = p \in R$. Hence $e = pe$ and $p = ep$. Since R is abelian, we get $e = p$.
(3) $\Rightarrow$ (1) This follows from Theorem 2.15. □

Corollary 2.19. *Let R be a $*$-ring. Then R is a π-regular UNJ-ring if and only if R is a regular UNJ-ring and $* = 1_R$.*

A $*$-ring R is called semi-$*$-regular if $R/J(R)$ is $*$-regular and idempotents lift modulo $J(R)$. According to [22], R is called a $*$-clean ring if every element of R is a sum of a projection and a unit.

Theorem 2.20. *The following are equivalent for a $*$-ring R:*

(1) *R is a semi-$*$-regular UNJ-ring.*
(2) *R is a $*$-clean UNJ-ring.*
(3) *$R/J(R)$ has the identity $x^2 = x = x^*$.*

Proof. (1) $\Rightarrow$ (2) By [9, Proposition 4.9], a $*$-ring R is $*$-clean if and only if $R/J(R)$ is $*$-clean and every projection of $R/J(R)$ can be lifted to a projection of R. Now assume that R is a semi-$*$-regular UNJ-ring. Then idempotents lift modulo $J(R)$ and every projection of $R/J(R)$ can be lifted to a projection of R. Note that every nonzero right ideal of R contains a nonzero idempotent. Then $R/J(R)$ is reduced, and hence $\bar{R} = R/J(R)$ is an abelian regular ring. Let $a \in \bar{R}$. By [19, Theorem 1], we obtain $a = e + u$ for some $e^2 = e \in \bar{R}$ and $u \in U(\bar{R})$. We next show that $\bar{R}$ is $*$-clean. Clearly, $\bar{R}$ is $*$-regular and hence $e\bar{R} = p\bar{R}$ for some $p^2 = p^* = p \in \bar{R}$. This implies $e = pe = ep = p$ since $\bar{R}$ is abelian.
(2) $\Rightarrow$ (3) This is clear.
(3) $\Rightarrow$ (1) This follows from Corollary 2.18 and Theorem 2.18. □

Remark 2.21. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. An involution $*$ of R is defined by $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. Clearly, $* \neq 1_R$. We have that $R/J(R) \cong \mathbb{Z}_2$ has the identity $x^2 = x = x^*$ and R is a semi-$*$-regular UNJ-ring.

Corollary 2.22. *Let R be a UNJ $*$-ring. Then R is semi-$*$-regular if and only if R is $*$-clean.*

3. Group rings

The following observation describes some particular cases of ring extensions which preserves the UNJ-property.

Lemma 3.1. *Let R and S be rings and $i : R \to S$, $\epsilon : S \to R$ be ring homomorphisms such that $\epsilon\iota = \mathrm{id}_R$.*

(1) $\epsilon(N(S)) = N(R)$, $\epsilon(U(S)) = U(R)$, $\epsilon(J(S)) \subseteq N(R)$.
(2) *If S is a UNJ-ring, then R is a UNJ-ring as well.*
(3) *If R is a UNJ-ring,* $\operatorname{Ker}\epsilon \subseteq N(S) + J(S)$ *and* $\epsilon(J(S)) = J(R)$, *then S is a UNJ-ring.*

Proof. (1) Clearly, $\epsilon(N(S)) \subseteq N(R)$ and $\epsilon(U(S)) \subseteq U(R)$. On the other hand, $N(R) = \epsilon\iota(N(R)) \subseteq \epsilon(N(S))$ and $U(R) = \epsilon\iota(U(R)) \subseteq \epsilon(U(S))$. Let I be a maximal ideal of R. Since ϵ is a homomorphism onto R, $\epsilon^{-1}(I)$ is a maximal ideal of S, hence $J(S) \subseteq \epsilon^{-1}(I)$ which implies that $\epsilon(J(S)) \subseteq J(R)$.
(2) Let S be a UNJ-ring. Then $U(S) = 1 + N(S) + J(S)$, hence by (1)

$$U(R) = \epsilon(U(S)) = 1 + \epsilon(N(S)) + \epsilon(J(S)) \subseteq 1 + N(R) + J(R)$$

(3) If R is a UNJ-ring we have

$$U(S) = \epsilon^{-1}(U(R)) = 1 + N(S) + J(S) + \operatorname{Ker}\epsilon = 1 + N(S) + J(S)$$

by (1). □

Remark 3.2. It is easy to see that Lemma 3.1 (2) and (3) hold also for UU and UJ-rings.

