# ON THE SCHRÖDER-BERNSTEIN PROPERTY FOR ABELIAN GROUPS

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ABSTRACT. A right $R$-module $M$ satisfies the Schröder-Bernstein property, if whenever direct summands, say $N$ and $K$, of $M$ are $d$-subisomorphic to each other (i.e. if $N$ is isomorphic to a direct summand of $K$ and $K$ is isomorphic to a direct summand of $N$), then $N \cong K$. The module $M$ is said to be ADS (Absolute Direct Summand) if for every decomposition $M = S \oplus T$ and every complement $A$ of $S$, we have $M = S \oplus A$. We primarily show that the question, whether ADS abelian groups satisfying the Schröder-Bernstein property, has a positive answer. Then we consider a related problem on the property C2 (a group $G$ is C2 if whenever $A$ is a summand of $G$ and $B$ is a subgroup of $G$ isomorphic to $A$, then $B$ is also a summand of $G$) and we present several sufficient conditions of C2 abelian groups to satisfy the Schröder-Bernstein property.

## 1. INTRODUCTION

In the set theory, the Schröder-Bernstein theorem states that if there exist injective functions $A \to B$ and $B \to A$ between the sets $A$ and $B$, then there exists a bijective function $A \to B$. This has been investigated in some branches of mathematics: In the module theory, Bumby [3] proved that the Schröder-Bernstein problem has a positive solution for homomorphism of modules which are invariant under endomorphisms of their injective envelopes. In [8], Dehghani et al. studied the Schröder-Bernstein property for several classes of modules. Two $R$-modules $N$ and $K$ are said to be *direct summand subisomorphic* to each other (or *$d$-subisomorphic*) if $N$ is isomorphic to a direct summand of $K$ and $K$ is isomorphic to a direct summand of $N$, and a module $M$ satisfies the *Schröder-Bernstein property*, or the "SB property" for short, if whenever direct summands $N$ and $K$ of $M$ are $d$-subisomorphic to each other, then $N \cong K$ ([8,

1991 *Mathematics Subject Classification.* 20K20, 20K21.

*Key words and phrases.* Schröder-Bernstein property, Absolute Direct Summand, C2 property.

Definitions 1.5 and 1.6]). They proved that over a Noetherian ring $R$, every extending module (defined by the property that every submodule of the module is essential in a direct summand) satisfies the Schröder-Bernstein property. In the theory of abelian groups, the following question was raised by Kaplansky [12] (known as Kaplansky's First Test Problem):

If $G$ and $H$ are abelian groups such that $G$ is isomorphic to a direct summand of $H$ and $H$ is isomorphic to a direct summand of $G$, are $G$ and $H$ necessarily isomorphic?

For more results on this direction, we refer to the papers [7], [9], [18].

The notion of the *absolute direct summand* was introduced by Fuchs in [10]. In [1] and [4], the authors introduced and studied the module-theoretical version of the absolute direct summand. A right $R$-module $M$ is said to be *Absolute Direct Summand* (*ADS*) if for every decomposition $M = S \oplus T$ and every complement $A$ of $S$, we have $M = S \oplus A$. Let $\mathbb{P}$ denote the set of all prime numbers, $A$ be an abelian group, and $p \in \mathbb{P}$. Following the terminology of [6], we say that $A$ is *$p$-automorphic* if the map $a \to pa$ is an automorphism of $A$, and $A$ is called *homococyclic* if there exist a cardinal $\lambda$, a value $k \in \mathbb{N} \cup \{\infty\}$ and $p \in \mathbb{P}$ such that $A \cong \mathbb{Z}_{p^k}^{\lambda}$. Recall the characterization of ADS abelian groups, which was first published by Rangaswamy in the paper [17] and independently proved in the recent paper [13]:

**Theorem 1.1** ([17, Corollary 3.8],[13, Theorem 3.1]). *An abelian group is ADS if and only if*

*(1) either it is divisible,*
*(2) or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*
*(3) or it is a torsion group such that $p$-components are homococyclic for all $p \in \mathbb{P}$.*

In view of the studies on the Schröder-Bernstein property in the theory of abelian groups, our main aim is to study the following problem.

