ON ABELIAN GROUPS WHOSE ENDOMORPHISM RINGS ARE VON NEUMANN REGULAR

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Abstract. The paper describes the structure of E-regular abelian groups, i.e. those abelian groups G such that $\mathrm{End}(G)g$ is a direct summand of G for each $g \in G$. Besides structural characterization of this class of groups, it is proven existence of a one-to-one correspondence between the isomorphism classes of reduced E-regular abelian groups with commutative endomorphisms ring and simple subrings of $\mathbb{Z}(p)/\bigoplus_{p\in Q}\mathbb{Z}(p)$.

1. Introduction

A ring R is said to be *von Neumann regular* if for each $x \in R$, there exists an element y in R such that $x = xyx$. Abelian groups whose endomorphism rings are von Neumann regular have been extensively investigated in the literature. The most notable and well-known of these is the question posed by Fuchs [4, Problem 50], and the answers given by Fuchs-Rangaswamy [6], Rangaswamy [9] and Ramamurthi-Rangaswamy [10]. If an abelian group G is regular over its endomorphism ring $E_G := \mathrm{End}(G)$, i.e. if for all $x \in G$, Ex is a direct summand of G as a left E-module, then we say that G is *E_G-regular* (in short, *E-regular*). Similarly, if H is an other group, then we say *"H is E_H-regular."* Since a subgroup $B \leq G$ is an E-submodule iff it is fully invariant, we have that G is E-regular iff for all $x \in G$ there is a fully invariant submodule $B \leq G$ such that $G = (E\,x) \oplus B$.

We begin with the observation that if G is E-regular, then T is semisimple (as a $\mathbb{Z}$-module), i.e. $pT_p = 0$ for all primes p (Proposition 2.1(b)), and if $G = T$ is torsion, then the converse of this holds as well (Corollary 2.2). Furthermore, it is established in Proposition 2.3 that a non-reduced group G is E-regular iff $G = A \oplus D$, where $D \neq 0$ is torsion-free divisible and $A = T$ is semisimple. Therefore, there is no loss of generality in assuming that G is reduced.

We next verify in Proposition 2.5 that a reduced E-regular group is an *sp-group*, i.e. G/T is divisible and $G \cong G'$, where $T \leq G' \leq P := \prod_p T_p$ (where again, T is semisimple – usually we will identify G with this copy, G', contained in the product). By the *support* of such a G, we mean the set of all primes p such that $T_p \neq 0$. Similarly, if $\mathbf{x} = (x_p)_p \in G \leq P$, let $\mathrm{supp}(\mathbf{x})$ be the set of primes p with $x_p \neq 0$, and $\pi_{\mathbf{x}} : P \to P$ be the natural projection onto $P[\mathbf{x}] := \prod_{p\in \mathrm{supp}(\mathbf{x})} T_p$; we further set $G[\mathbf{x}] := G \cap P[\mathbf{x}]$.

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We are then ready to present the first of the main results of section 2 (Theorem 2.14): Assume that G is an sp-group with $pT_p = 0$ for all primes p (i.e., T is semisimple). Then G is E_G-regular iff $E_G\,\mathbf{x} = G[\mathbf{x}] = \pi_{\mathbf{x}}(G)$ for all $\mathbf{x} \in G$. This implies that the collection of supports, $\{\operatorname{supp}(\mathbf{x}) : \mathbf{x} \in G\}$, is closed under intersections, set differences and unions (Corollary 2.15).

After exhibiting a few concrete examples of reduced E-regular groups, we discuss a series of consequences following from Theorem 2.14. For example, we say such a reduced E-regular group G is *almost E-indecomposable* if whenever $G = A \oplus B$ is an E-module decomposition, then either A or B has finite support, i.e., it is bounded. We observe (Corollary 2.24(b)) that such a G is almost E-indecomposable iff it is either bounded, or $\operatorname{supp}(G) \setminus \operatorname{supp}(\mathbf{x})$ is finite, whenever $\mathbf{x} \in G$ has infinite order. This leads us to an important decomposition result for such groups of finite rank (Theorem 2.26): If G is a reduced E-regular group of finite rank, then there is an E-module decomposition $G \cong G_1 \oplus \cdots \oplus G_k \oplus \widehat{T}$, where each G_i is almost E-indecomposable and $\widehat{T}$ is torsion. Furthermore, this decomposition is unique up to moving around a collection of bounded summands. We finish this section with a slightly strengthen version, Theorem 2.32, of Theorem 2.14 after we formulate an easy description of E_G-regular groups via central idempotents: Let G be a reduced E_G-regular group such that $\dim_{\mathbb{Q}}(G/T)$ is finite and let $\kappa_p := \dim(T_p)$ for each $p \in \operatorname{supp}(G)$. Then there exist a $k \leq 0$ and central idempotents $e_0, \ldots, e_k \in E_G$ such that (i) $G = \bigoplus_{i=0}^{k} e_i G$, (ii) $E_{\widehat{G}} \cong \prod_{i=1}^{k} E_{e_i(G)} \times \prod_{p \in \operatorname{supp}(e_0 G)} \operatorname{End}(\mathbb{Z}(p)^{(\kappa_p)})$, (iii) $\operatorname{supp}(G)$ is partitioned by the sets $\operatorname{supp}(e_0 G), \ldots, \operatorname{supp}(e_k G)$ and (iv) E_G/I_T is semisimple, where $I_T = \{\varphi \in E_G \mid \varphi(G) \subseteq T\}$.

In section 3, we focus on the case where E_G is commutative (if this holds we say that G is *E-commutative*). It is shown in Proposition 3.2 that a reduced sp-group G is E-commutative iff T_p is cyclic for all $p \in \operatorname{supp}(G)$, which means that commutativity significantly restricts the possible structure of G. We observe (Theorem 3.4) that a reduced, E-regular and E-commutative group G must have the structure of an E-ring (see, [5, Section 18.6])). This result has two important consequences: first, $\mathbf{y} \in G$ is a unit of a reduced, E-regular, E-ring G iff $\operatorname{supp}(\mathbf{y}) = \operatorname{supp}(G)$ (Corollary 3.6) and second, any two unitary subrings of P containing T that are isomorphic as groups are necessarily equal as sets (Corollary 3.7). Using these results, we are able to prove a type of classification for such groups: there is a one-to-one correspondence between the isomorphism classes of reduced, E-regular, E-commutative and almost E-indecomposable G with $\operatorname{supp}(G) := Q$ and the collection of (unitary) subrings of $L/S := \prod_{p \in Q} \mathbb{Z}(p) / \bigoplus_{p \in Q} \mathbb{Z}(p)$ that are fields (Theorem 3.11). Recall that a ring R is called *abelian* if every idempotent of R is central. In Proposition 3.14, we show that the endomorphism ring of an E_G-commutative group is necessarily abelian regular.

Concerning Theorem 3.11, it is important to note that for a given finite dimensional field extension R of $\mathbb{Q}$, there might be different unitary subrings of P/T that are each isomorphic to R, so that there may be different groups G such that $\overline{G} := \widehat{G}/S$ is *isomorphic* to R without begin *equal* to the same subring of L/S. Example 3.16 is one example of that phenomenon: There are reduced E-regular, E-cyclic (it is cyclic as an E-module), E-commutative and almost E-indecomposable groups, say G and G', such that $\operatorname{supp}(G) = \operatorname{supp}(G')$ and $\overline{G} \cong \overline{G}'$, but G is not isomorphic to G'. We finish the paper with the question whether the field $\overline{G} = \widehat{G}/S$ in Theorem 3.11 can be any finite degree extension of $\mathbb{Q}$. If $R \leq \mathbb{C}$ is a field of finite degree over $\mathbb{Q}$, we answer the question that there is a reduced E-regular, E-commutative and almost E-indecomposable group G such that $\overline{G} = \widehat{G}/S \cong R$ (Theorem 3.19). Of course, though this result is quite general, it remains somewhat abstract. So, we introduce more concrete examples (e.g., Example 3.20) of the techniques involved: Let $n \in \mathbb{N}$ be an odd integer, p be a prime that does not divide n and $\zeta_n = e^{2\pi i/n}$ be the n-th primitive root of 1. If $R = \mathbb{Q}(\sqrt[n]{p})$ or $R = \mathbb{Q}(\zeta_n)$, then there is a reduced, E-ring and E-regular group G such that $\operatorname{supp}(G) = Q$ and $\overline{G}/T = R$.

Throughout this paper,

- R is an associative ring with unit 1 and modules are usually stand for a unital right R-module,
- $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}_n$ is the set of rational numbers, the set of integers and the ring of integers modulo n, respectively,
- G is an abelian group with maximal torsion subgroup T, and if p is a prime, then $T_p \leq T$ will denote its p-torsion,
- we write $A \leq G$ when A is a subgroup of G and $A \leq^{\oplus} G$ when it is a direct summand of G,
- for a prime integer p, G is said to be p-group if $G = T_p$, i.e. the order of every element of G is a power of p,
- by the *rank* of G, we will mean it *torsion-free* rank, i.e. the $\mathbb{Q}$-dimension of $\mathbb{Q} \otimes G$,
- we will on occasion use the standard "iff" for "if and only if."

Furthermore, our notation and notions are all standard and may be found in the books [2, 3, 4, 5, 7, 8].

2. E_G-REGULAR GROUPS

Recall that a submodule N of a module M is called *fully invariant* if for every $h \in End_R(M)$, $h(N) \subseteq N$.

Proposition 2.1. *Let G be a group with torsion T and A be a direct summand of G (as an abelian group). If G is E_G-regular, then*

(a) A is also E_A-regular.

(b) T is semisimple, i.e. if p is a prime, then $pT_p = 0$.

