ON VIRTUALLY REGULAR MODULES AND RINGS

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ABSTRACT. A right R-module M is called (strongly) virtually regular if every cyclic (finitely generated) submodule is isomorphic to a direct summand of M, and M said to be completely virtually regular if every submodule of M is virtually regular. In this paper, we provide description of classes of (strongly) virtually regular rings, in particular for semiperfect rings. Furthermore, we completely characterize (strongly/completely) virtually regular Abelian groups, which generalizes recently published results.

1. INTRODUCTION

In a series of papers [4, 5], Facchini et all and, independently, Behboodi et all [9] introduced and studied structures of modules with chain conditions up to isomorphism, and they called a module M *virtually/iso semisimple* if every submodule of M is isomorphic to a direct summand of M. A non-zero indecomposable virtually semisimple right R-module is called *virtually/iso simple*. Recently, the authors of [10] introduced the concepts of virtually regular modules: a right R-module M *(strongly) virtually regular* if every cyclic (finitely generated) submodule is isomorphic to a direct summand of M, and a module M said to be *completely virtually regular* if every submodule of M is virtually regular. Let us recall that these terms generalize the following well-known concepts: A ring R is said to be *von Neumann regular* if for each $x \in R$, there exists an element y in R such that $x = xyx$. Various module theoretic versions of von Neumann regular rings have been considered by Azumaya [3], Fieldhouse [11], Ware [16], and Zelmanowitz [17], in particular, a module M is said to be

Zelmanowitz-regular if for each $x \in M$ there exists a homomorphism $f : M \to R$ such that $f(x)x = x$;

Azumaya-regular if every submodule of M is locally split in M (i.e., for any $x_0 \in N$ there exists a homomorphism $s : M \to N$ such that $s(x_0) = x_0$). Equivalently, every cyclic submodule of M is a direct summand?

Recalling a well-known characterization of regular rings as rings such that each principal left (or right) ideal is generated by an idempotent, we obtain in Proposition 2.2 similar result that R is a right virtually regular ring if and only if $r(a)$ is a summand of R for each $a \in R$ if and only if $r(a)$ is generated by an idempotent for each $a \in R$, where

2010 *Mathematics Subject Classification.* 16D50, 16D60, 18G25.

Key words and phrases. Virtually regular modules and rings, abelian groups.

$r(a)$ denotes the right annihilator of $a \in R$. This useful characterization leads to obtain the important closure property that the class of virtually regular rings is inherited by direct products (Theorem 2.3). Moreover, they yield that a local ring is right virtually regular if and only if it is a domain (Corollary 2.5) and, if R is an abelian semiperfect ring, then R is right virtually regular if and only if R is isomorphic to a product of finitely many local domains (Theorem 2.6). By the chart in [10], virtually/iso semisimple modules are strongly virtually regular (and hence, virtually regular). It is shown in Theorem 2.7 that the converse is true if R is a semiperfect right Kasch ring. Section 2 is closed by the following result: R is right virtually regular and satisfies the right C2-condition (i.e. every right ideal of R that is isomorphic to a direct summand of R is a direct summand of R) if and only if R is a von Neumann regular ring (Theorem 2.8).

In section 3, we obtained a new characterization of strongly virtually regular ring over semiperfect rings Rs, and it is shown in Theorem 3.5 that, R is right strongly virtually regular ring if and only if there exists natural numbers $k, n_1, \ldots, n_k$ and right chain domains S_i for all $i = 1, \ldots, k$ such that $R \cong \prod_{i=1}^{k} M_{n_i}(S_i)$. This yields that a commutative semiperfect ring is strongly virtually regular if and only if it is a finite product of valuation domains (Corollary 3.6).

The assertion of [10, Proposition 16] that a non-zero finite abelian p-group A is virtually regular iff $A \cong (\mathbb{Z}_p)^{a_1} \oplus (\mathbb{Z}_{p^2})^{a_2} \oplus \ldots \oplus (\mathbb{Z}_{p^k})^{a_k}$ for some positive integers $a_1, \ldots a_k$ is generalized in Proposition 4.1, which claims that a bounded abelian p-group A is virtually regular if and only if there exist non-zero cardinals α_i for each $i \leq n$ such that $A \cong \bigoplus_{i<\epsilon_p(A)} \mathbb{Z}_{p^i}^{(\alpha_i)}$. Since a torsion group is virtually regular if and only if each p-torsion component is virtually regular by [10, Proposition 17] and bounded torsion group is a direct sum of finitely many p-torsion components, we obtained Corollary 4.2. It is also shown in Corollary 4.4 that a torsion abelian group A is virtually regular if and only if A contains a subgroup $\bigoplus_{p\in\mathbb{P}} \bigoplus_{i<\epsilon_p(A)} A_i$ of A such that $A_{pi} \cong \mathbb{Z}_{p^i}$ is a pure subgroup of A for each $p \in \mathbb{P}$ and $i < \epsilon_p(A)$. Since a torsion-free module over a domain R is virtually regular if and only if it contains a direct summand isomorphic to R by [10, Proposition 11], we can say that virtually regular torsion-free groups are exactly those containing a free summand of rank 1. Thus, we have a characterization of virtually regular mixed abelian groups (i.e. groups which are neither torsion, nor torsion-free). Let A be a mixed abelian group. It is shown in Theorem 4.5 that A is virtually regular if and only if A contains a direct summand isomorphic to $\mathbb{Z}$ and a subgroup $\bigoplus_{p\in\mathbb{P}} \bigoplus_{i<\epsilon_p(A)} A_i$ such that $A_{pi} \cong \mathbb{Z}_{p^i}$ is a pure subgroup of A for each $p \in \mathbb{P}$ and $i < \epsilon_p(A)$.