Given a ring R and a group G, we denote the group ring of R over G by RG. An arbitrary element of RG, say $\alpha \in RG$, is of the form $\alpha = \sum_{g\in G} r_g g$ where $r \in R$.

Let us remark that if R is a ring and M is a monoid, then RM is a monoid ring which is defined in the same way as a group ring, using the monoid structure to get a multiplication (see e.g.[15, section II.3]). In particular, RM is a group ring if M is group or RM is isomorphic to a polynomial ring $R[X]$ for any set of variables for the additive monoid $M = \mathbb{N}_0^{(X)}(+)$.

Proposition 3.3. *Let R be a ring, M a monoid and RM a monoid ring. If RM is a UNJ-ring, then R is a UNJ-ring as well.*

Proof. Let us consider the inclusion $\iota : R \to RM$ (i.e. $i(r) = re$ for a monoid unit e) and $\epsilon : RM \to R$ is the augmentation homomorphism $\epsilon(\sum_{m\in M} r_m m) = \sum_{m\in M} r_m$ (cf. [15, Proposition II.3.1]). Then it is enough to apply Lemma 3.1(2). □

As a consequence we get that if the polynomial ring $R[X]$ is a UNJ-ring, then R is a UNJ-ring. For polynomial rings over commutative rings we can obtain more.

Remark 3.4. It is well-known that if R is a commutative ring with identity and $f = a_0 + a_1x + \cdots + a_nx_n \in R[x]$ is a polynomial, then

(1) f is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, a_2, \dots, a_n \in N(R)$ are nilpotent in R.
(2) f is a nilpotent in $R[x]$ if and only if all the coefficients are nilpotents.

From Remark 3.4, we have the following.

Proposition 3.5. *A polynomial ring $R[x]$ over a commutative ring R with identity is UNJ if and only if R is UNJ.*

$\Delta(RG)$ denotes the ideal of RG generated by the set $\{g-1|g \in G\}$.

Lemma 3.6. *Let G be a locally finite 2-group, R a ring and I an ideal of R. If R/I is UU-ring and $I \subseteq J(RG)$, then $\Delta(RG) \subseteq J(RG)$.*

Proof. Repeating parts of proofs of [11, Theorem 2.1(ii)] and [13, Theorem 2.22], observe that $2 \in N(R/I)$ by [10, Theorem 2.6(1)], hence $\Delta((R/I)G) \subseteq N((R/I)G)$ by [8, Corollary, p.682]. As $\Delta((R/I)G)$ is a nil ideal, it is contained in $J((R/I)G)$. Since $IG \subseteq J(RG)$ by the assumption, we get

$$J((R/I)G) \cong J(RG/IG) = J(RG)/IG.$$

Thus $\Delta(RG) \subseteq J(RG)$. □

Proposition 3.7. *Let R be a UNJ-ring and G be a locally finite 2-group. If $J(R) \subseteq J(RG)$, then for each $u \in U(RG)$ there exist $n \in N(R)$ and $j \in J(RG)$ such that $u = 1 + n + j$. In particular, RG is a UNJ-ring.*

Proof. Since $R/J(R)$ is a UU-ring by Proposition 2.5 and $J(R) \subseteq J(RG)$, we get that $\Delta(RG) \subseteq J(RG)$ by Lemma 3.6.