Proof. (a) Let $G := A \oplus B$. We remark that, since an E_G-submodule is fully invariant, we obtain that if C is any E_G-submodule of G, then $C = C_A \oplus C_B$, where C_A is an E_A-submodule of A and C_B is an E_B-submodule of B. And if $x \in A$, then $E\,x =$

$(E_A\, x) \oplus (\mathrm{Hom}(A,B)\, x)$. Turning to our proof, since G is E_G-regular, there is an E_G-submodule $C \leq^{\oplus} G$ such that $E\, x \oplus C = G$. It follows that $C = C_A \oplus C_B$, which implies that $A = E_A\, x \oplus C_A$, as required.
(b) Suppose that $pT_p \neq 0$ for some prime p and $x \in T_p[p]$ have minimal p-height α (so either $\alpha < \omega$ or $\alpha = \infty$, and in later case, T_p is divisible). Then $E\, x = T_p[p]$ and $E\, x$ will not even be a direct summand of G as an abelian group since $T_p[p^2] \setminus T_p[p] \neq 0$. □

A group is said to be *elementary* if the order of each of its elements is not divided by the square of an integer.

Recall from [6] that if $G = T$ is a torsion group, then E_G is regular iff G is an elementary group. We provide a proof of this important fact for the aid of the reader.

Corollary 2.2. *Suppose that G is a torsion group. Then G is E-regular iff G is semisimple.*

Proof. Suppose that G is semisimple (and torsion). If $x \in G$ has order $n \in \mathbb{N}$ (so that n is square-free), then $G = E\, x \oplus B$, where $E\, x = \bigoplus_{p|n} T_p$ and $B = \bigoplus_{p \not| \, n} T_p$ for some prime p.

The converse follows from Proposition 2.1. □

A group is said to be *reduced* if it has no divisible subgroups other than 0.

If G is a nonreduced group, then the ring E_G is regular if and only if G is a direct sum of an elementary group and a divisible torsion-free group (see [6]). We again provide a proof of this result for the aid of the reader.

Proposition 2.3. *Suppose that $G = A \oplus D$, where $D \neq 0$ is divisible and A is reduced. The following statements are equivalent:*

- *(a) Then G is E_G-regular,*
- *(b) D is torsion-free and A is semisimple, i.e. $A = T$ is torsion and E_A-regular.*

Proof. (a) ⇒ (b). It follows immediately from Proposition 2.1 that D is torsion-free. Since we are assuming that D is non-zero, there is an $0 \neq x \in D$, i.e. $\langle x \rangle \cong \mathbb{Z}$. It is easy to check that $E_G\, x = D$, which implies that $G = B \oplus D$ for some E_G-submodule B of G.
Claim. B is torsion: If not, it is easy to see that there is a non-zero group homomorphism $\phi : B \to D$. However, we can think of this ϕ as an element of E_G, and this contradicts that B is an E_G-submodule of G (since it will not be fully invariant). Therefore B must be torsion.

Now, by Proposition 2.1, A must be semisimple and $B = T = A$.
(b) ⇒ (a). Suppose that D is torsion-free (and divisible) and A is torsion (and reduced). It is easily seen that A and D will be fully-invariant subgroups of G. This, and the readily observed fact that D is E_D-regular (and that A is E_A-regular), easily leads to the conclusion that G is E_G-regular. □

Remark 2.4. Because of Proposition 2.3, in our discussion of E-regular groups, there is no loss of generality in assuming that G is reduced.

Groups that lie between the direct sums and direct products of their p-components are said to be *sp-groups* provided that G/T is divisible; the term is derived from the words "sum" and "product".

Proposition 2.5. *Reduced E-regular groups are sp-groups.*

Proof. Suppose that G is a reduced E_G-regular group. Since $pT_p = 0$ for each prime p, we can conclude that there exists a decomposition $G = T_p \oplus H_p$. Consider the multiplication by p.

Claim 1. The multiplication by p is an injection when restricted to $H_p \to H_p$: This follows from the fact that H_p contains no p-torsion.

Claim 2. The multiplication by p is an automorphism of H_p : If this fails, then we could find $y \in H_p$ of p-height 0. It follows that $x := py \in H_p$ will have p-height exactly 1. Since $x \in pG$, we obtain that $E\,x \leq pG = pH_p$. Hence if $E\,x$ is a direct summand of G, then it is a direct summand of H_p, i.e. $H_p \cong (E\,x) \oplus (H_p/(E\,x))$. However, $y + E\,x$ will be a non-zero element of $H_p/(E\,x)$ with $p(y + E\,x) = 0$ contradicting that H_p has no p-torsion. Therefore, $pG = p^\omega G = H_p$.

Claims (1) and (2) imply that T_p is dense in G in the p-adic topology. So, we can conclude that T is dense in G in the p-adic topology for all primes p, i.e. it is dense in G in the $\mathbb{Z}$-adic topology. In particular, G/T must be divisible.

Note, furthermore, that $P = \prod_p T_p$ will be the completion of T in the $\mathbb{Z}$-adic topology. Hence there is a homomorphism $G \to \prod_p T_p$ extending $T \hookrightarrow P$, whose the kernel is the first Ulm subgroup of G, i.e.

$$G^1 = \cap_{n\in\mathbb{N}}\, nG.$$

Claim 3. G^1 is divisible, and, since G is reduced (this means that $G^1 = \{0\}$), enabling the desired embedding: Suppose that $x \in G^1 \leq pG \leq H_p$. Because the multiplication by p is an automorphism of H_p by Claim (2), there exists a $y \in H_p$ such that $py = x$. Since $y \in H_p$, we have that $y \in p^\omega G$ and, since $(p, q) = 1$ for all primes $q \neq p$, x and y will have the same (infinite) q-height. Finally, $y \in G^1$, i.e. G^1 is p-divisible for all primes p, and hence $\{0\}$, as claimed, finishing the proof. □

Remark 2.6. For any sp-group G with $T = \bigoplus_p T_p \leq G \leq P = \prod_p T_p$ as above, since T is pure in P and G/T is divisible, it will be pure in P/T. Therefore, G will be pure in P.

Notation 2.7. Let $\mathcal{S}$ be the collection of all subsets of the collection of primes and $\mathcal{F}$ be the ideal of $\mathcal{S}$ consisting of all finite subsets.

- The sets $X, Y \in \mathcal{S}$ will be denoted by $X \sim_{\mathcal{F}} Y$, namely *almost equal*, if the symmetric difference of the sets X and Y is finite.
- The sets $X, Y \in \mathcal{S}$ are said to be almost disjoint if $X \cap Y \in \mathcal{F}$.
- The set $X \in \mathcal{S}$ is said to be almost contained in $Y \in \mathcal{S}$ if $X \setminus Y \in \mathcal{F}$.

Remark 2.8. We notice that the partial ordering of inclusion on $\mathcal{S}$ induces a partial order on $\mathcal{S}/\mathcal{F}$ and $\mathcal{S}/\mathcal{F}$ has meets and joins, which are basically the same are ordinary intersections and unions, interpreted in terms of equivalence classes.

Notation 2.9. Suppose that G is a reduced sp-group and $T = \bigoplus_p T_p \le G \le P = \prod_p T_p$, with G/T divisible. Let $\operatorname{supp}(G)$ be the collection of all primes such that $T_p \neq 0$.

Certainly, if $\operatorname{supp}(G)$ is finite, then G is a torsion group with $T_p = 0$ for almost all primes p; this case being so simple, we may pass over it without comment, and implicitly assume $\operatorname{supp}(G)$ is infinite.

Similarly, if $\mathbf{x} \in P := \prod_p T_p$ and $\mathbf{x} = (x_p)_p$, then $\operatorname{supp}(\mathbf{x})$ denotes the set of all primes p such that $x_p \neq 0$; in other words, $p \in \operatorname{supp}(\mathbf{x})$ exactly when $\mathbf{x}$ has p-height $\neq \infty$.

If $\mathbf{x} \in G$, then let

$$P[\mathbf{x}] := \prod_{p \in \operatorname{supp}(\mathbf{x})} T_p \quad \text{and} \quad \pi_{\mathbf{x}} : P \to P[\mathbf{x}]$$

be the natural projection. Finally, we can also think of $P[\mathbf{x}]$ as a subgroup of P and we let

$$G[\mathbf{x}] = G \cap P[\mathbf{x}] = \{\mathbf{y} \in G : \operatorname{supp}(\mathbf{y}) \subseteq \operatorname{supp}(\mathbf{x})\}.$$

Remark 2.10. Note that if $\mathbf{x} \in P$ and $\mathbf{y} \in T$, then $\operatorname{supp}(\mathbf{x})$ and $\operatorname{supp}(\mathbf{x}+\mathbf{y})$ are almost equal. In addition, if $0 \neq z \in \mathbb{Z}$, then $\operatorname{supp}(\mathbf{x})$ is almost equal to $\operatorname{supp}(z\mathbf{x})$. It follows that if $\mathbb{Q}(\mathbf{x}+T) = \mathbb{Q}(\mathbf{y}+T)$ in P/T, then $\operatorname{supp}(\mathbf{x})$ and $\operatorname{supp}(\mathbf{y})$ are almost equal.

Notation 2.11. We will repeatedly use the following important factoid: If G is a reduced sp-group, then any endomorphism $\phi : G \to G$ is uniquely determined by the collection of restrictions $\phi_p := \phi \restriction : T_p \to T_p$ for all primes p (we write this as $\phi = (\phi_p)_p$). In particular, this means that G is *extending* in the sense that this ϕ will, in a natural way, uniquely extend to an endomorphism of P.

Proposition 2.12. *Suppose that G is a (reduced) sp-group with $T = \bigoplus_p T_p \le G \le P = \prod_p T_p$. The following statements holds:*

(a) Group decompositions $G = A \oplus B$ correspond to decompositions $T = T_A \oplus T_B$ such that the projection $\pi_A : T \to T_A$ extends (uniquely) to an endomorphism of G, i.e. $\pi(G) \le G$.

(b) E_G-module decompositions $G = A \oplus B$ correspond to partitions of $\operatorname{supp}(G)$ *into disjoint sets* $\operatorname{supp}(A)$ *and* $\operatorname{supp}(B)$ *such that the corresponding $\pi : T \to T_A$ satisfies $\pi(G) \le G$. In other words, an abelian group decomposition is an E-module decomposition precisely when, for each prime p, either $T_p = (T_A)_p$ (and $(T_B)_p = 0$) or $T_p = (T_B)_p$ (and $(T_A)_p = 0$).*

(c) If T_p is cyclic for each $p \in \operatorname{supp}(G)$, then any abelian group decomposition $G = A \oplus B$ is also an E_G-module decomposition.