Suppose A is a torsion-free abelian group. Let $\{f_i\}_{\in I}$ be the collection of all homomorphisms $A \to \mathbb{Z}$, and let A^+ be $\cap_{i\in I}\ker(f_i)$ (recall *Arnold-Vinsonhaler invariants* [1, 2]). For each ordinal α, we inductively define a subgroup A_α, as follows:

$$A_0 := A;$$

if $\alpha > 0$ and A_β has been defined for all $\beta < \alpha$;

$$A_\alpha := \cap_{\beta<\alpha}(A_\beta)^+.$$

We say A is *free-reduced* if there exists some α such that $A_\alpha = 0$. Section 4 is devoted to the characterization of strongly virtually regular abelian groups. Let G be an abelian group with torsion T and $A := G/T$ be the corresponding torsion-free quotient. It is obtained in Theorem 4.6 that T is completely virtually regular iff $pT_p = 0$ for all primes p; A is completely virtually regular iff it is free-reduced, and, finally, G is completely virtually regular iff T and A are completely virtually regular. We will close section 4 with some characterizations of torsion-free completely virtually regular abelian groups. As an early result, we discuss situations in which a free-reduced group must actually be free (Proposition 4.11). Recall that a finite rank torsion-free group is called a *Butler group* if it can be embedded as a pure subgroup of a group that is completely decomposable (of finite rank), which holds exactly when it is a homomorphic image of such a (finite rank) completely decomposable group. It is shown in Theorem 4.12 that if A is a torsion-free group such that there is a composition series and (not necessarily Butler) finite-rank subgroups C_γ $(0 \le \gamma < \lambda)$ such that each $B_{\gamma+1} = B_\gamma + C_\gamma$, then A is free-reduced iff if it is free. This theorem yield the following interesting results: If A is a B_2-group (it is a generalization of Butler groups of finite-rank which coincide in the case of groups of countable rank) , then A is free-reduced iff it is free (Corollary 4.13), a (torsion-free) group of countable rank is free-reduced iff it is free (Corollary 4.14), and if $A = G/T$ is either of countable rank or is a B_2-group, where G is a mixed group G with torsion T, then G is completely virtually regular iff G has a decomposition $G \cong T \oplus A$ such that T is semisimple and A is free (Corollary 4.15).

In order to obtain a surprising connection between free-reduced groups and the continuum hypothesis, we use scalar products on the Baer-Specker group P. Recall that $P = \prod_\omega \mathbb{Z}$ is the set of all elements $x = \sum_{i\in\omega} x_i e_i$ with $x_i \in \mathbb{Z}$ and $e_i \in P$ defined by the Kronecker symbol, addition is defined component-wise. We also remark that the *continuum hypothesis* in [12], in short CH, means $2^\omega = \omega_1$. Note that the Baer-Specker group P has the subgroup S of all elements x of finite support, i.e. $x_i = 0$ for almost all $i \in \omega$. Now, it is shown in Theorem 4.17 that the continuum hypothesis is logically equivalent that every (torsion-free) free-reduced group A of cardinality $|A| < 2^\omega$ is necessarily free. We also produce some relevant examples.

Throughout,

(i) R is an associative ring with unit 1 and modules M_R are usually stand for a unital right R-module and $\mathbb{Z}_n$ is the ring of integers modulo n.

(ii) $J(R)$ denotes the Jacobson radical of R,

(iii) $X \le^\oplus M$ means X is a direct summand of M,

(iv) a ring R is said to be domain if $r_R(a) = 0$ for each non-zero $a \in R$, where $r_R(a)$ denotes the right annihilator of $a \in R$, denote such a group with $T \le G$ its torsion

(v) A is an abelian group with torsion T an groups and if p is a prime, then $T_p \leq T$ will denote its p-torsion,

(vi) for a prime integer p, an abelian group A is said to be p-group if the order of every element of A is a power of p.

(vii) the expression *rank of* A will refer to its torsion-free rank.

$(viii)$ for a prime p, the *p-rank* of G will be the dimension of the *p-socle*, $G[p]$, as a vector space over $\mathbb{Z}(p)$.

Furthermore, our notation and notions are all standard and may be found in the books [14, 15, 6, 7, 8].

2. Virtually regular rings

We begin with the following elementary examples of virtually regular rings.

Example 2.1. (1) The ring $\mathbb{Z}_n$, $n \geq 2$, is virtually regular if and only if n is square-free.

(2) The homomorphic images of a virtually regular ring need not virtually regular: $\mathbb{Z}$ is virtually regular, but $\mathbb{Z}_4$ is not.

Recall that a ring R is said to be *right Rickart* if $r_R(a)$ is generated by an idempotent for each $a \in R$.

We introduce the easy extension of the characterization [10, Proposition 1(1)].

Proposition 2.2. *The following conditions are equivalent for a ring R:*

(1) R is right virtually regular,
(2) R is right Rickart,
(3) every principal right ideal of R is projective.

Proof. (1)⇔(3) is proved in [10, Proposition 1(1)].
(2)⇔(3) is proved in [15, Proposition 7.48]. □

The class of virtually regular rings is closed under taking the direct product:

Theorem 2.3. *The ring $\prod_i R_i$ is right virtually regular if and only if each ring R_i is right virtually regular.*

Proof. Denote by $(e_i)_i$ the set of central orthogonal idempotents of R such that $e_iR \cong R_i$ and $R \cong \prod_i e_iR$, i.e. we may identify rings R_i and e_iR, and consider R_i as an ideal in R generated by a central idempotent e_i.
(:⇒) Let $a \in R_i$ for a fixed i. Note that $e_ia = a$ and $e_ja = 0$ for all others $j \neq i$. Since there exists an idempotent, say $f \in R$, such that $r_R(a) = fR$ by Proposition 2.2, we have that $r_{R_i}(a) = e_ifR$. Furthermore, as each e_i is a central idempotent, we obtain that e_if is an idempotent of R_i again by Proposition 2.2. Thus each R_i is a right virtually regular ring.
(⇐:) Let $a \in R$. Then there exists an idempotent $f_i \in R_i$ such that $r_{R_i}(e_ia) = f_iR_i$ by Proposition 2.2. Let $f := (f_i) \in R$. Then it is easy to see that $f^2 = f \in R$ and $r_R(a) = fR$ as desired by Proposition 2.2. □

Recall that any commutative domain is right and left virtually regular by [10, Example 2(2)]. We show that it is the only kind of examples right virtually regular rings which are indecomposable as modules.

Proposition 2.4. *Assume that a ring R contains only trivial idempotents. Then R is right (left) virtually regular if and only if it is a domain.*

Proof. (:$\Rightarrow$) If R is right virtually regular and $a \in R$ is non-zero, then $r(a)$ is a direct summand of R by Proposition 2.2. On the other hand, since $r_R(a) \neq R$ is generated by an idempotent by Proposition 2.2, we get that $r_R(a) = 0$. Hence $a \cdot b \neq 0$ for each non-zero $b \in R$.
($\Leftarrow$:) If R is a domain and $a \in R$ is non-zero, then $r_R(a) = 0$. It means that $r_R(a)$ is generated by the idempotent 0 and the assertion follows from Proposition 2.2, cf. [10, Example 2(2)]. □

Now, Proposition 2.4 yields the following consequence.

Corollary 2.5. *A local ring is right virtually regular if and only if it is a domain.*

Recall that a ring is called *abelian* provided each its idempotent is central.

Theorem 2.6. *Let R be an abelian semiperfect ring. The following conditions are equivalent:*

(1) R is right virtually regular,
(2) R is isomorphic to a product of finitely many local domains.