**Problem 1.2.** *Characterize ADS abelian groups satisfying the Schröder-Bernstein property.*

We will answer Problem 1.2 in Section 2. First we prove the next theorem.

**Theorem 1.3.** *If $A$ and $B$ are $d$-subisomorphic ADS abelian groups, then $A$ and $B$ are isomorphic.*

Since each direct summand of an ADS module is ADS, we obtain the following direct consequence.

**Corollary 1.4.** *Every ADS abelian group satisfies the Schröder-Bernstein property.*

A direct consequence of Corollary 1.4 and Theorem 1.1 will be the following result for automorphism-invariant abelian groups.

**Corollary 1.5.** *Let $G$ be an abelian group. If $G$ is either divisible or is a torsion group with homococyclic components, then $G$ satisfies the Schröder-Bernstein property.*

Recall that a module $M$ which is invariant under automorphisms of its injective hull is called an *automorphism-invariant* module ([14]). Since an abelian group is automorphism-invariant iff it is pseudo-injective iff it is is either divisible or torsion with homococyclic components by [5, Theorem 2.1], we have the following result of Corollary 1.5:

**Corollary 1.6.** *Every automorphism-invariant (pseudo-injective) abelian group satisfies the Schröder-Bernstein property.*

A group is *reduced* if it contains no nonzero divisible subgroup. Recall that every abelian group $A$ contains a maximal divisible subgroup, say $D$, and a reduced subgroup, say $R$, such that $A = D \oplus R$.

A group $G$ is *C2* if whenever $A$ is a direct summand of $G$ and $B$ is a subgroup of $G$ isomorphic to $A$, then $B$ is also a direct summand of $G$ [6]. Since, by [6];

$(i)$ every divisible group is injective (so quasi-injective) hence C2,

$(ii)$ a torsion-free group is C2 iff it is divisible,

$(iii)$ the only indecomposable C2 groups are the cocyclic groups and $\mathbb{Q}$,

$(iv)$ a torsion group is C2 iff it has homococyclic,

it is natural to raise the following problem.

**Problem 1.7.** *Characterize C2 abelian groups satisfying the Schröder-Bernstein property.*

We will partially answer Problem 1.7 in Sections 3 and 4. In particular, we formulate several structural conditions under which C2 groups satisfy the Schröder-Bernstein property.

It might be noted in the text that any class of groups that is closed under summands and which is classified by cardinal invariants that respect direct sum decompositions will have the property that any two $d$-subisomorphic groups are isomorphic; so that in particular, any group in the class will satisfy the SB-property.

Throughout this paper, $R$ is an associative ring with unity and all modules over $R$ are unitary right modules. $r_R(x)$ denotes a right annihilator of an element $x$ over a ring $R$. We also write $M_R$ to indicate that $M$ is a right $R$-module. For a submodule $N$ of $M$, we use $N \leq M$. We write $\mathbb{Z}$ and $\mathbb{N}$ for the ring of integers and for the set of all positive integer numbers, respectively. For any group $G$, as usually $X \subseteq G$ shows $X$ is a subset of $G$ but $X \leq G$ is used only for a subgroup $X$ of $G$. For unexplained notions and results, we refer the reader to [10].

## 2. ADS ABELIAN GROUPS WITH SB PROPERTY

Let us formulate a well-known observation about fully invariant modules and its easy consequence.

**Lemma 2.1.** *Let $A$ be a fully invariant submodule of a module $M$ and $B$ a direct summand of $M$. Then $B \cap A$ is a direct summand of $A$ and $(B + A)/A$ is a direct summand of $M/A$ .*

*Proof.* By the hypothesis, the natural projection $M \to B$ can be represented as an idempotent $\epsilon \in \operatorname{End}(M)$ satisfying $\epsilon(M) = B$ and $(1 - \epsilon)(M) \oplus \epsilon(M) = M$. Since $A$ is fully invariant, both images $\epsilon(A)$ and $(1 - \epsilon)(A)$ are submodules of $A$. Thus $\epsilon(A) = A \cap B$ and $A = \epsilon(A) \oplus (1 - \epsilon)(A)$. Similarly, $\tilde{\epsilon}(m + A) = \epsilon(m) + A$ presents a correctly defined idempotent endomorphism of the module $M/A$, hence $M/A = \tilde{\epsilon}(M/A) \oplus (1 - \tilde{\epsilon})(M/A)$ with $\tilde{\epsilon}(M/A) = (B + A)/A$. □