Proof. Regarding (a), a decomposition of G clearly restricts to a decomposition of T.

Conversely, a decomposition of T determines a unique idempotent homomorphism on T. This idempotent clearly uniquely extends to an idempotent on P, which gives a decomposition of G precisely when it is an endomorphism of G.

Turning to (b), suppose first that $G = A \oplus B$ is an E_G-module decomposition. If p is a prime such that $(T_A)_p \neq 0 \neq (T_B)_p$, then A has a direct summand of the form $\mathbb{Z}(p^j)$ and B has a direct summand of the form $\mathbb{Z}(p^k)$. There is clearly a non-zero homomorphism $\mathbb{Z}(p^j) \to \mathbb{Z}(p^k)$ which will extend to a non-zero homomorphism $A \to B$. However, if A is an E_G-submodule of G, it must be fully invariant, and this contradiction completes this implication.

Conversely, suppose that either $(T_A)_p = 0$ or $(T_B)_p = 0$ for all $p \in \operatorname{supp}(G)$. It is easy to see that that $\operatorname{Hom}(A, B) = \operatorname{Hom}(B, A) = 0$, which implies that A and B are fully invariant in G, i.e. they are E_G-submodules of G. Therefore, our decomposition is actually one of E_G-modules.

Finally, (c) follows directly from (b), since a cyclic p-group will always be indecomposable as an abelian group. □

We also state the following result for a general sp-group.

Proposition 2.13. *Suppose that G is a reduced sp-group with $T \leq G \leq P$ as above, where T is semisimple. If B is any E_G-submodule of G (i.e. fully invariant subgroup), then B is pure in G.*

Proof. Suppose that $\mathbf{x} \in B$, $\mathbf{y} \in G$ and $p^k\mathbf{y} = \mathbf{x}$ where $k \in \mathbb{N}$. There is a decomposition $G = T_p \oplus H_p$, where $H_p = \{\mathbf{z} \in G : p \notin \operatorname{supp}(\mathbf{z})\}$. Since $k \geq 1$, we clearly have that $\mathbf{x} \in H_p$. Note that p^k is an automorphism of H_p. Hence if $\phi : G \to G$ is multiplication by p^{-k} on H_p and 0 on T_p, then $\phi \in E_G$. Now, as B is fully invariant, we obtain that $\phi(\mathbf{x}) \in B \cap H_p$, i..e $p^k\phi(\mathbf{x}) = \mathbf{x}$, completing the proof. □

If $\phi : G \to G$ is an endomorphism and $\mathbf{x} \in G$, then clearly $\operatorname{supp}(\phi(\mathbf{x})) \subseteq \operatorname{supp}(\mathbf{x})$. The next result shows that this statement is intimately related to such an sp-group being E-regular.

Theorem 2.14. *Assume that G is an sp-group with $pT_p = 0$ for all primes p where $T \leq G \leq P$ such that G/T is divisible and T is semisimple as above. The following statements are equivalent:*

- *(a) Then G is E_G-regular,*
- *(b) For all $\mathbf{x} \in G$, the map $\pi_{\mathbf{x}}$ restricts to an endomorphism of G and we have $E_G\,\mathbf{x} = \pi_{\mathbf{x}}(G)$. In other words, we have that*

$$E_G\,\mathbf{x} = G[\mathbf{x}] = \pi_{\mathbf{x}}(G)$$

for all $\mathbf{x} \in G$.

Proof. $(a) \Rightarrow (b)$. Let $\mathbf{x} \in G$. Certainly, for any prime p, the restriction of $\pi_p : P \to T_p$ to G is in E_G, and $\pi_p(\mathbf{x}) \neq 0$ iff $p \in \operatorname{supp}(\mathbf{x})$. Then, we obtain that

$$T[\mathbf{x}] = \bigoplus_{p \in \operatorname{supp}(\mathbf{x})} T_p \leq E_G\,\mathbf{x}.$$

Since $P[\mathbf{x}]$ is fully invariant in P and any $\phi : G \to G$ in E_G extends to an endomorphism of P, we get that

$$T_{E_G\,\mathbf{x}} = T[\mathbf{x}].$$

Therefore, if there exists a decomposition $G = (E_G\,\mathbf{x}) \oplus B$ (leading to a decomposition $T = T[\mathbf{x}] \oplus T_B$) and $\pi : G \to E_G\,\mathbf{x}$ is the associated projection onto the first direct summand, then we have the followings:

- if $\pi = (\pi_p)_p$ when $p \in \operatorname{supp}(\mathbf{x})$, then we can conclude that $\pi_p = 1_{T_p}$, and
- $\pi_p = 0_{T_p}$, otherwise.

In other words, we must have $\pi = \pi_{\mathbf{x}}$, which yields that

$$G[\mathbf{x}] \subseteq \pi_{\mathbf{x}}(G) = E_G\,\mathbf{x} \subseteq G[\mathbf{x}].$$

$(b) \Rightarrow (a)$. Assume that $\pi_{\mathbf{x}}$ is an element of E_G and $E_G\,\mathbf{x} = G[\mathbf{x}] = \pi_{\mathbf{x}}(G)$. Since $\pi_{\mathbf{x}}$ is, in fact, an E_P-module idempotent on P, we obtain that it restricts to an E_G-module idempotent on G, i.e. there is a decomposition of E_G-modules

$$G = \pi_{\mathbf{x}}(G) \oplus \operatorname{Ker}(\pi_{\mathbf{x}}) = (E_G\,x) \oplus \left(G \cap \prod_{p \notin \operatorname{supp}(\mathbf{x})} T_p \right),$$

as desired. ☐

Corollary 2.15. *Assume that G is an sp-group with $pT_p = 0$ for all primes p and $T \leq G \leq P$ such that G/T is divisible and T is semisimple as above. If G is E-regular, then $\mathcal{G} = \{\operatorname{supp}(\mathbf{x}) : \mathbf{x} \in G\}$ is closed under intersections, set differences and unions.*

Proof. If $\mathbf{x}$ and $\mathbf{y} \in G$, then it is easy to check that

$$\begin{aligned} \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) &= \operatorname{supp}(\pi_{\mathbf{y}}(\mathbf{x})) \in \mathcal{G} \\ \operatorname{supp}(\mathbf{x}) \setminus \operatorname{supp}(\mathbf{y}) &= \operatorname{supp}(\mathbf{x} - \pi_{\mathbf{y}}(\mathbf{x})) \in \mathcal{G} \\ \operatorname{supp}(\mathbf{x}) \cup \operatorname{supp}(\mathbf{y}) &= \operatorname{supp}(\mathbf{x} + \mathbf{y} - \pi_{\mathbf{y}}(\mathbf{x})) \in \mathcal{G}, \end{aligned}$$

as desired. ☐

Remark 2.16. Since any finite set $F \subseteq \operatorname{supp}(G)$ is the support of some (torsion) element of such a group G, we obtain that if $\mathbf{x} \in G$ and $S \subseteq \operatorname{supp}(G)$ is almost equal to $\operatorname{supp}(\mathbf{x})$, then $S \in \mathcal{G}$, i.e. $S = \operatorname{supp}(\mathbf{y})$ for some $\mathbf{y} \in G$.

We now present a few concrete examples of reduced E-regular groups.

Example 2.17. The algebraically compact group $G = P := \prod_p T_p$, where $pT_p = 0$ for all primes, will always be E-regular.

Proof. Let $\mathbf{x}$ be any element of G. Then clearly $\pi_{\mathbf{x}}$ is an element of E and it is easy to compute that $E\,\mathbf{x} = P[\mathbf{x}] = \pi_{\mathbf{x}}(G)$. $\square$

A bit more complicated than the last result is the following, whose proof we leave to the interested reader:

Example 2.18. Suppose that $\{Q_n\}_{n\in\mathbb{N}}$ is a partition of the collection of all primes into a countable collection of countably infinite subsets $Q_n = \{q_{1,n}, q_{2,n}, q_{3,n}, \dots\}$. For $n \in \mathbb{N}$, let $P_n = \prod_m \mathbb{Z}(q_{m,n})$ and $S_n = \bigoplus_m \mathbb{Z}(q_{m,n})$. Then $\bigoplus_n P_n$ and $\prod_n S_n$ are E-regular.

The above examples all have cardinality of the continuum. For something in the opposite extreme, we have the following:

Example 2.19. Suppose that G is an sp-group with $pT_p = 0$ for all primes p and $T \leq G \leq P$ such that $G/T \cong \mathbb{Q}$ has rank 1. Then G will be E-regular.

Proof. Let $\mathbf{x} \in G$.
Case 1. Suppose that $\mathbf{x}$ has finite order. It is easily checked that $\operatorname{supp}(x)$ must be finite and that $E\,\mathbf{x} = T[\mathbf{x}] = \oplus_{p\in \operatorname{supp}(x)} T_p$ is a bounded subgroup of G and hence a summand. Therefore, there will be an E-module decomposition

$$G = E\,\mathbf{x} \oplus \left(G \cap \prod_{p\notin \operatorname{supp}(\mathbf{x})} T_p \right),$$

as needed.
Case 2. Suppose next that $\mathbf{x}$ has infinite order and $B = \bigoplus_{p\notin \operatorname{supp}(x)} T_p$. By Proposition 2.13, $E\,\mathbf{x}$ is a pure subgroup of G and, since $G/\langle \mathbf{x}\rangle$ is torsion, so is $G/(E\,\mathbf{x})$. Therefore,

$$G/(E\,\mathbf{x}) = [T + E\,\mathbf{x}]/(E\,\mathbf{x}) \cong T/[T \cap E\,\mathbf{x}] = T/T[\mathbf{x}] \cong \bigoplus_{p\notin \operatorname{supp}(\mathbf{x})} T_p = B,$$

which is semisimple and hence a direct sum of cyclic subgroups. Therefore, $G = (E\,x) \oplus B$, and both terms are fully-invariant, i.e. E-submodules. $\square$

The following example shows that the above observations do not extend to the case where G/T is (torsion-free divisible) of rank 2.

Example 2.20. There is an sp-group with semisimple torsion $T \leq G \leq P$ such that G/T has rank 2, that is not E-regular.