Proof. (1)$\Rightarrow$(2). Let R be right virtually regular. Since R is an abelian semiperfect ring, there exists a finite sequence $(e_i \mid i \leq n)$ of orthogonal central idempotents such that $\sum_i e_i = 1$ and e_iR is a local ring by [14, Theorem 23.6]. Then $R \cong \prod_i e_iR$, where each e_iR is a virtually regular ring by Theorem 2.3. Hence e_iR is a domain by Corollary 2.5 for each i.
(2)$\Rightarrow$(1). The claim follows from Proposition 2.2 and Theorem 2.3. □

Recall that, a ring R is called *right Kasch* if every simple right R-module embeds into R.

Proposition 2.7. *Let R be a semiperfect right Kasch ring. The following conditions are equivalent:*

(1) R is right virtually regular,
(2) R is semisimple.

Proof. (1)$\Rightarrow$(2). Since R is a semiperfect ring, there exists an orthogonal sequence $(e_i \mid i \leq n)$ of idempotents such that $\sum_i e_i = 1$ and e_iR is an indecomposable right ideal by [14, Theorem 23.6]. Let $i \leq n$. As R is a right Kasch and right virtually regular ring, we obtain that the simple module $e_iR/e_iJ(R)$ is isomorphic to a right ideal of R, and hence it is projective. This implies that $e_iJ(R)$ is a direct summand in e_iR and so

$e_iJ(R) = 0$ because e_iR is indecomposable. We have proved that $e_iJ(R) = 0$ for each i. Hence $J(R) = \sum e_iJ(R) = 0$ and $R \cong R/J(R)$ is semisimple.
(2)⇒(1). The claim is clear by the definition. □

A ring R satisfies *the right C2-condition* if every right ideal of R that is isomorphic to a direct summand of R is a direct summand of R.

Theorem 2.8. *The following are equivalent for a ring R:*

(1) R is von Neumann regular,
(2) R is right virtually regular and satisfies the right C2-condition,
(3) R is strongly right virtually regular and satisfies the right C2-condition.

Proof. (1)⇒(3) Since the strong virtually regularity is obvious, we need to show that R satisfies the right C2-condition. Let I be an ideal of R such that $I \cong eR$ for some $e^2 = e \in R$. Then $I = aR$ for some $a \in R$. Since R is regular, we obtain that $I = aR$ is a direct summand of R, as desired.
(3)⇒ (2) This is obvious.
(2)⇒ (1) Let I be a principal ideal of R. Since R is a right virtually regular ring, we have that $I \cong eR$ for some $e^2 = e \in R$. Now, by the right C2-condition, $I = aR$ is a direct summand of R which means that R is a regular ring. □

3. Strongly virtually regular rings

Recall that a module M is *chain* if the lattice of submodules of M is linearly ordered (equivalently, if, for each $a, b \in M$, either $aR \subseteq bR$ or $bR \subseteq aR$).

Example 3.1. Let R be a right chain domain.
Claim. The matrix ring $M_n(R)$ is semiperfect right strongly virtually regular: Since every finitely generated right ideal of R is principal, it is isomorphic to R_R, which means that R is local, right strongly virtually regular, and so right semihereditary. Furthermore $M_n(R)$ is right semihereditary as well by [15, Theorem 7.62] and every finitely generated submodule of R_R^n is isomorphic to R^k for some natural k by [15, Theorem 2.29]. Since R is right Ore, there exists the division ring of right fractions Q which is flat as left R-module by [15, Proposition 4.4]. If we have a monomorphism $R^k \hookrightarrow R^n$, it induces an embedding

$$Q_Q^k \cong R_R^k \otimes_R Q_Q \hookrightarrow R_R^n \otimes_R Q_Q \cong Q_Q^n,$$

thus $k \leq n$, i.e. R^n contains only submodules isomorphic to R^k for suitable $k \leq n$. Then any right ideal of $\operatorname{End}(R_R^n)$ generated by finitely many endomorphisms $\sigma_1, \ldots, \sigma_s$ is generated by arbitrary epimorphism $R^n \to \sum_i \operatorname{im}(\sigma_i)$, which exists as $\sum_i \operatorname{im}(\sigma_i) \cong R^k$ for $k \leq n$, This implies that the ring $\operatorname{End}(R_R^n) \cong M_n(R)$ is right Bezout and right semihereditary, and hence it is a right strongly virtually regular ring (cf. [10, Proposition 1(2)]). Finally, note that $M_n(R)$ is semiperfect by [14, Corollary 23.9].

In the rest of the section, we show that semiperfect right strongly virtually regular rings are necessarily of the form presented in Example 3.1.

Let us denote by $l(M)$ a composition length of any semisimple module M, i.e. the number $l(M) = l$ such that there exists simple modules $S_1, \dots S_l$ for which $M = \bigoplus_{i=1}^{l} S_i$.

Lemma 3.2. *If R is a semiperfect right strongly virtually regular ring, then every central idempotent of $R/J(R)$ can be lifted to a central idempotent of R.*

Proof. Let $\overline{e} \in R/J(R)$ be a central idempotent. Since R is semiperfect, $\overline{e}$ can be lifted to an idempotent $e \in R$.
Claim. $(1-e)Re = 0$: Let $r \in R$ and put $c := (1-e)re$ and $M =: cR + eR$. Then $M = cR \oplus eR$ is a principal right ideal as R is strongly virtually regular, and $M/J(M) \cong cR/J(cR) \oplus eR/eJ$. Hence there exists an epimorphism of $R_R = eR \oplus (1-e)R$ onto $M/J(M)$. Note that $\mathrm{Hom}((1-e)R/(1-e)J(R), eR/eJ(R)) = 0$ and $\mathrm{Hom}((1-e)R, eR/eJ) = 0$ since $e + J = \overline{e}$ is a central idempotent of $R/J(R)$. Moreover cR is a homomorphic image of eR, which means that $\mathrm{Hom}((1-e)R, cR/J(cR)) = 0$. Then

$$\begin{aligned} \mathrm{Hom}(R, M/J(M)) &= \mathrm{Hom}(eR, M/J(M)) \oplus \mathrm{Hom}((1-e)R, M/J(M)) \\ &= \mathrm{Hom}(eR, M/J(M)) \cong \mathrm{Hom}(eR/eJ, M/J(M)), \end{aligned}$$

and hence

$$l(M/J(M)) = l(cR/J(cR)) + l(eR/eJ(R)) \leq l(eR/eJ(R)),$$

which already proves that $cR/J(cR) = 0$ and so $(1-e)re = c = 0$. We have shown that $(1-e)Re = 0$ and the symmetric argument for $1-e$ gives us that $eR(1-e) = 0$, i.e. e is central. □

We need the following easy observation.