Now, let $u \in U(RG)$. Then $\epsilon(u) = 1 + \epsilon(u-1) \in U(R)$ by Lemma 3.1(1) used for the augmentation map ϵ and the inclusion ι. Since R is a UNJ-ring, there exist $n \in N(R)$ and $j \in J(R)$ such that $\epsilon(u) = 1 + n + j$. Now we can easily compute using Lemma 3.1(1) again that $\epsilon(u - 1 - n - j) = 0$, and so $u - 1 - n - j \in \Delta(RG) \subseteq J(RG)$. As $j \in J(R) \subseteq J(RG)$, we can see that $u-1-n \in J(RG)$. Thus $u = 1+n+(u-1-n)$ presents a desired decomposition of U. □

Lemma 3.8. *Let G be a locally finite 2-group, R be a ring and I and ideal such that $I \subseteq J(R) \cap J(RG)$ and R/I is a UU-ring. Then $J(R) \subseteq J(RG)$.*

Proof. Observe that $\Delta(RG) \subseteq J(RG)$ by Lemma 3.6. Then we obtain

$$\frac{RG}{J(RG)} \cong \frac{RG/\Delta(RG)}{J(RG)/\Delta(RG)} \cong \frac{R}{J(R)},$$

which implies that $J(R) \subseteq J(GR)$. □

Now we are able to formulate an analogue of [11, Theorem 2.1(ii)] and [13, Theorem 2.22] which characterizes UNJ-rings over locally finite groups.

Theorem 3.9. *Let G be a locally finite group and R be a ring. Put $H = G \cap 1 + J(RG)$. Then H is a normal subgroup of G. Furthermore, RG is a UNJ-ring if and only if G/H is a 2-group, $J(R) \subseteq J(R(G/H))$ and R is a UNJ-ring.*

Proof. Obviously, G is a subgroup of the group $U(RG)$. Consider a group homomorphism $\varphi : G \to U(RG/J(RG))$ defined by $\varphi(g) = g + J(RG)$. Then $\operatorname{Ker}\varphi = \{g \in G| \mid g - 1 \in J(RG)\} = H$ which implies that H is a normal subgroup of the group G and G/H is isomorphic to a subgroup of the group $U(RG/J(RG))$.

($\Rightarrow$) By Proposition 2.5, $RG/J(RG)$ is a UU-ring. Put $I = R \cap J(RG)$. Applying Lemma 3.1(1) for the augmentation map $\epsilon : RG \to R$ and the inclusion $\iota : R \to RG$, we see that $I = \epsilon\iota(I) \subseteq \epsilon(J(RG)) \subseteq J(R)$. Then we have a natural embedding $R/I \to RG/J(RG)$ which means that R/I is a subring of the UU-ring $RG/J(RG)$. So it is a UU-ring as well by [10, Theorem 2.6].

Since $U(RG/J(RG))$ is a 2-group by [10, Theorem 3.4] and G/H is isomorphic to a subgroup of $U(RG/J(RG))$, it is a 2-group as well. Finally, applying Lemma 3.8 on the group G/H and the ideal I, we obtain $J(R) \subseteq J(R(G/H))$. Finally, R is a UNJ-ring by Proposition 3.3.

($\Leftarrow$) Let us denote by $\Delta(H)$ an ideal of RG generated by the set $\{h-1|h \in H\}$ and recall that $RG/\Delta(H) \cong R(G/H)$ by [20, Corollary 3.3.5]. Furthermore, let us observe that $h-1 \in J(RG)$ for every $h \in H$ by the definition of the subgroup H. Hence $\Delta(H) \subseteq J(RG) \cap \Delta(RG)$ which implies $J(RG/\Delta(H)) = J(RG)/\Delta(H)$.

Let $u \in U(RG)$. Then $u+\Delta(H) \in U(RG/\Delta(H)) \cong U(R(G/H))$ and we obtain from Proposition 3.7 applied on the locally finite 2-group G/H and the ring R that there exist $n \in R$ and $j + \Delta(H) \in J(RG)/\Delta(H) = J(RG/\Delta(H)$ such that $u + \Delta(H) = 1 + n + j + \Delta(H)$. Since $\Delta(H) \subseteq J(RG)$ there exists $\tilde{j} \in\subseteq J(RG)$ for which $u = 1 + n + r + \tilde{j}$ with $n \in N(R) \subseteq N(RG)$ and $j + \tilde{j} \in J(RG)$, which finishes the proof. □

Acknowledgment. M. Tamer Koşan was supported by a grant (117F070) from TUBITAK of Turkey. T. Cong Quynh has been partially founded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant number 101.04-2017.22.

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