Proof. Consider a partition the set of all primes into three disjoint infinite subsets $P_1 \cup P_2 \cup P_3$ and for each prime p and let $T_p = \mathbb{Z}(p)$. Inside of $P = \prod_p T_p$, let $\mathbf{x} = (x_p)_p$, where $x_p = 1$ if $p \in P_1 \cup P_2$ and $x_p = 0$ if $p \in P_3$. Similarly, let $\mathbf{y} = (y_p)_p$, where $y_p = 1$ if $p \in P_2 \cup P_3$ and $y_p = 0$ if $p \in P_1$. Finally, let G be the pure closure of $T + \langle \mathbf{x}, \mathbf{y}\rangle$ in P. Clearly G/T has rank 2 and if $\mathbf{z} \in G$ has infinite order, then either P_1

or P_2 is almost contained in $\operatorname{supp}(\mathbf{z})$. Since $\operatorname{supp}(\pi_{\mathbf{x}}\mathbf{y}) = P_2$, we obtain that $\pi_{\mathbf{x}}$ will not be in E. Therefore, by Theorem 2.26, G is not E-regular. □

Definition 2.21. We say a group G is *E-cyclic* if it is cyclic as an E-module, i.e. we have $E\mathbf{x} = G$ for some $\mathbf{x} \in G$.

Corollary 2.22. *Suppose that G is a reduced and E-regular group. Then G is E-cyclic if and only if there is an $\mathbf{x} \in G$ such that $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(G)$.*

Proof. This follows from Theorem 2.14 since if $\mathbf{x} \in G$, then $E\,\mathbf{x} = G[\mathbf{x}] = \{\mathbf{y} \in G : \operatorname{supp}(\mathbf{y}) \subseteq \operatorname{supp}(\mathbf{x})\}$. □

We have already observed in Corollary 2.22 that it is sufficient to find an $\mathbf{x} \in G$ such that $\operatorname{supp}(\mathbf{x})$ is almost equal to $\operatorname{supp}(G)$.

Definition 2.23. We call a reduced E-regular group G *torsion E-indecomposable* (respectively, *almost E-indecomposable*) if for every E-module decomposition $G = A \oplus B$, then either A or B is torsion (respectively, either $\operatorname{supp}(A)$ or $\operatorname{supp}(B)$ is finite).

Clearly, if G is almost E-indecomposable, then it is torsion E-decomposable.

The following results are direct consequences of Theorem 2.14:

Corollary 2.24. *Suppose that G is a reduced E-regular group. The following statements hold:*

- *(a) G is torsion E-indecomposable iff all elements of infinite order have supports that are almost equal iff whenever $\mathbf{x} \in G$ has infinite order, then $G = G[\mathbf{x}] \oplus \widehat{T}$, where $\widehat{T} \leq T$.*
- *(b) G is almost E-indecomposable iff either $\operatorname{supp}(G)$ is finite or whenever $\mathbf{x} \in G$ has infinite order, $\operatorname{supp}(x)$ is almost equal to $\operatorname{supp}(G)$.*

Corollary 2.25. *If G is a reduced, E-regular and E-almost indecomposable, then it is E-cyclic.*

This leads us to the following important reduction result for reduced E-regular groups such that G/T has finite rank:

Theorem 2.26. *Suppose that G is a reduced and E-regular group, $T \leq G \leq P$ as above, with T semisimple and G/T divisible of finite rank n. Then there are $\mathbf{x}_1, \ldots, \mathbf{x}_k \in G$ of infinite order such that there is a partition $\operatorname{supp}(G) = \operatorname{supp}(\mathbf{x}_1) \cup \cdots \cup \operatorname{supp}(\mathbf{x}_k) \cup \widehat{\mathcal{P}}$ so that if $\widehat{T} := \bigoplus_{p \in \widehat{\mathcal{P}}} T_p$, then there is an E-decomposition into fully invariant direct summands $G = G_1 \oplus \cdots \oplus G_k \oplus \widehat{T}$ such that $G_j := E\,\mathbf{x}_j$ is almost E-indecomposable for each such j.*

In other words, the supports of all elements of infinite order in G_j are almost equal to $\operatorname{supp}(\mathbf{x}_j)$, and hence almost equal to each other. Furthermore, if $G = G'_1 \oplus \cdots \oplus G'_{k'} \oplus \check{T}$ is another such decomposition, then $k = k'$, and after possibly reordering, $\operatorname{supp}(G_i)$ and $\operatorname{supp}(G'_i)$ are almost equal for $i = 1, \ldots, k$.

Proof. There is clearly no loss of generality in assuming that $n > 0$, i.e. that G is not a torsion group. By a simple induction on the rank n, we obtain that G is E-isomorphic to a decomposition $A_1 \oplus \cdots \oplus A_m$ such that each A_i is not torsion, but is torsion E-indecomposable. If $\mathbf{x}_i \in A_i$ $(i = 1, \ldots, m)$ have infinite orders, then $A_i = G[\mathbf{x}_i] \oplus \widehat{T}_i$ where T_i must be torsion by Corollary 2.24(a). Let $\widehat{T} := \widehat{T}_1 \oplus \cdots \widehat{T}_m$. Then we have the desired decomposition $G = G_1 \oplus \cdots \oplus G_k \oplus \widehat{T}$.

The uniqueness statement follows from the fact that

$$G'_i = (G_1 \cap G'_i) \oplus \cdots \oplus (G_k \cap G'_i) \oplus (\widehat{T} \cap G'_i)$$

for $i = 1, \ldots, k'$. Now, there must be a j in $\{1, \ldots, k\}$ such that all of the terms in this decomposition, other than $G_j \cap G'_i$ have finite support, i.e. $\operatorname{supp}(G'_i) \sim_{\mathcal{F}} \operatorname{supp}(G_j)$, as needed. □

Remark 2.27. By Corollary 2.25, all of the terms $G_1, \ldots, G_k$ in Theorem 2.26 are necessarily E-cyclic.

We formulate an easy description of E_G-regular groups via central idempotents.

Lemma 2.28. *The following statements are equivalent for a group G:*

(a) G is E_G-regular.
(b) For each $x \in G$, there exists a central idempotent $e \in E_G$ such that $eG = E_G x$.

Proof. $(a) \Rightarrow (b)$. Since there exists a submodule B of the left E_G-module G such that $G = E_G x \oplus B$ by the assumption, there exists an idempotent $e \in E_G$ given by the projection of G onto $E_G x$ along B satisfying $eG = E_G x$ and $(1-e)G = B$. As both $E_G x$ and B are left E_G-modules, we obtain that

$$(1-e)E_G e(G) = (1-e)e(G) = 0 \quad \text{and} \quad eE_G(1-e)(G) = eB = e(1-e)G = 0.$$

Thus $(1-e)E_G e = 0 = eE_G(1-e)$, which means that e is central.
$(b) \Rightarrow (a)$. If $x \in G$ and $e \in E_G$ is a central idempotent satisfying $eG = E_G x$, then, obviously, we have a decomposition

$$G = eG \oplus (1-e)G = G = E_G x \oplus (1-e)G,$$

where $(1-e)G$ is E_G is an E_G-module (as $1-e$ is again a central idempotent). □

Notation 2.29. *For an E_G-regular group G with the torsion part T, denote*

$$I_T := \{\varphi \in E_G \mid \varphi(G) \subseteq T\}.$$

It is easy to obtain that, $I_T = G$ if and only if G is torsion.

Lemma 2.30. *Suppose that G is a reduced E_G-regular group and $\varphi \in I_T$. The following statements hold:*

(a) I_T is an ideal.
(b) $\ker \varphi$ is a direct summand of G and $\ker \varphi + T = G$.
(c) φ is a regular element in I_T iff $\varphi(G)$ is a direct summand of G.

Proof. Denote $K := \ker\varphi$.
(a) The claim follows from the fact that T is a fully invariant semisimple subgroup of G.
(b) Since $\varphi(G)/\varphi(T)$ is divisible and torsion, we can observe that $\varphi(G)/\varphi(T) = 0$ which implies that $\varphi(G) = \varphi(T)$. Suppose that $g \in G \setminus T$. Then there exists a $t \in T$ such that $\varphi(g) = \varphi(t)$. Hence $g - t \in K$, which proves that $K + T = G$. As T is semisimple, there exists a $U \leq G$ such that $T = U \oplus (K \cap T)$, and hence it is easy to see that $G = K + T = U \oplus K$.
(c) If $\varphi \in I_T$ is a regular element, then there exists a $\psi \in I_T$ such that $\varphi\psi\varphi = \varphi$. Since $e = \varphi\psi$ is an idempotent, we get that $e\varphi(G) = \varphi(G)$. Hence $e(G) = \varphi(G)$ and $G = e(G) \oplus (1 - e)(G)$.

For the converse, let $\varphi(G)$ be a direct summand of G. Then there exists an $H \leq G$ such that $G = \varphi(G) \oplus H$ and, by (b), there exists a $U \leq T$ such that $G = K \oplus U$, which implies that the restriction of φ onto U induces an isomorphism $U \cong \varphi(G)$. Hence there exists an inverse isomorphism $\tilde{\psi} : \varphi(G) \to U$ which can be extended to $\psi \in E_G$ by the rule $\psi(f + h) = \tilde{\psi}(f)$ for each $f \in \varphi(G)$ and $h \in H$. Since $\psi(G) \in I_T$ and $\varphi\psi\varphi = \varphi$, the element φ is regular as desired. □

Proposition 2.31. *Let G be a reduced E_G-regular group such that $\dim_{\mathbb{Q}}(G/T)$ is finite and G/T is E_G-simple. Then there exists natural number m satisfying $E/I_T \cong (G/T)^m$ as left E_G-modules and, in particular, E/I_T is semisimple.*

Proof. First observe that there is nothing to prove if G is torsion. So we may suppose that $\dim_{\mathbb{Q}}(G/T) > 0$. Note that G/T is torsion free of finite rank $n = \dim_{\mathbb{Q}}(G/T)$. Thus we have isomorphisms of abelian groups

$$\mathrm{Hom}(G, G/T) \cong \mathrm{End}(G/T) \cong L_n(\mathbb{Q}) \cong \mathbb{Q}^{n^2},$$

which gives that $\mathrm{Hom}(G, G/T)$ is torsion free of rank n^2. If we apply the functor $\mathrm{Hom}(G, -)$ on the natural short exact sequence

$$0 \longrightarrow T \longrightarrow G \longrightarrow G/T \longrightarrow 0,$$

we derive the following exact sequence of groups

$$0 \longrightarrow \mathrm{Hom}(G, T) \xrightarrow{\iota} \mathrm{Hom}(G, G) \longrightarrow \mathrm{Hom}(G, G/T).$$

Observe that $I_T = \iota(\mathrm{Hom}(G, T))$ because ι is a monomorphism and I_T is an ideal by Lemma 2.30(a). Furthermore, from the exact sequence and the first isomorphism theorem follows existence of a group embedding of E_G/I_T into $\mathrm{Hom}(G, G/T) \cong \mathbb{Q}^{n^2}$, we obtain that E_G/I_T is a torsion-free group a rank less or equal to n^2.