Lemma 3.3. *If R is a semilocal ring and C is a cyclic module, then $l(C/J(C)) \leq l(R/R(J)) < \infty$.*

Now, we are ready to describe semiperfect strongly virtually regular rings with only one simple module up to isomorphism.

Lemma 3.4. *Let R be a semiperfect right strongly virtually regular ring. If $R/J(R)$ is an indecomposable ring, then there exists a right chain domain S such that $R \cong M_n(S)$ for $n = l(R/J(R))$*

Proof. Since R is semiperfect and $R/J(R)$ is indecomposable, we obtain that
• there exists a complete orthogonal sequence $(e_i \mid i \leq n)$ of idempotents such that $\sum_i e_i = 1$,
• $e_i R \cong e_j R$ are indecomposable right ideals for all i, j, and
• $S \cong e_i R/e_i J(R)$ is up to isomorphism unique right simple module over R by [14,

Theorem 23.6], which implies that $R \cong \mathrm{End}(\bigoplus_i e_i R) \cong M_n(e_1 R e_1)$.

Claim 1. e_1R is a chain right ideal: Assume to contrary that there exist $a, b \in e_1R$ such that $aR \not\subseteq bR$ and $bR \not\subseteq aR$. Then $C = aR \cap bR$ is a proper submodule of both right ideals aR and bR and $aR + bR/C \cong aR/C \oplus bR/C$. Hence there exists a right ideal D such that $C \leq D \leq aR + bR$ and $aR + bR/D \cong S^2$. As

$$E = aR + bR + (1 - e_1)R = aR + bR \oplus (1 - e_1)R,$$

there exists an epimorphism of $E/J(E)$ onto

$$E/(D \oplus ((1 - e_1)J)) \cong S^2 \oplus (1 - e_1)R/(1 - e_1)J \cong S^{n+1}.$$

Since E is finitely generated and R is right strongly virtually regular ring, it is cyclic and we obtain in such way $l(E/J(E)) > n$ which contradicts to $l(R/J(R)) = n$ by Lemma 3.3. We have proved that e_1R is a chain right ideal, and hence e_1Re_1 is a right chain ring.

Claim 2. e_1Re_1 is a domain: It suffices to show that $r_R(e_1re_1) \cap e_1R = 0$ for each $e_1re_1 \neq 0$. Let $e_1re_1 \neq 0$. There exists an idempotent $f \neq 1$ such that $r_R(e_1re_1) = fR$ by Proposition 2.2. Clearly, $(1 - e_1) \in fR$, which implies that $(1 - e_1)R \subseteq fR$. Since $R/(1-e_1)R \cong e_1R$ and $fR/(1-e_1)R$ is isomorphic to a cyclic submodule of eR which is projective by Proposition 2.2 as R_1 is right virtually regular, we obtain that there exists a right ideal A for which

$$fR \cong A \oplus (1 - e_1)R \quad \text{and} \quad R \cong (1 - f)R \oplus A \oplus (1 - e_1)R.$$

Furthermore, as $l((1 - f)R/(1 - f)J) > 0$, $l((1 - e_1)R) = n - 1$ and

$$\begin{aligned} n = l(R/J(R)) &= l((1 - f)R/(1 - f)J \oplus A/AJ \oplus (1 - e_1)R/(1 - e_1)J) \\ &= l((1 - f)R/(1 - f)J) + l(A/AJ) + l((1 - e_1)R/(1 - e_1)J), \end{aligned}$$

we get that $A = 0$. Finally, we can see that $fR = (1 - e_1)R$ and $fR \cap e_1R = 0$, which proves that e_1Re_1 a domain. □

Theorem 3.5. *Let R be a semiperfect ring. The following are equivalent:*

(1) R is right strongly virtually regular ring,

(2) there exists natural numbers k, $n_1, \dots, n_k$ and right chain domains S_i for all $i = 1, \dots, k$ such that $R \cong \prod_{i=1}^k M_{n_i}(S_i)$.

Proof. (1) ⇒ (2). Since $R/J(R) \cong \prod_{i=1}^k M_{n_i}(D_i)$ for some k, $n_1, \dots, n_k$ and division rings D_i by Wedderburn–Artin theorem, there exists central idempotents $e_i \in R$ such that $R \cong \prod_{i=1}^k e_iR$ and $e_iR/J(e_iR) \cong M_{n_i}(D_i)$ by Lemma 3.2. Observe that e_iR is a right strongly virtually regular ring by Theorem 2.3. Hence Lemma 3.4 implies that there exists a right chain domain S_i satisfying $e_iR \cong M_{n_i}(S_i)$ for each $i = 1, \dots, k$, which finishes the proof.

(1) ⇒ (2). The implication follows from Example 3.1. □

Note that the Jategaonkar's example [13, Theorem 4.6] of a left chain domain which is not right chain witnesses that the notion of a strongly virtually regular ring is not right-left symmetric.

In commutative case we obtain the following immediate consequence.

Corollary 3.6. *A commutative semiperfect ring is strongly virtually regular if and only if it is a finite product of valuation domains.*

4. Virtually regular Abelian groups

Denote by $\mathbb{P}$ the set of all prime numbers. If A is an abelian group and $p \in \mathbb{P}$, denote by

$$\epsilon_p(A) = \sup\{k+1 \in \mathbb{N} \cup \{0\} \mid \exists a \in A : \mathrm{ord}(A) = p^k\}$$

and observe that $\epsilon_p(A) = \omega$ precisely if the p-torsion component A_p is unbounded, and $k = p^{\epsilon_p(A)-1}$ is the least natural number such that $kA_p = 0$ whenever A_p is bounded.

We can extend [10, Proposition 16]:

Proposition 4.1. *Let A be a bounded p-group. The following are equivalent:*

(1) A is virtually regular,

(2) there exist non-zero cardinals α_i for each $i \leq n$ such that $A \cong \bigoplus_{i<\epsilon_p(A)} \mathbb{Z}_{p^i}^{(\alpha_i)}$.