**Lemma 2.2.** *Let $A$ be a $d$-subisomorphic to an abelian group $B$, let $E$, $F$ be maximal divisible subgroups of $A$ and $B$ respectively, and $S \subseteq \mathbb{P}$. If $A_S = \bigoplus_{p \in S} A_p$ and $B_S = \bigoplus_{p \in S} B_p$, then*

*(1) $A_S$ is $d$-subisomorphic to $B_S$,*
*(2) $A/A_S$ is $d$-subisomorphic to $B/B_S$,*
*(3) $E$ is $d$-subisomorphic to $F$,*
*(4) $A/E$ is $d$-subisomorphic to $B/F$.*

*Proof.* Let us denote by $C_S = \bigoplus_{p \in S} C_p$ for an arbitrary abelian group and remark that $C_S$ is a fully invariant submodule of $C$. Suppose that $D$ is a direct summand of $B$ which is isomorphic to $A$.

(1) Since $A_p \cong D_p = D \cap B_p$, it is easy to see that $A_S \cong D_S = D \cap B_S$, which is a direct summand of $B_S$ by Lemma 2.1.
(2) Note that $A/A_S \cong D/D_S = D/(D \cap B_S) \cong (D + B_S)/B_S$ by the hypothesis. Then the conclusion follows since $(D + B_S)/B_S$ is a direct summand of $B/B_S$ by Lemma 2.1.
(3) Denote by $G$ the maximal divisible subgroup $G$ of $D$. Since $G$ is a direct summand of $B$ and it is isomorphic to $E$, the assertion is clear.
(4) Similarly as in (2), we get $A/E \cong D/G = D/(D \cap F) \cong (D + F)/F$ by Lemma 2.1, where the last group is a direct summand of $B/F$, as $F$ is fully invariant. □

**Lemma 2.3.** *If $A$ and $B$ are subisomorphic ($d$-subisomorphic) homococyclic abelian groups, then $A \cong B$.*

*Proof.* By the hypothesis, there exist $k, l \in \mathbb{N} \cup \{\infty\}$ and cardinals $\kappa, \lambda$ such that $A \cong \mathbb{Z}_{p^k}^{(\kappa)}$ and $B \cong \mathbb{Z}_{p^l}^{(\lambda)}$.

First, assume that $A$ is $d$-subisomorphic to $B$. Then $k = l$ and $\kappa \leq \lambda$. The symmetric argument says that $\kappa = \lambda$.

Now, we assume that $A$ and $B$ are subisomorphic abelian groups. Since $A_{\mathbb{Z}}$ and $B_{\mathbb{Z}}$ are quasi-injective and so continuous, we get that $A \cong B$ by [15, Proposition 10]. □

We are now ready to give the proof of Theorem 1.3.

**Proof of Theorem 1.3.**

Assume that $A$ and $B$ are $d$-subisomorphic ADS abelian groups. Then they are either divisible, or a direct sum of an indecomposable torsion-free group and a divisible torsion group, or torsion groups such that each $p$-component is homococyclic by Theorem 1.1. We show that $A \cong B$ in all these cases.

If $A$ is divisible, then $B$ is divisible as well, and so $A$ and $B$ are isomorphic by [3, Theorem]. Suppose that $A = F \oplus D$ for a nonzero indecomposable torsion-free group $F$ and a divisible torsion group $D$. Since $A$ is $d$-subisomorphic to $B$, the group $B$ is a proper mixed ADS group, hence it is of the same form $B = \tilde{F} \oplus \tilde{D}$ where $\tilde{F}$ is a nonzero indecomposable torsion-free and $\tilde{D}$ is a divisible torsion group. As $D = \bigoplus_{p \in \mathbb{P}} A_p$ and $\tilde{D} = \bigoplus_{p \in \mathbb{P}} B_p$, both subgroups $D$, $\tilde{D}$ are fully invariant and $D$ and $\tilde{D}$ are $d$-subisomorphic by Lemma 2.2(1). Thus $D$ and $\tilde{D}$ are isomorphic by the argument of the first part of the proof. Similarly, $F \cong A/D$ and $\tilde{F} \cong A/\tilde{D}$ are $d$-subisomorphic pairs of groups by

Lemma 2.2(2). Hence $F$ and $\tilde{F}$ are isomorphic because $F$ contains no proper direct summand.