Let us denote $\mathcal{G} := G/T \setminus \{0\}$. Then we obtain from the hypothesis G/T is E_G-simple that

$$\bigcap_{x \in \mathcal{G}} \mathrm{Ann}(x) = \mathrm{Ann}(G/T) = \{\varphi \in \mathrm{End}(G) \mid \varphi(G) + T = 0 + T\} = I_T,$$

an hence E_G/I_T is embeddable into $\prod_{x\in\mathcal{G}} E_G/\operatorname{Ann}(x) \cong (G/T)^{\mathcal{G}}$ as an E_G-module. Since $(G/T)^{\mathcal{G}}$ is a torsion free group for each $K \subseteq \mathcal{G}$ and $\operatorname{rank}_{\mathbb{Z}}(E_G/I_T) \leq n^2$, there exists a finite k such that E_G/I_T embeds into $(G/T)^k$ as an E_G-module. As $(G/T)^k$ is semisimple, we get that E_G/I_T is isomorphic to a direct summand of $(G/T)^k$, which finishes the proof. □

Now, we can slightly strengthen the Theorem 2.26.

Theorem 2.32. *Let G be a reduced E_G-regular group such that $\dim_{\mathbb{Q}}(G/T)$ is finite and let $\kappa_p := \dim(T_p)$ for each $p \in \operatorname{supp}(G)$. Then there exist a $k \leq 0$ and central idempotents $e_0, \dots, e_k \in E_G$ such that*

- *(a) $G = \bigoplus_{i=0}^{k} e_iG$,*
- *(b) $E_{\widehat{G}} \cong \prod_{i=1}^{k} E_{e_i(G)} \times \prod_{p\in\operatorname{supp}(e_0G)} \operatorname{End}(\mathbb{Z}(p)^{(\kappa_p)})$,*
- *(c) e_0G is torsion, e_iG is E_G-cyclic, e_iG/e_iT is E_G-simple, and $e_iE_G \cong E_{e_i}$ for each $i = 1, \dots, k$,*
- *(d) $\operatorname{supp}(G)$ is partitioned by the sets $\operatorname{supp}(e_0G), \dots, \operatorname{supp}(e_kG)$,*
- *(e) E_G/I_T is semisimple.*

Proof. If $\dim_{\mathbb{Q}}(G/T) = 0$, then G is torsion and we put $k = 0$ and $\widehat{T} = T = G$. Now we suppose that $\dim_{\mathbb{Q}}(G/T) > 0$. Since for each $x \in G \setminus T$ there exists a central idempotent $e \in E_G$ satisfying $eG = E_Gx$ by Lemma 2.28 and eG/eT is divisible torsion free by Proposition 2.5, G/T is of finite length. Thus $\operatorname{Soc}(G/T) \neq 0$ and there exist an element $x_1 \in G \setminus T$ and a central idempotent $e_1 \in E_G$ such that $e_1G = E_Gx_1$ and e_1G/e_1T is E_G-simple. Obviously,

$$G/T \cong e_1G/e_1T \oplus (1-e_1)G/(1-e_1)T,$$

which implies that $\dim_{\mathbb{Q}}((1-e_1)G/(1-e_1)T) < \dim_{\mathbb{Q}}(G/T)$. Then, proceeding by induction as in the proof of Theorem 2.26, we obtain a sequence of central idempotents $e_1, \dots, e_k \in E_G$ such that

- $e_{i+1} \in (1-e_i)E_G$,
- e_iG is E_G-cyclic,
- e_1G/e_1T is E_G-simple and
- $G = T + \sum_{i=1}^{k} e_iG$.

Now, clearly, $e_1, \dots, e_k$ is an orthogonal sequence of central idempotents and if we put $e_0 = 1 - \sum_{i=1}^{k} e_i$, we can see that $G = \bigoplus_{i=0}^{k} e_iG$ and the conditions $(a), (b), (c)$ hold.

Note that the condition (d) follows immediately from Proposition 2.12(b), I_T contains e_0E since e_0G is a semisimple direct summand of G, and

$$E_{e_0G} = \operatorname{End}\Big(\bigoplus_{p\in\operatorname{supp}(e_0G)} T_p\Big) \cong \prod_{p\in\operatorname{supp}(e_0G)} \operatorname{End}(\mathbb{Z}(p)^{(\kappa_p)})$$

by (c).

Since $E_G/I_T \cong \bigoplus_{i=1}^{k} e_iE_G/e_iI_T$, it remains to observe that $e_iE_G/e_iI_T \cong E_{e_iG}/I_{e_iT}$ is semisimple for each $i = 1, \dots, k$, which follows from Proposition 2.31. □

Remark 2.33. Another approach to the above discussion refers to classical structure results of the ring theory:

- Suppose that G is reduced, E-regular and almost E-indecomposable. It is easy to see from Theorem 2.14 that $\overline{G} := G/T$ is a simple and faithful (left) $\overline{E} := E/I_T$ module. Therefore, $D := \mathrm{H}om_{\overline{E}}(\overline{G}, \overline{G})$ is a division ring, and $\overline{G}$ is also a (right) D-module.
- If G has finite rank, i.e. $\overline{G}$ has finite dimension over $\mathbb{Q}$ (as holds for the terms G_i in Theorem 2.26), then $\overline{G}$ also has finite rank over D. Therefore, $\overline{G}$ is D-isomorphic to $D^{(n)}$ for some n and $\overline{E}$ is ring isomorphic to the matrix ring $M_n(D)$ by the Jacobson density theorem. In particular, it follows that $\overline{E}$ is $\overline{E}$-isomorphic to $\overline{G}^{(n)}$ so that it is a semisimple ring.

We also note that the general case in Theorem 2.26 just follows by applying the above ideas to each G_i.

Remark 2.34. One thing that is not clear is the nature of the division ring D in Remark 2.33. In the next section, we will see that each field that is finite dimensional over $\mathbb{Q}$ can appear in such a role, but it is not clear whether D must have such a form, i.e. whether it must be commutative.

3. The commutative case

Definition 3.1. We say G is *E-commutative* if E_G is commutative.

Proposition 3.2. *Suppose that G is a reduced sp-group. The following statements are equivalent:*

(a) G is E-commutative,
(b) T_p is cyclic for all $p \in \operatorname{supp}(G)$.

Proof. $(a) \Rightarrow (b)$. Clearly, if $p \in \operatorname{supp}(G)$, then $G = T_p \oplus H_p$ for some $H_p \leq G$ and E_{T_p} can easily be seen to embed in E. Hence E_{T_p} is commutative, which can only happen if T_p is cyclic since T_p is reduced.
$(b) \Rightarrow (a)$. By the assumptions, any endomorphism $\phi \in E$ can be seen to extend uniquely to an endomorphism $P \to P$. If each T_p is isomorphic to, say $\mathbb{Z}(p^{k_p})$, then it readily follows that E_P is isomorphic to the product ring, $\prod_p \mathbb{Z}(p^{k_p})$, which is clearly commutative. □

We now want to consider the structure of reduced E-regular groups G with support Q that are both E-commutative and E-cyclic. There is clearly one quite concrete example:

Let $L_Q := \prod_{p\in Q} \mathbb{Z}(p)$ whose torsion is $S_Q := \oplus_{p\in Q} \mathbb{Z}(p)$. Note that if Q is clear from context, we may abbreviate these to S and L, respectively. It is easy to see that

- L is a unitary ring;
- S is an ideal of L (so that, L/S is also a ring).

Our strategy is to embed all such abstract groups satisfying these conditions as unitary subrings of this particular fixed example. We will then show that two such abstract groups will be isomorphic iff they correspond to the same subring of L.

We separate the following observation for the future use:

Lemma 3.3. *Suppose that G is a reduced, E-regular and E-commutative group. If G is E-cyclic and $E\,\mathbf{x} = G$ (i.e. $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(G) := Q$), then there is a unique embedding $G \cong \widehat{G}_{\mathbf{x}}$, where $S := \oplus_{p\in Q}\mathbb{Z}(p) \le \widehat{G}_{\mathbf{x}} \le \prod_{p\in Q}\mathbb{Z}(p) =: L$ with*

$$\mathbf{x} \mapsto \mathbf{1}_L = (1_p)_{p\in Q} \in \widehat{G}_{\mathbf{x}}.$$

Proof. If $\mathbf{x} = (x_p)_{p\in Q}$, then there is an isomorphism $\gamma : T \to \oplus_{p\in Q}\mathbb{Z}(p)$ such that $\gamma(x_p) = 1_{\mathbb{Z}(p)}$ for each $p \in Q$. Since G/T is divisible, this uniquely extends to an embedding $\gamma : G \to P$ with $\gamma(\mathbf{x}) = \mathbf{1}_L$, as stated. □

Recall that a (unitary) ring R is said to be an *E-ring* if, for every (group) endomorphism $\phi : R \to R$, there is an $\alpha \in R$ such that $\phi(x) = \alpha x$ for all $x \in R$. For example, $\mathbb{Z}$ and $\mathbb{Q}$ are E-rings, as is $\mathbb{Z}(n)$, for $n \in \mathbb{N}$.