Proof. (1)⇒(2). Let A be virtually regular. Since A is a bounded p-group, it is a direct sum of cyclic groups of order p^i for some $i < \epsilon_p(A)$ by [8, Theorem 3.5.1]. Thus $A \cong \bigoplus_{i<\epsilon_p(A)} \mathbb{Z}_{p^i}^{(\alpha_i)}$ for some cardinals α_i. It remains to show that $\alpha_i > 0$ for each $i < \epsilon_p(A)$. Since A contains an element a of the order $p^{\epsilon_p(A)-1}$, the subgroup $\langle a^{p^{\epsilon_p(A)-i-1}} \rangle \cong \mathbb{Z}_{p^i}$ is isomorphic to a direct summand of A. Hence we obtain that $\alpha_i > 0$ by the Krull-Schmidt-Azumaya theorem as desired.
(2)⇒(1). Let $A = \bigoplus_{i<\epsilon_p(A)} \mathbb{Z}_{p^i}^{(\alpha_i)}$. Then each cyclic subgroup is isomorphic to $\mathbb{Z}_{p^i}$ for some $i < \epsilon_p(A)$ which is a direct summand of A, which means that A is virtually regular. □

Since a torsion group is virtually regular if and only if each p-torsion component is virtually regular by [10, Proposition 17] and a bounded torsion group is a direct sum of finitely many p-torsion components, we get the following consequence:

Corollary 4.2. *The following are equivalent for a bounded (torsion) group A:*

(1) A is virtually regular,

(2) A is isomorphic to a group $\bigoplus_{i\leq n} \bigoplus_{j\leq e_i} \mathbb{Z}_{p_i^j}^{(\alpha_{ij})}$ for non-negative integers n, e_i, prime numbers p_i, and non-zero cardinals α_{ij}, where $i \leq n$ and $j \leq e_i$.

Now, we describe the structure of general virtually regular p-groups.

Lemma 4.3. *Let A be an abelian p-group. The following are equivalent:*

(1) A is virtually regular,

(2) *there exists a subgroup* $\bigoplus_{i<\epsilon_p(A)} A_i$ *of* A *such that* $A_i \cong \mathbb{Z}_{p^i}$ *is a pure subgroup of* A *for each* $i < \epsilon_p(A)$.

Proof. (1)⇒(2). This inclusion follows from the fact that any p-basic subgroup $B \leq A$ (which will be pure), has for each $i < \epsilon_p(A)$, a cyclic summand of order p^i.
(2)⇒(1). Assume that A contains a subgroup $\bigoplus_{i<\epsilon_p(A)} A_i$ such that $A_i \cong \mathbb{Z}_{p^i}$ is a pure in A for each $i < \epsilon_p(A)$. Let $a \in A$. Then there exists $i < \epsilon_p(A)$ such that $\operatorname{ord}(a) = p^i$. Hence $\langle a \rangle \cong \mathbb{Z}_{p^i} \cong A_i$. Since A_i is pure in A, it is a direct summand of A by [8, Lemma 5.2.1]. □

The following follows immediately from [10, Proposition 17], Lemma 4.3 and the fact that a torsion group is a direct sum of its p-torsion components.

Corollary 4.4. *The following conditions are equivalent:*

(1) *A torsion abelian group* A *is virtually regular,*
(2) A *contains a subgroup* $\bigoplus_{p\in\mathbb{P}} \bigoplus_{i<\epsilon_p(A)} A_{pi}$ *of* A *such that* $A_{pi} \cong \mathbb{Z}_{p^i}$ *is a pure subgroup of* A *for each* $p \in \mathbb{P}$ *and* $i < \epsilon_p(A)$.

Recall that a torsion-free module over a domain R is virtually regular if and only if it contains a direct summand isomorphic to R by [10, Proposition 11], so virtually regular torsion-free groups are exactly those containing a free summand of rank 1. Now we can sum all our observation to characterize virtually regular mixed abelian groups, i.e. groups which are neither torsion, nor torsion-free.

Theorem 4.5. *The following conditions are equivalent a mixed abelian group* A*:*

(1) A *is virtually regular,*
(2) A *contains a direct summand isomorphic to* $\mathbb{Z}$ *and a subgroup* $\bigoplus_{p\in\mathbb{P}} \bigoplus_{i<\epsilon_p(A)} A_{pi}$ *such that* $A_{pi} \cong \mathbb{Z}_{p^i}$ *is a pure subgroup of* A *for each* $p \in \mathbb{P}$ *and* $i < \epsilon_p(A)$.

Proof. (1)⇒(2). Let A be virtually regular and let $T(A)$ denote the torsion part of A. Then $A/T(A)$ and $T(A)$ are virtually regular by [10, Corollary 3]. Then $A/T(A)$ contains a free summand of rank 1 by [10, Proposition 11] and there exists an epimorphism $A \to \mathbb{Z}$. Thus there exists a direct summand of A isomorphic to $\mathbb{Z}$. As $T(A)$ is virtually regular, there exists a subgroup $\bigoplus_{p\in\mathbb{P}} \bigoplus_{i<\epsilon_p(A)} A_i$ of $T(A)$ for which $A_{pi} \cong \mathbb{Z}_{p^i}$ is a pure subgroup of $T(A)$ for $p \in \mathbb{P}$ and $i < \epsilon_p(A)$ by Corollary 4.4. Since $T(A)$ is pure in A, we obtain that A_{pi} is pure in A, which finished the proof.
(2)⇒(1). The claim follows from Corollary 4.4and [10, Corollary 3 and Proposition 11]. □

We have following description of completely virtually regular abelian groups.

Theorem 4.6. *Let* G *be a group with torsion* T *and* $A := G/T$ *be the corresponding torsion-free quotient. The following statements hold:*

(a) T *is completely virtually regular if and only if* $pT_p = 0$ *for all primes* p*;*

(b) *A is completely virtually regular if and only if it is free-reduced;*
(c) *G is completely virtually regular if and only if T and A are completely virtually regular.*

Proof. Regarding (a), suppose first that T is completely virtually regular. If $pT \neq 0$ for some prime p, then there is an $x \in T$ of order p^2. Then $S = \langle x \rangle \cong \mathbb{Z}(p^2)$ has a cyclic subgroup isomorphic to $\mathbb{Z}(p)$, but S has no a direct summand of that order, i.e. S is not virtually regular. Therefore, T is not completely virtually regular.

Conversely, suppose that $pT_p = 0$ for all primes p. If $S \leq T$, then $pS_p = 0$ for all primes p as well. Since S is semisimple, we have that all of its subgroups, and in particular, all its cyclic subgroups, are direct summands, i.e. S is virtually regular. Therefore, T is completely virtually regular.

Turning to (b), suppose first that A is completely virtually regular. Define, for all ordinals α, the descending subgroups A_α in the definition contained in the introduction. If β is any ordinal, $A_\beta \neq 0$ and $\alpha = \beta + 1$, it will suffice to show that $A_\alpha \neq A_\beta$: Let $0 \neq x \in A_\beta \neq 0$. Since A is torsion-free, we must have $\langle x \rangle \cong \mathbb{Z}$. Now, the complete virtually regularity of A gives that A_β is virtually regular, and so there exists a decomposition $A_\beta = Z \oplus Y$, where $Z \cong \mathbb{Z}$. It follows that $A_\alpha \leq Y < A_\beta$, i.e. $A_\alpha \neq A_\beta$, completing this implication.