Finally, let $A = \bigoplus_{p\in\mathbb{P}} A_p$ and $B = \bigoplus_{p\in\mathbb{P}} B_p$ be sums of homococyclic $p$-components. Then $A_p$ and $B_p$ are $d$-subisomorphic by Lemma 2.2(1) and so are isomorphic by Lemma 2.3. This proves that $A$ and $B$ are isomorphic.

Recall that a ring $R$ is called right *pure-semisimple* if every right $R$-module is a direct sum of finitely generated $R$-modules. The next example illustrates that modules satisfying the SB-property are not necessarily ADS.

**Example 2.4.** *Let $R$ be a Dedekind domain and $I$ a nonzero ideal of $R$. Then $R/I$ is a commutative Artinian principal ideal ring by* [2, Theorems 9.3 and 8.5, Exercise 9.7, p. 99]*, and so it is pure-semisimple by* [11, Theorem 4.3]*. By* [8, Theorem 4.2]*, every right $R$-module has the SB-property. But it is not ADS by* [16, Theorem 2.4].

## 3. C2-ABELIAN GROUPS WITH SB-PROPERTY

**Example 3.1.** *(1) $\mathbb{Z}$ as a $\mathbb{Z}$-module is ADS (since $\mathbb{Z}$ is indecomposable) which does not satisfy C2.*

*(2) The localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime $p$ is ADS but it is not C2.*

**Example 3.2.** *Let $p$ be a prime integer number and let $M$ be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p)\oplus\mathbb{Q}$. Then $M$ is not an ADS module. However, $M$ satisfies C2.*

We list some basic properties of reduced C2 abelian groups.

**Lemma 3.3.** *Let $A$ be a reduced C2 abelian group and $E = \mathrm{End}(A)$. Then*

1. *for each $p \in \mathbb{P}$, there exist $n_p \in \mathbb{N}$, a cardinal $\kappa_p$ and a central idempotent $e_p \in E$ such that $e_p(A) = A_p \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$ and $(1 - e_p)(A)$ is $p$-divisible,*
2. *$A/t(A)$ is torsion free divisible,*
3. *the map $\varepsilon : E \to \prod_{p\in\mathbb{P}} e_pE$ given by $\varepsilon(r) = (e_pr)_{p\in\mathbb{P}}$ is a ring embedding and $\bigoplus_{p\in\mathbb{P}} e_pE$ is an ideal of the ring $\varepsilon(E)$,*
4. *if $s, e \in E$ and $a \in A$ such that $e$ is an idempotent, $se(a) = 0$ and $e(a) \neq 0$, then there exists $g \in E$ for which $seg = 0$ and $eg \neq 0$.*

*Proof.* (1) By [6, Theorem 8], $A_p$ is homococyclic and there exists a $p$-divisible subgroup, say $D_p$, of $A$ such that $A = A_p \oplus D_p$. This implies the existence of an idempotent, say $e_p \in E$, with $e_p(A) = A_p$ and $(1 - e_p)(A) = D_p$, which is central since $\mathrm{Hom}(A_p, D_p) = 0 = \mathrm{Hom}(D_p, A_p)$. Finally, as $A$ is reduced and $A_p$