Theorem 3.4. *Suppose that G is a reduced and E-regular group and $Q = \operatorname{supp}(G)$. The following statements are equivalent:*

- *(a) G is E-commutative and E-cyclic,*
- *(b) Each T_p (for $p \in Q$) is cyclic and there is an $\mathbf{x} \in G$ such that $\operatorname{supp}(\mathbf{x}) = Q$,*
- *(c) A ring structure on G can be defined that turns G into an E-ring.*

Proof. $(a) \Leftrightarrow (b)$. The equivalence follows directly from Corollary 2.22 and Proposition 3.2.
$(b) \Rightarrow (c)$. Suppose that (b) holds, i.e. each T_p is cyclic and there is an $\mathbf{x} \in G$ with $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(G)$. Using Lemma 3.3 and replacing G by $\widehat{G}_{\mathbf{x}}$, there is clearly no loss of generality in assuming that $\mathbf{1}_L \in G \le L$. Since, in the natural way, L is an E-ring, it will suffice to show the following claims.
Claim 1. G is a subring of L: Let $\mathbf{y} \in G$. Since G is E-cyclic and $\operatorname{supp}(\mathbf{y}) \subseteq Q = \operatorname{supp}(\mathbf{1}_L)$, we obtain that there exists a $\phi \in E$ such that $\phi(\mathbf{1}_L) = \mathbf{y}$. However, ϕ must extend to an endomorphism of L (and so, in L) and ϕ must be multiplication by $\mathbf{y}$. This implies that $\mathbf{y}G \subseteq G$ for all $\mathbf{y}$, i.e. G is a subring of L.
Claim 2. G is an E-ring: If ϕ is an endomorphism of G, then it extends to an endomorphism of P. So, if $\mathbf{y} = \phi(\mathbf{1}_L)$, then we must have $\mathbf{y} \in G$ since $\mathbf{1}_L \in G$. Therefore, for all $\mathbf{x} \in G$, $\phi(\mathbf{x}) = \mathbf{y}\mathbf{x}$, i.e. G is an E-ring.
$(c) \Rightarrow (a)$. Suppose that G is an E-ring. Since $G = G\cdot 1_G = E\cdot 1_G$, we can conclude that G is E-cyclic. And, since any E-ring must be commutative (by [5, Theorem 18.6.5]), the proof is complete. □

We separate out one of the points of the proof Theorem 3.4:

Corollary 3.5. *Suppose that a group G is reduced, E-regular E-commutative and E-cyclic group with support Q. If $\mathbf{x} \in G$ has $\operatorname{supp}(x) = Q$, then $\widehat{G}_{\mathbf{x}}$ is a unitary subring of L that is isomorphic as a group to G and isomorphic as a ring to E.*

Corollary 3.6. *Suppose that a group G is reduced, E-regular E-commutative and E-cyclic. Then as an E-ring, $\mathbf{y} \in G$ is a unit of G if and only if $\operatorname{supp}(\mathbf{y}) = \operatorname{supp}(G)$.*

Proof. Clearly, $\mathbf{y}$ is a unit iff $E\,\mathbf{y} = G$ iff $G[\mathbf{y}] = G$ iff $\operatorname{supp}(\mathbf{y}) = \operatorname{supp}(G)$. $\square$

Proposition 3.7. *Suppose that G and G' are reduced, E-regular E-commutative and E-cyclic groups with support Q. If $\mathbf{x} \in G$ and $\mathbf{x}' \in G'$ both have support Q, then $G \cong G'$ (i.e. G and G' are isomorphic as groups) iff we have $\widehat{G}_{\mathbf{x}} = \widehat{G}'_{\mathbf{x}'}$ (i.e. they correspond to the same subring of L).*

Proof. Certainly, if $\widehat{G}_{\mathbf{x}} = \widehat{G}'_{\mathbf{x}'}$, then $G \cong \widehat{G}_{\mathbf{x}} = \widehat{G}'_{\mathbf{x}'} \cong G'$.

Conversely, suppose $G \cong G'$. Replacing G by $\widehat{G}_{\mathbf{x}}$ and G' by $\widehat{G}'_{\mathbf{x}'}$, there is no loss of generality in assuming that $S \leq G, G' \leq L$ with $\mathbf{1}_L \in G \cap G'$ and $\gamma : G \to G'$ is an isomorphism. Clearly γ restricts to an automorphism of S, so that it will also extend to an automorphism of L. Hence, for some $\mathbf{y} \in L$, we have that $\gamma(\mathbf{x}) = \mathbf{y}\mathbf{x}$ for all $\mathbf{x} \in L$. Since $\mathbf{1}_L \in G$, we obtain that $\mathbf{y} = \mathbf{y}\mathbf{1}_L = \gamma(\mathbf{1}_L) \in G'$. Since, clearly, $\mathbf{y}$ must have support Q, it follows that it must be a unit of G'. Therefore,

$$G = \gamma^{-1}(G') = \mathbf{y}^{-1}G' = G',$$

as required. $\square$

Theorem 3.8. *Suppose that G and G' are reduced, E-regular E-commutative and E-cyclic groups. Then G and G' are isomorphic as groups iff their endomorphism rings, E_G and $E_{G'}$ are isomorphic as rings.*

Proof. It is clear that if either the groups are isomorphic or their endomorphism rings are isomorphic, then they must have the same support; call this Q.

Certainly two isomorphic groups have isomorphic endomorphism rings.

For the converse, suppose that E_G is isomorphic to $E_{G'}$. If $\mathbf{x} \in G$ and $\mathbf{x}' \in G'$ have support Q, then $\widehat{G}_{\mathbf{x}}$ and $\widehat{G}'_{\mathbf{x}'}$ are isomorphic unitary subrings of L (denote this isomorphism by ϕ). Clearly, ϕ extends to an automorphism of L. Note that, since we have that $\phi(\mathbf{1}_{\widehat{G}_{\mathbf{x}}}) = \mathbf{1}_{\widehat{G}'_{\mathbf{x}'}}$, we obtain have $\phi(\mathbf{1}_L) = \mathbf{1}_L$. This, however, easily implies that ϕ is the identity on L. Therefore, $\widehat{G}_{\mathbf{x}} = \widehat{G}'_{\mathbf{x}'}$, which shows that $G \cong G'$. $\square$

Remark 3.9. Note that by Corollary 3.7, if G is as above (reduced, E-regular E-commutative and E-cyclic), then the subring $\widehat{G}_{\mathbf{x}}$ of L does not depend upon which element $\mathbf{x}$ of support Q is chosen (though, the particular embedding $G \to \widehat{G}_{\mathbf{x}} \leq L$ does depend upon which $\mathbf{x} \mapsto \mathbf{1}_L$).

Notation 3.10. *We denote the common subring in Remark 3.9 by $\widehat{G}$.*

We now pay particular attention to these groups and their embeddings in the case of groups of finite rank, as in Theorem 2.26, where it is shown that such groups naturally decompose into ones that are almost E-indecomposable:

Theorem 3.11. *The isomorphism classes of reduced, E-regular, E-commutative and almost E-indecomposable groups G with* $\operatorname{supp}(G) := Q$ *are in one-to-one correspondence with the collection of (unitary) subrings of*

$$L/S = \prod_{p\in Q} \mathbb{Z}(p) / \bigoplus_{p\in Q} \mathbb{Z}(p)$$

that are fields.

Proof. If G is any reduced, E-regular, E-commutative and E-cyclic group, then we can associate it with $\overline{G} := \widehat{G}/S \leq L/S$. By Corollary 3.7, any two such groups are isomorphic iff they correspond to the same subring. So, our argument is completed by two claims. The first shows that this operation always associates almost E-indecomposable groups with subfields of L/S:

Claim 1. If G is almost E-indecomposable, then $\overline{G}$ is a field: Clearly $\overline{G}$ is unitary and commutative. Let $0 \neq \mathbf{y} + S \in \overline{G}$. Then, by Lemma 2.24(b) and Corollary 3.6, there is a $\mathbf{z} \in S$ such that $\mathbf{y} + \mathbf{z}$ is a unit of $\widehat{G}$. Therefore, $\mathbf{y} + S = \mathbf{y} + \mathbf{z} + S$ is a unit of $\overline{G}$, i.e. $\overline{G}$ is a field.

Our second claim shows that any subfield of L/S is in the image of this operation:

Claim 2. If R is a (unitary) subfield of L/S, then

$$G := \{\mathbf{x} \in L : \mathbf{x} + S \in R\},$$

is E-regular, E-commutative and almost E-indecomposable: Clearly, since R is a subring of L/S, G will be a subring of L. Next, to show that G is E-regular, we will use Theorem 2.14.

• If $\mathbf{x} \in G$ is any element of infinite order, then since $G/S \cong R$ is a field, there is a $\mathbf{y} \in G$ and $\mathbf{z} \in S$ such that $\mathbf{1}_L = \mathbf{x}\mathbf{y} + \mathbf{z}$. Therefore,

$$Q \supseteq \operatorname{supp}(\mathbf{x}) \supseteq \operatorname{supp}(\mathbf{x}\mathbf{y}) = \operatorname{supp}(\mathbf{1}_L - \mathbf{z}) \supseteq Q \setminus \operatorname{supp}(\mathbf{z})$$

which implies that $\operatorname{supp}(\mathbf{x}) \sim_{\mathcal{F}} Q$ (let's remember Notation 2.7). But, this clearly implies that $\pi_{\mathbf{x}} \in E$.

• If $\mathbf{w} \in \pi_{\mathbf{x}}(G) = G[\mathbf{x}] \leq G$, then since the multiplication by $\mathbf{y}\mathbf{w}$ is an element of E and $\mathbf{z}\mathbf{w} \in S \cap L[\mathbf{x}] = S[\mathbf{x}] \leq E\,\mathbf{x}$, we have

$$\mathbf{w} = \mathbf{1}_L \cdot \mathbf{w} = (\mathbf{x}\mathbf{y} + \mathbf{z})\mathbf{w} = (\mathbf{y}\mathbf{w})\mathbf{x} + \mathbf{z}\mathbf{w} \in E\,\mathbf{x} + E\,\mathbf{x} = E\,\mathbf{x}.$$

Therefore, $E\,\mathbf{x} = G[\mathbf{x}] = \pi_{\mathbf{x}}(G)$, so that G is E-regular by Theorem 2.14.