Conversely, suppose that $A_\alpha = 0$ for some ordinal α and $0 \neq x \in B < A$. Clearly, $\langle x \rangle \cong \mathbb{Z}$. Let $\beta < \alpha$ be the smallest ordinal such that $B \leq A_\beta$. Therefore, B will not be contained in $A_{\beta+1}$ which means that there is a homomorphism $\phi : A_\beta \to \mathbb{Z}$ such that $\phi(B) \neq 0$. If $K = \ker(\phi) \cap B$, then it follows that $B = Z \oplus K$, where $Z \cong \mathbb{Z} \cong \langle x \rangle$. This shows that B is virtually regular, i.e. A is completely virtually regular.

Finally, regarding (c), suppose first that G is completely virtually regular. Since it is clear that an arbitrary subgroup of a completely virtually regular group inherits that property, the group T is completely virtually regular. Turning to $A = G/T$, if A failed to be completely virtually regular, then there would be a non-zero $B \leq A$ such that every homomorphism $B \to \mathbb{Z}$ is 0. If $H/T = B$, then it would follow that H has elements of infinite order. But, since any homomorphism $H \to \mathbb{Z}$ factors through $B = H/T \to \mathbb{Z}$, we could conclude that every homomorphism $H \to \mathbb{Z}$ is 0. Therefore, H is not virtually regular, i.e. G is not completely virtually regular.

Conversely, suppose that T and A are completely virtually regular and $0 \neq x \in B \leq G$. If x has infinite order, then $0 \neq \overline{B} := [B+T]/T \leq A$. Since we are assuming that A is completely virtually regular, there must be a non-zero homomorphism $\overline{B} \to \mathbb{Z}$. But then, $B \to \overline{B} \to \mathbb{Z}$ will also be non-zero, i.e. B will have the required infinite cyclic summand. Now, if x has finite order, say n, then $x \in T_B := B \cap T$. Since T is assumed to be completely virtually regular, we obtain that T_B is virtually regular. Therefore, T_B has a cyclic direct summand of order n, and since T_B is pure in B, this (bounded pure) subgroup will also be a summand of B. Therefore, each such B will be virtually regular, i.e G is completely virtually regular. □

We now want to present some results and examples related to groups that are are not free-reduced, i.e., torsion-free completely virtually regular.

In the following, unless specifically noted otherwise, we are assuming all groups are torsion-free.

We include a few easy to verify observations, whose proofs we leave to the reader (for $\alpha = 1$, one uses the definition, and then a natural transfinite induction gives the result for arbitrary α):

Fact 4.7. *If $\phi : A \to B$ is a group homomorphism, then we conclude that $\phi(A_\alpha) \leq B_\alpha$ for every ordinal α.*

Fact 4.8. *From Fact 4.7, we can conclude that if $\{A^i\}_{i\in I}$ is a collection of groups and α is an ordinal, then*

$$\left(\bigoplus_{i\in I} A^i\right)_\alpha = \bigoplus_{i\in I}(A^i)_\alpha.$$

Fact 4.9. *The following statements hold for a group A:*

(a) *If $\eta \leq \alpha$, then $(A/A_\alpha)_\eta = A_\eta/A_\alpha$.*
(b) *If $\eta \geq \alpha$, then $(A/A_\alpha)_\eta = 0$.*

Example 4.10. If A has a rank-1 subgroup C that is not cyclic, then A is not free-reduced. On the other hand, there is a group A such that every rank-1 subgroup $C \leq A$ is cyclic (i.e., *A is homogeneous of type* $\mathbf{0}$), but A is not free-reduced.

Proof. Assume that A has such a rank-1 subgroup C that is not cyclic. Then it is clear that C is not be virtually regular, which implies that A is not completely virtually regular, a contradiction.

On the other hand, by the standard constructions, there are *indecomposable* groups A of rank exceeding 1 such that every rank-1 subgroup is cyclic. Since such a group is not virtually regular, which implies, in particular, that A is not completely virtually regular. □

We now discuss some situations in which a free-reduced group A must actually be free.

Proposition 4.11. *The following statements are equivalent for a torsion-free group A of finite rank:*

(a) *A is free-reduced,*
(b) *A is free.*

Proof. $(a) \Rightarrow (b)$. Suppose that A is free-reduced. By an obvious induction on its rank, we obtain a decomposition $A \cong C_1 \oplus \cdots \oplus C_k$ such that each C_i is indecomposable. By Fact 4.8, we have that each C_i is free-reduced. Since an indecomposable group is virtually regular if and only if it is cyclic, we can conclude that A is free.
$(b) \Rightarrow (a)$. The claim is clear. □

Recall that there are two generalizations of this Butler groups of infinite rank, which coincide in the case of groups of countable rank. One of these is the following: The (torsion-free) group A is a *B_2-group* if and only if there is an ascending sequence of pure subgroups

$$0 = B_0 \leq B_1 \leq B_2 \leq \cdots \leq B_\lambda = A \tag{†}$$

such that whenever $0 \leq \gamma < \lambda$, there is a finite rank Butler group $C_\gamma \leq A$ such that $B_{\gamma+1} = B_\gamma + C_\gamma$. We use the idea of such a composition series in the following.

Theorem 4.12. *Suppose that A is a torsion-free group such that there is a composition series (†) and there exist (**arbitrary**, i.e. not necessarily Butler) finite-rank subgroups C_γ $(0 \leq \gamma < \lambda)$ such that each $B_{\gamma+1} = B_\gamma + C_\gamma$. Then the following statements are equivalent:*

(a) A is free-reduced,
(b) A is free.

Proof. $(a) \Rightarrow (b)$. Suppose that A is free-reduced, i.e. completely virtually regular. Since an arbitrary subgroup of a completely virtually regular group retains that property, using the above notation, we obtain that each $C_\gamma \leq A$ is free-reduced. Since C_γ is of finite rank, it is free by Proposition 4.11, and hence, it is finitely generated. Now, for each such γ, we have that

$$B_{\gamma+1}/B_\gamma = [B_\gamma + C_\gamma]/B_\gamma \cong C_\gamma/[B_\gamma \cap C_\gamma].$$

Since these quotients are clearly finitely generated and torsion-free, we obtain that they are free. Therefore,

$$A \cong \bigoplus_{0 \leq \gamma < \lambda} (B_{\gamma+1}/B_\gamma)$$

is free.
$(b) \Rightarrow (a)$. The claim is clear. □

Corollary 4.13. *Then the following statements are equivalent for a B_2-group A:*

(a) A is free-reduced,
(b) A is free.

Corollary 4.14. *Then the following statements are equivalent for a (torsion-free) group A of countable rank:*

(a) A is free-reduced,
(b) A is free.