is homococyclic, there exist $n_p \in \mathbb{N}$ and a cardinal $\kappa_p$ for which$A_p \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$.
(2) Clearly, $A/t(A) = A/\bigoplus_{p\in\mathbb{P}} A_p$ is torsion free and it is $p$-divisible for each $p \in \mathbb{P}$ by (1).
(3) It is easy to see that $\varepsilon$ is a ring homomorphism, so it is enough to show that it is injective. Let $\varepsilon(f) = 0$. Then $f(A_p) = 0$ for each $p \in \mathbb{P}$, and hence $f(t(A)) = 0$. Note that $A/t(A)$ is divisible by (2). Now $f(A)$ is isomorphic to a factor divisible group. Therefore $f(A)$ is a divisible subgroup of a reduced group $A$ which implies $f(A) = 0$.
(4) As there exists $p \in \mathbb{P}$ satisfying $e_p(e(a\mathbb{Z})) \neq 0$ by (3) and $e_p(A) \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$ is a free module over the ring $\mathbb{Z}_{p^{n_p}}$ we may chose $g \in e_pE$ for which $g(A) = e_p(e(a\mathbb{Z}))$. Now $eg \neq 0$ since $eg(A) = ee_pe(a\mathbb{Z}) = e_pe(a\mathbb{Z}) \neq 0$. Similarly, $seg(A) = e_pse(a\mathbb{Z}) = 0$, and so $seg = 0$. □

We recall the well known fact that the central idempotents $e_p$ of $\mathrm{End}(A)$ are uniquely determined by the $p$-component.

**Proposition 3.4.** *Let $A$ and $B$ be $d$-subisomorphic C2 abelian groups and $A_p$ be finite for every $p \in \mathbb{P}$ such that $A_p$ is a non-divisible $p$-component. Then $A$ and $B$ are isomorphic.*

*Proof.* First, note that $A_p$ and $B_p$ are homococyclic by [6, Theorem 8] and $d$-subisomorphic by Lemma 2.2(1) for each $p \in \mathbb{P}$.

Let $E$ and $F$ be maximal divisible subgroups of $A$ and $B$, respectively. Then $E$ and $F$ are $d$-subisomorphic by Lemma 2.2(3), and hence $E \cong F$ by [3, Theorem]. Note that $E$ and $F$ are direct summands of $A$ and $B$ respectively and $A/E$ and $B/F$ are $d$-subisomorphic groups by Lemma 2.2(3) containing no nonzero divisible subgroup. Thus we may suppose without loss of the generality that $A$ is reduced and $A_p$ is finite for all $p \in \mathbb{P}$ and it remains to show that $A \cong B$ for such $A$.

Suppose that $\varphi : B \to A$ is a monomorphism such that $D := \varphi(B)$ is a direct summand in $A$, which is isomorphic to $B$. Then, $D_p \subseteq A_p$ is finite, $D_p = \varphi(B_p) \cong B_p$, and $A_p \cong B_p$ by Lemma 2.3, which shows that $A_p = D_p$ for each $p \in \mathbb{P}$. Since there exists a direct summand $X$ of $A$ satisfying $X \oplus D = A$ and $\bigoplus_{p\in\mathbb{P}} A_p \subseteq D$, we get that

$$A/\bigoplus_{p\in\mathbb{P}} A_p \cong X \oplus (D/\bigoplus_{p\in\mathbb{P}} A_p),$$

where the term on the right side is divisible by Lemma 3.3(2). Hence $X$ is a divisible subgroup of $A$. As $A$ is reduced, $X = 0$ and so $A = D = \varphi(B)$. □

As in the case of ADS groups, also any direct summand of C2 group is C2, which allows us to formulate the following consequence:

**Corollary 3.5.** *Every C2 abelian group which has every non-divisible $p$-component finite satisfies the Schröder-Bernstein property.*

We note that, for abelian groups with C2 property, the notions of ″subisomorphic″ and ″$d$-subisomorphic″ coincide ([8, Theorem 3.3]).

**Proposition 3.6.** *Let $A$ and $B$ be $d$-subisomorphic C2 abelian groups. If $A$ is reduced and there are only finitely many primes $p$ for which $A_p$ is infinite, then $A$ and $B$ are isomorphic.*

*Proof.* Denote by $p_1, p_2, \dots, p_n$ all primes such that $A_{p_i}$ is infinite. Since $A_{p_i}$ and $B_{p_i}$ are homococyclic by Lemma 3.3 and non-divisible by the hypothesis, there exist $k_1, \dots, k_n \in \mathbb{N}$ such that $p_i^{k_i} A_{p_i} = 0$ for each $i$. Furthermore $A_{p_i}$ and $B_{p_i}$ are $d$-subisomorphic by Lemma 2.2(1), and hence they are isomorphic by Lemma 2.3 which implies $p_i^{k_i} B_{p_i} = 0$.