Finally, G is E-commutative by Proposition 3.2 and almost E-indecomposable by Corollary 2.24(b). □

Remark 3.12. Suppose that G is an sp-group and T' is a torsion semisimple group with $\mathrm{supp}\,(G) \cap \mathrm{supp}\,(T') = \emptyset$. If $G' := G \oplus T'$, then $\overline{G} = G/T$ and $\overline{G'} = G'/(T \oplus T')$ are naturally isomorphic, and they can be considered identical.

Remark 3.12 and Theorem 2.26 yield the following:

Corollary 3.13. *Suppose that G is a reduced, E-regular and E-commutative group and $\dim_{\mathbb{Q}}(G/T)$ is finite. Then $G = G_1 \oplus \cdots \oplus G_k \oplus \widehat{T}$, where*

(a) $Q := \mathrm{supp}\,(G)$ is partitioned by the sets $Q_1 := \mathrm{supp}\,(G_1), \ldots, Q_k := \mathrm{supp}\,(G_k)$, $\mathrm{supp}\,(\widehat{T})$,

(b) $\widehat{T}$ is torsion, each G_i is almost E-indecomposable and if $L_i := L_{Q_i}$, $S_i := S_{Q_i}$, then the $\overline{G}_i \leq L_i/S_i$ are fields.

Further, such a decomposition is unique, up to replacing each G_i with a corresponding G_i', such that $\mathrm{supp}\,(G_i)$ and $\mathrm{supp}\,(G_i')$ are almost equal. In particular, such replacements will not alter the corresponding subfields of $L_i/S_i \cong L_i'/S_i'$.

Now, we show that the endomorphism ring of an E_G-commutative group is necessarily (abelian) regular.

The following is essentially a restatement of Corollary 3.13 using the language of endomorphism rings.

Proposition 3.14. *Suppose that G is a reduced E_G-regular E_G-commutative group and $\dim_{\mathbb{Q}}(G/T)$ is finite. Then E_G is Abelian regular ring and E_G/I_T is isomorphic to a finite product of finite-degree extensions of $\mathbb{Q}$.*

Proof. Note that E_G/I_T isomorphic to a finite product of finite-degree extensions of $\mathbb{Q}$ by Theorem 2.32(e) since E_G is commutative. Thus, by [7, Lemma 1.3] and Theorem 2.32, it is enough to prove that I_T is a von Neumann regular ideal.

Let $\varphi \in I_T$, i.e. $\varphi(G) \subseteq T$. Then $K := \ker(\varphi)$ is a direct summand of G and $K + T = G$ by Lemma 2.30(b). Hence there exists a $U \leq T$ such that $G = U \oplus K$ and $U \cong \varphi(U)$. Since E_G is commutative, we obtain that $T \cong \bigoplus_{p \in \mathrm{supp}\,(T)} \mathbb{Z}(p)$ and $U \cong \bigoplus_{p \in \mathrm{supp}\,(U)} \mathbb{Z}(p)$, which implies $\varphi(G) = \varphi(U \oplus K) = \varphi(U) = U$ is a direct summand of G. Thus φ is a regular element of I_T by Lemma 2.30(c) as desired. □

Remark 3.15. In the above discussions, it is important that for a finite dimensional field extension R of $\mathbb{Q}$, there might be many groups G such that $\overline{G} := G/T$ is isomorphic to R without being equal to the same subring of L/S; in particular, they will not be isomorphic as groups.

Here is one example of that phenomenon in Remark 3.15:

Example 3.16. There are reduced E-regular, E-cyclic, E-commutative and almost E-indecomposable groups, say G and G', such that $\mathrm{supp}\,(G) = \mathrm{supp}\,(G')$ and $\overline{G} \cong \overline{G}'$, but G is not isomorphic to G'.

Proof. Let Q be the collection of all primes p congruent to ± 1 modulo 8, i.e. those primes for which 2 is a quadratic residue modulo p. By Dirichelet's Theorem, Q is infinite. For each $p \in Q$, let $x_p \in \mathbb{Z}(p)$ satisfy $x_p^2 = 2$ in $\mathbb{Z}(p)$.

In L, if $\mathbf{x} = (x_p)_{p \in Q}$, then we have that $\mathbf{x}^2 = 2 \cdot \mathbf{1}_L \in L$ by the construction. Clearly, there exists a unitary ring homomorphism

$$\kappa : \mathbb{Z}[\sqrt{2}] \to L/S \quad \text{such that} \quad \kappa(\sqrt{2}) = \mathbf{x} + S$$

and it extends to a ring homomorphism

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q} \cdot (\mathbb{Z}[\sqrt{2}]) \to L/S.$$

Since $\mathbb{Q}(\sqrt{2})$ is a field and this homomorphism is clearly not 0, its kernel must be 0, i.e. it must be an embedding. If $R \cong \mathbb{Q}(\sqrt{2})$ is its image, then we can let

$$G := \{\mathbf{y} \in L : \mathbf{y} + S \in R\}.$$

For $p \in Q$, let

$$\left\{ \begin{array}{lll} x'_p & = x_p, & \text{if } \; p \cong 1 \pmod 8 \\ x'_p & = -x_p, & \text{if } \; p \cong -1 \pmod 8 \end{array} \right.$$

and $\mathbf{x}' := (x'_p)_{p \in Q}$. Hence, again, there exists a ring embedding

$$\kappa' : \mathbb{Q}(\sqrt{2}) \to L/S \quad \text{such that} \quad \kappa'(\sqrt{2}) = \mathbf{x}' + T.$$

Let R' be the image of κ' and

$$G' = \{\mathbf{y} \in L : \mathbf{y} + S \in R'\}.$$

Clearly,

$$\operatorname{supp}(G) = \operatorname{supp}(G') = Q$$

and

$$\overline{G} = G/S \cong \mathbb{Q}(\sqrt{2}) \cong G'/S = \overline{G}'.$$

Claim. G is not isomorphic to G': Assume on contrary that G is isomorphic to G'. Then $G = G'$, which implies that $\mathbf{y} := \mathbf{x} + \mathbf{x}' \in G = G'$. However, it is easy to check that $\operatorname{supp}(\mathbf{y}) = \{p \in Q : p \cong 1 \pmod 8\} := Q'$, but Q' is neither finite, nor almost equal to Q, which contradicts Lemma 2.24(b). □

Remark 3.17. In fact, using the above techniques and an epsilon worth of set theory, it can be shown that there are $c = 2^{\aleph_0}$ non-isomorphic such G' with $G'/S \cong R \cong G/S$. Again, the reason these examples are not isomorphic is that even though the *quotients* $\overline{G}$ and $\overline{G}'$ are *isomorphic fields*, they are not *equal* in L/S, so that G and G' will not be isomorphic as groups. In other words, L/S will have potentially many distinct copies of a given field contained in it, and all of these copies will correspond to groups that fail to be isomorphic.

We now aim to show that in Theorem 3.11, the field $\overline{G} = G/T$ can be any finite degree extension of $\mathbb{Q}$.

We begin this construction with an important observation, which the alert reader will immediately recognize as a generalization of Euclid's famous proof of the infinitude of the prime numbers.

Lemma 3.18. *Suppose* $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ *is a non-constant polynomial (i.e. of degree* $n > 0$*). Then for an infinite number of primes* p *there is an* $m \in \mathbb{Z}$ *(depending of course on* p*) for which* $x = m$ *is a solution to*

$$f(x) \equiv 0 \pmod{p}$$

Proof. Let $\mathcal{P}$ be the set of all primes, and for any $a(x) \in \mathbb{Z}[x]$, let

$$\mathcal{P}_{a(x)} = \{p \in \mathcal{P} : a(x) \equiv 0 \pmod{p} \text{ has a solution } x = m \in \mathbb{Z}\}.$$

So we need to show that $\mathcal{P}_{f(x)}$ is infinite.

If $a_0 = 0$, then for all primes $p \in \mathcal{P}$, $f(p) \equiv 0 \pmod{p}$. In other words, $\mathcal{P}_{f(x)} = \mathcal{P}$, which is clearly infinite. We may assume, therefore, that $a_0 \neq 0$.

Let

$$g(x) = a_n a_0^{n-1} x^n + \cdots + a_2 a_0 x^2 + a_1 x + 1,$$

which, since $a_0 \neq 0$, also has degree n. Clearly, $a_0 g(x) = f(a_0 x)$, which easily implies that $\mathcal{P}_{g(x)} \subseteq \mathcal{P}_{f(x)}$. Therefore, if $\mathcal{P}_{g(x)}$ is infinite, then so is $\mathcal{P}_{f(x)}$; so replacing $f(x)$ by $g(x)$, there is no loss of generality in assuming that $a_0 = 1$.

Arguing indirectly, suppose $\mathcal{P}_{f(x)}$ is finite. If $\mathcal{P}_{f(x)} = \emptyset$, let $\pi = 1$ and if $\mathcal{P}_{f(x)} = \{p_1, \ldots, p_k\}$ is non-empty (and finite), let

$$\pi := p_1 p_2 \cdots p_{k-1} p_k \in \mathbb{N}\,.$$

Since $f(x)$ is a non-constant polynomial, there is an $N \in \mathbb{N}$ such that if $x > N$, then

$$|f(x)| > 1.$$

Choose $m > N$ such that $\pi | m$; so $|f(m)| > 1$. Therefore there is a prime $p \in \mathcal{P}$ such that

$$p | f(m), \text{ i.e. } f(m) \equiv 0 \pmod{p}.$$

It follows that $p \in \mathcal{P}_{f(x)}$, so that $p = p_i$ for some $i \in \{1, \ldots, k\}$. Therefore, $p | \pi$, and since $\pi | m$, we can conclude $p | m$. Consequently,

$$p | [f(m) - a_n m^n - a_{n-1} m^{n-1} - \cdots - a_1 m] = 1,$$

which clearly is not true. This contradiction shows that $\mathcal{P}_{f(x)}$ must in fact be infinite, as stated. □

Theorem 3.19. *Suppose* $R \leq \mathbb{C}$ *is a field of finite degree over* $\mathbb{Q}$*. Then there is a reduced* E*-regular,* E*-commutative and almost* E*-indecomposable group* G *such that* $\overline{G} = \widehat{G}/S \cong R$*.*

Proof. By the classical Primitive Element Theorem ([1, Theorem 26]), $R \cong \mathbb{Q}(\alpha)$ for some (algebraic) element $\alpha \in \mathbb{C}$. Let $f(x) \in \mathbb{Q}[x]$ be the minimal (monic) polynomial for $f(x)$ (over $\mathbb{Q}$). Multiplying by some constant, we may assume $f(x) \in \mathbb{Z}[x]$.