Proof. It is easy to obtain a decomposition series as in (†) such that $\lambda = \omega$ and each B_γ has finite rank. For each $\gamma < \omega$, if we let $C_\gamma = B_{\gamma+1}$, then the result is an immediate consequence of Proposition 4.12 □

Corollary 4.15. *Suppose that G is a mixed group with torsion T and $A := G/T$. If A is either of countable rank or is a B_2-group, then the following statements are equivalent:*

(a) G is completely virtually regular,
(b) G is torsion splitting, i.e. G has a decomposition $G \cong T \oplus A$ such that T is semisimple and A is free.

Proof. $(a) \Rightarrow (b)$. If A is either of countable rank or is a B_2-group, then A must be free-reduced by Theorem 4.6(b or c). Hence, we can conclude that A must be free. Therefore, the splitting necessarily happens, as indicated.
$(b) \Rightarrow (a)$. The claim is an immediate consequence of Theorem 4.6(c). □

We recall the following well-known (and straightforward to verify) observation.

Fact 4.16. *If $P = \prod_\omega$ is the Baer-Specker group and $\overline{S}$ is the $\mathbb{Z}$-adic closure of $S := \bigoplus_\omega \mathbb{Z} \leq P$ in P, then $|\overline{S}| = 2^\omega = c$ and $\overline{S}/S$ is divisible.*

Let P denote the *Baer-Specker group*, $\prod_\omega \mathbb{Z}$. We now present a possibly surprising connection between free-reduced groups and the *continuum hypothesis*, or CH, i.e., $2^\omega = \omega_1$.

Theorem 4.17. *The following two statements are logically equivalent (and so, both independent of ZFC):*

(a) The continuum hypothesis,
(b) Every (torsion-free) free-reduced group, say A, of cardinality $|A| < 2^\omega$ is necessarily free.

Proof. $(a) \Rightarrow (b)$. Suppose that CH holds. Let A be a free-reduced group with $|A| < 2^\omega$. Then A must have countable rank. So, by Corollary 4.14, we can conclude that A is free, i.e. (b) holds.
$(b) \Rightarrow (a)$. Suppose on contrary that (a) fails, i.e. $\omega_1 < 2^\omega$. By Fact 4.16, there is a $A \leq \overline{S}$ with $S \leq A$ and A/S divisible of rank ω_1. Clearly, A is free-reduced. By (b), we assume that A is free. An elementary argument gives us that A has a decomposition $A = M \oplus N$ such that $S \leq M$ and M is countable. Since N is free of rank ω_1 and maps to a direct summand of $A/S \cong (M/S) \oplus N$, this contradicts that $A/S \leq^\oplus \overline{S}/S$ is divisible. □

We now construct an auxiliary lemma that will allow us to produce some relevant examples.

Lemma 4.18. *If $P := \prod_\omega \mathbb{Z}$ is the Baer-Specker group and E is a countable group that is not cotorsion, then* $\mathrm{Ext}(P, E)$ *has (torsion-free) rank $2^{2^\omega} = 2^c$, where c is the continuum.*

In particular, this group of extensions has (non-zero) elements of infinite order.

Proof. Let $S := \bigoplus_\omega \mathbb{Z}$. Then we have a short exact sequence

$$0 \to S \to P \to H \to 0$$

where $H = P/S$. Let $\overline{S} \leq P$ be the closure of S in P in the $\mathbb{Z}$-adic topology so that $\overline{S}/S \cong \bigoplus_c \mathbb{Q} \leq^\oplus P/S = H$. If we apply $\mathrm{Ext}(-, E)$ to the long exact sequence, we obtain that

$$0 \to \mathrm{Hom}(H, E) \to \mathrm{Hom}(P, E) \to \mathrm{Hom}(S, E) \xrightarrow{\delta} \mathrm{Ext}(H, E) \to \mathrm{Ext}(P, E) \to 0,$$

where $\mathrm{Ext}(S, E) = 0$ since S is free. Note that

$$\mathrm{Hom}(S, E) \cong \mathrm{Hom}(\bigoplus_\omega \mathbb{Z}, E) \cong \prod_\omega \mathrm{Hom}(\mathbb{Z}, E) \cong \prod_\omega E$$

will have cardinality $|E|^\omega = c$. On the other hand, we will also have that

$$\prod_c \mathrm{Ext}(\mathbb{Q}, E) \cong \mathrm{Ext}(\bigoplus_c \mathbb{Q}, E) \cong \mathrm{Ext}(\overline{S}/S, E) \leq^\oplus \mathrm{Ext}(H, E).$$

Since $\mathrm{Ext}(\mathbb{Q}, E)$ is divisible of (torsion-free) rank c, we obtain that

$$|\prod_c \mathrm{Ext}(\mathbb{Q}, E)| = c^c = 2^c > c.$$

Therefore, $\mathrm{Ext}(H, E)$ will have rank 2^c, which implies that

$$\mathrm{Ext}(P, E) \cong \mathrm{Ext}(H, E)/\delta(\mathrm{Hom}(S, E))$$

will also have (torsion-free) rank 2^c, giving the result. □

By Corollary 4.15, one may conjecture that every completely virtually regular mixed group is torsion-splitting. The following example eliminates this possibility.

Example 4.19. Let $E = \bigoplus_p \mathbb{Z}(p)$ be semisimple and P be the Baer-Specker group. Clearly, E is countable and it is not cotorsion. Now, by Lemma 4.18, there is a non-splitting short exact sequence $0 \to E \to G \to P \to 0$. By Theorem 4.6, G is completely virtually regular and it is not torsion-splitting by the construction.

We now want to verify that there are examples of (torsion-free) completely virtually regular groups A such that a chain of subgroups A_α is arbitrarily long. We begin with the first step.

Example 4.20. Let P denote the Baer-Specker group. There exists a free-reduced group A such that $A/A_1 \cong P$ and $A_1 \cong \mathbb{Z}$, so that $A_2 = 0$.