Put $r := \prod_{i=1}^{n} p_i^{k_i}$. Then $A \cong rA \oplus \bigoplus_{i=1}^{n} A_{p_i}$ and $B \cong rB \oplus \bigoplus_{i=1}^{n} B_{p_i}$. By Lemma 2.2(2), $rA$ and $rB$ are $d$-subisomorphic groups with finite $p$-components for all $p \in \mathbb{P}$. Hence $rA \cong rB$ by Proposition 3.4. $\square$

**Corollary 3.7.** *Every C2 abelian group which has only finitely many non-zero $p$-components satisfies the Schröder-Bernstein property.*

**Proposition 3.8.** *If $A$ and $B$ are $d$-subisomorphic C2 abelian groups such that there are only finitely many primes $p$ for which each $A_p$ is non-divisible infinite, then $A$ and $B$ are isomorphic.*

*Proof.* It is easy to say that $A = R_A \oplus D_A$ and $A = R_B \oplus D_B$ for a pair of reduced groups ($R_A$, $R_B$) and a pair of divisible groups ($D_A$, $D_B$) where the both pairs ($R_A$, $R_B$) and ($D_A$, $D_B$) are $d$-subisomorphic by Lemma 2.2(3),(4). Then $R_A \cong R_B$ by Proposition 3.6 and $D_A \cong D_B$ by [3, Theorem]. $\square$

**Corollary 3.9.** *Every C2 abelian group containing only finitely many non-divisible infinite $p$-components satisfies the Schröder-Bernstein property.*

## 4. More on reduced abelian groups and the $C2$-condition

Recall that a ring $R$ is said to be right C2 if the module $R_R$ is C2. Let us formulate an elementary description of such a ring.

**Lemma 4.1.** *A ring $R$ is right C2 if and only if the right ideal $seR$ is generated by an idempotent for every $e, s \in R$ such that $e$ is an idempotent and $r_R(se) = (1-e)R$.*

*Proof.* Note that a right ideal $I$ is a direct summand in $R_R$ if and only if $I = eR$ for an idempotent $e$.

Let $R_R$ have $C2$-property and $r_R(se) = (1-e)R$ where $e^2 = e \in R$ and $s \in R$. It is clear that $seR \cong R/r_R(se) \cong eR$. Therefore by $C2$-property of $R_R$, $seR$ should be a direct summand of $R_R$, as desired.

For the converse, we assume that $\varphi : eR \to R$ is an embedding. Then there exists $s \in R$ such that $s = \varphi(e)$. Since $r_R(se) = (1-e)R$, we get an idempotent generating the image $seR$ by the hypothesis. □

**Proposition 4.2.** *The following conditions are equivalent for a reduced abelian group $A$ and $E = \mathrm{End}(A)$:*

*(1) $A$ is C2,*
*(2) $E$ is right C2,*
*(3) For each $p \in \mathbb{P}$ there exist a central idempotent $e_p \in E$, $n_p \in \mathbb{N}$, and a cardinal $\kappa_p$ such that $e_p(A) = A_p \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$, the map $\varepsilon : E \to \prod_{p\in\mathbb{P}} e_pE$ given by $\varepsilon(r) = (e_pr)_{p\in\mathbb{P}}$ is a ring embedding, and for every $e, s \in E$ such that $e$ is an idempotent and $r_{e_pE}(e_pse) = e_p(1-e)E$ for all $p \in \mathbb{P}$, there exist idempotents $f_p \in e_pE$ satisfying $f_pe_pE = e_pseE$ for $p \in \mathbb{P}$ such that $(f_p)_{p\in\mathbb{P}} \in E$.*