Using the notation of Lemma 3.18, we obtain that

$$Q := \mathcal{P}_{f(x)} = \{p : f(x) \equiv 0 \pmod p \text{ has a solution}\}$$

is an infinite set of primes. If $Q = \{p_1, p_2, p_3, \dots\}$, then we suppose, for each $i \in \mathbb{N}$, that $x = m_i \in \mathbb{Z}$ is a solution for $f(x) \equiv 0 \pmod{p_i}$.

Let $T = \oplus_{i\in\mathbb{N}} \mathbb{Z}(p_i) (= S)$ and $L = \prod_{i\in\mathbb{N}} \mathbb{Z}(p_i)$. Identifying each m_i with its corresponding equivalence class in $\mathbb{Z}(p_i)$, define

$$\mathbf{a} := (m_i)_{i\in\mathbb{N}} \in L.$$

There is clearly a (unitary) ring homomorphism $\phi : \mathbb{Z}[x] \to L/S$ such that

$$\phi(x) = \mathbf{a} + S.$$

Since L/S is a $\mathbb{Q}$-algebra, this extends to a ring homomorphism $\phi : \mathbb{Q}[x] \to L/S$ again with $\phi(x) = \mathbf{a} + S$. Since

$$\phi(f(x)) = f(\mathbf{a}) + S = (f(m_i))_{i\in\mathbb{N}} + S = 0 + S,$$

it follows that ϕ induces a ring homomorphism

$$\widehat{\phi} : \mathbb{Q}[x]/(f(x)) \to L/S$$

defined by $x + (f(x)) \mapsto \mathbf{a} + S$. However, $\mathbb{Q}[x]/f(x) \cong \mathbb{Q}(\alpha)$ is a field, so that this (clearly non-zero) homomorphism must be injective. Note that in the isomorphisms

$$\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(f(x)) \cong \phi(\mathbb{Q}[x]),$$

we have that

$$\alpha \leftrightarrow x + (f(x)) \leftrightarrow \mathbf{a} + S.$$

If $G = \{\mathbf{y} \in L : \mathbf{y} + S \in \phi(\mathbb{Q}[x])\}$, then we obtain that

$$\overline{G} = G/S = \phi(\mathbb{Q}[x]) \cong \mathbb{Q}(\alpha) \cong R,$$

such that G has the stated properties by Theorem 3.11. □

We now include some additional concrete examples of the above construction.

Example 3.20. Let $n \in \mathbb{N}$ be an odd integer, p be a prime that does not divide n and $\zeta_n = e^{2\pi i/n}$ be the n-th primitive root of 1. If

(a) $R = \mathbb{Q}(\sqrt[n]{p})$, or
(b) $R = \mathbb{Q}(\zeta_n)$,

then there is a reduced, E-ring and E-regular group G such that $\operatorname{supp}(G) = Q$ and $\overline{G}/T = R$.

Proof. (a) Let $f(x) = x^n - p \in \mathbb{Q}[x]$. By the Eisenstein's irreducibility condition, $f(x)$ is irreducible over $\mathbb{Q}$. So, if $\alpha = \sqrt[n]{p}$, then $R = \mathbb{Q}(\alpha)$ is a field with $(R : \mathbb{Q}) = n$.

Let Q be the set of primes that are congruent to 2 modulo n (other than possibly p). Since 2 and n are relatively prime, we obtain that Q is be an infinite set by Dirichlet's Theorem. If $q \in Q$, then $q - 1 \equiv 1 \pmod{n}$, which gives that n is relatively prime to $q - 1$. Recall that, under the multiplication, the non-zero elements of $\mathbb{Z}(q)$ (i.e., $\mathbb{Z}(q)^*$) are a cyclic group (of order $q - 1$). It follows that $z \mapsto z^n$ is an automorphism of $\mathbb{Z}(q)^*$. Hence, for each $q \in Q$, there is a unique $a_q \in \mathbb{Z}(q)$ such that $a_q^n = p$ in $\mathbb{Z}(q)$.

Again, as usual, let $P := \prod_Q \mathbb{Z}(q)$, $T := T_P$, and $\mathbf{a} = (a_q)_{q \in Q}$. Then $1 \mapsto \mathbf{1}_P + T$ and $\alpha \mapsto \mathbf{a} + T$ extends to a ring homomorphism $R \to P/T$. Since R is a field and this map is clearly non-zero, it must be an embedding map. Identifying R with its image, we can again allow $G := \{\mathbf{y} \in P : \mathbf{y} + T \in R\}$, which will clearly satisfy our requirements.
(b) Let and $t_n(x) \in \mathbb{Z}[x]$ be the n-th cyclotomic polynomial. Then $t_n(x)$ is an irreducible polynomial of degree $\varphi(n)$, and hence $(R : \mathbb{Q}) = \varphi(n)$. Note that $t_n(x)$ decomposes to a product of linear polynomials over modulo prime q iff $q \equiv 1 \pmod{n}$ and there exists infinitely many such primes by Dirichlet's Theorem. Hence, we can take an infinite set Q of such primes and, for each $q \in Q$, we choose $a_q \in \mathbb{Z}(q)$ a root of t_n modulo q. Now if we put $\mathbf{a} := (a_q)_{q \in Q}$, then, by the same construction as in (a), we construct a group G satisfying the requirements. □

We now want to extend the above construction from the case of reduced E-regular and almost E-indecomposable groups that are E-commutative, to produce non-E-commutative examples that are closely related to them.

Theorem 3.21. *Suppose that $R \leq \mathbb{C}$ is a field of finite degree over $\mathbb{Q}$ and $n \in N$. Then there is a reduced E-regular and almost E-indecomposable group G such that $\overline{G} = \widehat{G}/S \cong R^{(n)}$, which is, in a natural way, a simple module over $E/I_T \cong M_n(R)$.*

Proof. Let $S = \oplus_{p \in Q} Z(p)$ and $S \leq H \leq L = \prod_{p \in Q} Z(p)$ with $H/S \cong R$ as in Theorem 3.19. In a natural way, we define $G = H^{(n)}$ with $T = S^{(n)}$.
Claim: $\overline{G} = G/T \cong R^{(n)}$: This is clear.

Since H is an E-ring, we can identify it with E_H in such a way that I_S is identified with S. More generally, E_G can therefore naturally be identified with $M_n(H)$, and in so doing, I_G is identified with $M_n(S)$. It follows that

$$E_G/I_T \cong M_n(H)/M_n(S) \cong M_n(H/S) \cong M_n(R).$$

So, the only point to be verified is that
Claim: G is E-regular and almost E-indecomposable: To show that G is E-regular, we use Theorem 2.14. So suppose that $\mathbf{x} \in G$ has infinite order. If $\mathbf{x} = \mathbf{u}_1 + \cdots + \mathbf{u}_n$ ($\mathbf{u}_i \in H$ in the i'th direct summand of G), then for some such i, $\mathbf{u}_i$ must have infinite order. Therefore, $\operatorname{supp}(\mathbf{x}) \supseteq \operatorname{supp}(\mathbf{u}_i)$ must be cofinite in Q. If F is the finite set $Q \setminus \operatorname{supp}(\mathbf{x})$, then $G = T_F \oplus G[\mathbf{x}]$, where $T_F := \oplus_{p \in F} T_p$ and $\mathbf{x} \in G[\mathbf{x}]$. Since both of these direct summands are fully invariant in G (and hence E-module direct summands),

it suffices to show that $G[\mathbf{x}] = E\,\mathbf{x}$. To this end, it suffices to assume that $G = G[\mathbf{x}]$ and $\operatorname{supp}(\mathbf{x}) = Q$.

Let $\mathbf{y} \in G$. We need to find $\gamma \in E$ such that $\gamma(\mathbf{x}) = \mathbf{y}$. Now, since G/S is a simple E/I_S-module, there is a $\phi \in E$ such that $\mathbf{y} = \phi(\mathbf{x}) + \mathbf{z}$, where $\mathbf{z} \in T$. There is clearly a finite set of primes $F \subseteq Q$ such that $\mathbf{z} \in T_F := \oplus_{p \in F} T_p$. Again, there is a decomposition $G = T_F \oplus G'$, where both terms are E-submodules, i.e. fully invariant. If $\mathbf{x} = \mathbf{x}_F + \mathbf{x}'$ and $\mathbf{y} = \mathbf{y}_F + \mathbf{y}'$, then we have that $\phi(\mathbf{x}') = \mathbf{y}'$. Since $\operatorname{supp}(\mathbf{x}_F) = F$, it follows that there is an endomorphism $\gamma : T_F \to T_F$ such that $\gamma(\mathbf{x}_F) = \mathbf{y}_F$). Setting $\gamma = \phi$ on G', we can conclude that $\gamma \in E$ and $\mathbf{y} = \gamma(\mathbf{x})$. Therefore, $G = E\,\mathbf{x}$, as claimed.

If G failed to be torsion E-indecomposable, there would be an E-module decomposition $G = A \oplus B$ where neither A nor B is torsion. This, however, would contradict that G/S is simple as an E/I_S-module. Finally, if G failed to be almost E-indecomposable, there would be an E-module decomposition $G = A \oplus B$ where B is a torsion group of infinite support. This, however, would contradict the fact that G has elements $\mathbf{x}$ of support equal to Q, completing the proof. □

We end with the following question:

Question 3.22. *Can the groups $G_1, \ldots, G_k$ in Theorem 2.26 be assumed to be constructed as was the group G in Theorem 3.21 (for different disjoint sets of primes Q_i and different finite field extensions R_i of $\mathbb{Q}$)?*

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