Proof. Since $\mathbb{Z}$ is certainly countable and not cotorsion, we have, by Lemma 4.18, a short exact sequence $0 \to \mathbb{Z} \to A \to P \to 0$ that represents an element of infinite order in $\mathrm{Ext}(P, \mathbb{Z})$. Let $\mathbb{Z}^1$ be the image of the left injection, so that $\mathbb{Z}^1 \cong \mathbb{Z}$.
Claim. $A_1 = \mathbb{Z}^1$: Certainly, if $x \in A \setminus \mathbb{Z}^1$, then $x + \mathbb{Z}^1$ is a non-zero element of P. Hence, there is a homomorphism $P \to \mathbb{Z}$ such that $x + \mathbb{Z}^1 \mapsto w \neq 0$. Therefore, the composition $\phi : A \to P \to \mathbb{Z}$ satisfies $\phi(x) \neq 0$. Ergo, $A_1 \leq \mathbb{Z}^1$. For the converse

inclusion, suppose that $z \in \mathbb{Z}^1$. We want to show $z \in A_1$. Assume on contrary that $z \notin A_1$, Then there exists a homomorphism $\gamma : A \to \mathbb{Z}$ such that $\gamma(\mathbb{Z}^1) \neq 0$. Suppose that the composite of $\mathbb{Z} \cong \mathbb{Z}^1$ with γ is multiplication by a non-zero $n \in \mathbb{Z}$. Then there is a commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z} & \to & A & \to & P \to 0 \\ & \cdot n \downarrow & \gamma \swarrow & & & \\ & \mathbb{Z} & & & & \end{array}$$

However, the existence of such a diagram would imply that $0 \to \mathbb{Z} \to A \to P \to 0$ was an element of finite order $|n| \neq 0$ in $\mathrm{Ext}(P, \mathbb{Z})$, contrary to its construction.

It follows that $A_1 = \mathbb{Z}^1 \cong \mathbb{Z}$, so that $A_2 = 0$. □

We now use an induction to generalize Example 4.20 to construct a group A whose corresponding descending chain A_α is as long as desired.

Example 4.21. If α is an ordinal, then there is a group A^α such that $(A^\alpha)_\alpha \cong \mathbb{Z}$, so that $(A^\alpha)_\eta = 0$ for all $\eta > \alpha$.

Proof. Certainly, if $\alpha = 0$, we can just let $A^0 = \mathbb{Z}$. Example 4.20 shows that the result is true if $\alpha = 1$. So, suppose we can construct such groups A^η for all $\eta < \alpha$, where $\alpha \geq 2$.

Case 1. α is a limit ordinal: For each $\eta < \alpha$, we can find an A^η such that

$$(A^\eta)_\eta := \mathbb{Z}^\eta \cong \mathbb{Z} \text{ (and } (A^\eta)_{\eta+1} = 0).$$

Let

$$K := \{\mathbf{x} = (x_\eta)_{\eta<\alpha} \in \bigoplus_{\eta<\alpha} \mathbb{Z}^\eta : \sum_{\eta<\alpha} x_\eta = 0\}.$$

Now, we let $A^\alpha = (\bigoplus_{\eta<\alpha} A^\eta)/K$ (in other words, we are simply identifying the subgroups $(A^\eta)_\eta = \mathbb{Z}^\eta$ for all $\eta < \alpha$) and we let $\mathbb{Z}^\alpha := (\bigoplus_{\eta<\alpha} \mathbb{Z}^\eta)/K \cong \mathbb{Z}$. Then, for each $\eta < \alpha$, we can identify A^η with a subgroup of A^α so that $\mathbb{Z}^\alpha = \mathbb{Z}^\eta$. For every $\eta < \alpha$, by Fact 4.7, we obtain that

$$\mathbb{Z}^\alpha = \mathbb{Z}^\eta = (A^\eta)_\eta \leq (A^\alpha)_\eta,$$

which implies that $\mathbb{Z}^\alpha \leq \cap_{\eta<\alpha} (A^\alpha)_\eta = (A^\alpha)_\alpha$.

For the converse, by Facts 4.7, 4.8 and 4.9, we obtain that

$$(A^\alpha)_\alpha \mapsto [A^\alpha/\mathbb{Z}^\alpha]_\alpha \cong [\bigoplus_{\eta<\alpha} A^\eta/\mathbb{Z}^\eta]_\alpha = \bigoplus_{\eta<\alpha} [A^\eta/\mathbb{Z}^\eta]_\alpha = 0,$$

where $A^\alpha \mapsto A^\alpha/\mathbb{Z}^\alpha$ is the natural epimorphism. Hence $(A^\alpha)_\alpha \leq \mathbb{Z}^\alpha$, as desired.

Case 2. $\alpha = \eta + 1$ is not a limit ordinal: Let H be the group from Example 4.20 with $H_1 \cong \mathbb{Z}$. For each non-zero $\mathbf{y} \in H$, let $A^\mathbf{y}$ be a copy of A^η, and let $(A^\mathbf{y})_\eta =: \mathbb{Z}^\mathbf{y} \cong \mathbb{Z}$. Consider the epimorphism with L kernel

$$\bigoplus_{\mathbf{y} \in H} \mathbb{Z}^\mathbf{y} \to H$$

which sends each $1 \in \mathbb{Z}^{\mathbf{y}}$ to $\mathbf{y} \in H$. Let

$$A^{\alpha} := [\bigoplus_{\mathbf{y}\in H} A^{\mathbf{y}}]/L.$$

Hence, we can think of H as a subgroup of A^{α}. Now, for every $\mathbf{y} \in H$, we also can also think of $A^{\mathbf{y}}$ as a subgroup of A^{α}. Therefore, for every $\mathbf{y} \in H$, by Fact 4.7, we have that

$$\mathbf{y} \in \mathbb{Z}^{\mathbf{y}} = (A^{\mathbf{y}})_{\eta} \leq (A^{\alpha})_{\eta},$$

i.e. $H \leq (A^{\alpha})_{\eta}$.

For the converse, using Facts 4.7, 4.8 and 4.9, and considering the natural epimorphism $A^{\alpha} \mapsto A^{\alpha}/H$, we obtain that

$$(A^{\alpha})_{\eta} \mapsto [A^{\alpha}/H]_{\eta} \cong [\bigoplus_{\mathbf{y}\in H} A^{\mathbf{y}}/\mathbb{Z}^{\mathbf{y}}]_{\eta} = \bigoplus_{\mathbf{y}\in H}[A^{\mathbf{y}}/\mathbb{Z}^{\mathbf{y}}]_{\eta} = 0,$$

which implies that $(A^{\alpha})_{\eta} \leq H$.

Therefore, $(A^{\alpha})_{\eta} = H$, $(A^{\alpha})_{\alpha} = H_1 \cong \mathbb{Z}$, completing the proof. □

Remark 4.22. Note that if A^{α} is the group constructed in Example 4.21 and $B^{\alpha} := A^{\alpha}/(A^{\alpha})_{\alpha}$, then $(B^{\alpha})_{\alpha} = 0$ and $(B^{\alpha})_{\eta} \neq 0$ for all $\eta < \alpha$, so we can think of B^{α} as having *free-reduced length* exactly α. It can also be seen that whenever $\eta < \alpha$, then $(A^{\alpha})_{\eta}/(A^{\alpha})_{\eta+1}$ will be a direct sum of copies of the Baer-Specker group.

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