*Proof.* (1)⇒(3) The properties of $A_p$, $p \in \mathbb{P}$ and $\varepsilon$ follow from Lemma 3.3(1) and (3). Note that $e(A)$ is a direct summand of the C2 group $A$ and the restriction of the endomorphism $s \in \mathrm{End}(A)$ to $e(A)$ forms a homomorphism $e(A) \to A$. If $s(e(a)) = 0$ for $e(a) \neq 0$, then there exists $g \in E$ such that $eg \neq 0$ and $seg = 0$ by Lemma 3.3(4). This implies that $0 \neq eg \in r_E(se)$ which contradicts to the hypothesis (i.e. $r_E(se) = (1-e)E$). Therefore $se(A)$ is a monomorphic image of $e(A) = B$, which is a direct summand of $A$ as $A$ is C2. Thus there exists an idempotent $f \in E$ such that $f(A) = se(A)$ which implies that $fE = seE$. Now it remains to put $f_p = e_pf$ for each $p \in \mathbb{P}$.
(3)⇒(2) This follows immediately from Lemma 4.1 where the desired idempotent is of the form $(f_p)_{p\in\mathbb{P}}$.
(2)⇒(1) Let $B$ be a direct summand of $A$ and $\varphi : B \to A$ be an embedding. Then there exist $s \in E$ and an idempotent $e \in E$ satisfying $B = e(A)$ and $s(a) = \varphi(e(a))$. Clearly, $r_E(se) = (1-e)E$, which implies the existence of an idempotent $f \in E$ such that $fE = seE$ by Lemma 4.1. Now, $f(A) = se(A) = \varphi(B)$ is a direct summand of $A$, which proves that $A$ is C2. □

Note that the equivalence of the first two conditions does not hold for general abelian groups.

**Example 4.3.** *Let $p \in \mathbb{P}$ and $A = \mathbb{Z}_{p^\infty}$. Note that $A$ is divisible and so is C2. Then $\operatorname{End}(A) = \hat{\mathbb{Z}}_p$ is the ring of $p$-adic integers which is not C2 by Lemma 4.1 since $\hat{\mathbb{Z}}_p$ is a non-trivial local domain.*

We formulate two consequences of Proposition 4.2.

**Corollary 4.4.** *Let $A$ be an abelian group and $D$ be the maximal divisible subgroup of $A$. The following conditions are equivalent:*

*(1) $A$ is C2,*
*(2) $\operatorname{End}(A/D)$ is right C2.*

*Proof.* (2)⇒(1) Since direct summands of C2 groups are C2 and so $A/D$ is a reduced group which is isomorphic to the direct summand of $A$, the claim follows from Proposition 4.2.
(1)⇒(2) Let us remark that $A \cong t(D) \oplus D_f \oplus A/D$ where $t(D)$ is torsion divisible, $D_f$ is torsion-free divisible and $A/D$ is $t(D)$-automorphic. Hence $\oplus D_f \oplus A/D$ is $t(D)$-automorphic. Now it remains to apply [6, Lemma 11]. □

**Corollary 4.5.** *Suppose $A$ is a reduced abelian group, $E = \operatorname{End}(A)$ and there exists a central idempotent $e_p \in E$ such that $A_p = e_p(A)$ is homococyclic for every $p \in \mathbb{P}$. If $\varepsilon : E \to \prod_{p\in\mathbb{P}} e_pE$, given by $\varepsilon(r) = (e_pr)_{p\in\mathbb{P}}$, is an isomorphism, then $A$ is C2.*

*Proof.* It is enough to check the hypothesis of Proposition 4.2(3). Let $e, s \in E$, where $e$ is an idempotent, and $r_{e_pE}(e_pse) = e_p(1-e)E$ for each $p \in \mathbb{P}$. Since $e_pse$ induces a monomorphism $B = e_pe(A) \to e_p(A)$, where $B$ is a projective $\mathbb{Z}_{p^{n_p}}$-module, we obtain $e_pse(B)$ is a projective module over the Frobenius ring $\mathbb{Z}_{p^{n_p}}$. Thus $e_pse(B)$ is injective, hence there exists an idempotent $f_p \in e_pE$ satisfying $f_p(e_p(A)) = e_pse(B)$ for each $p \in \mathbb{P}$. Since $\varepsilon(E) = \prod_{p\in\mathbb{P}} e_pE$, we get $(f_p)_{p\in\mathbb{P}} \in E$, and hence $A$ is C2 by Proposition 4.2. □

Recall that $e_p$ denotes the uniquely defined central idempotent such that $e_p(A) = A_p$. Furthermore, we will identify $E = End(A)$ with its image $\varepsilon(E)$ in the ring $\prod_{p\in\mathbb{P}} e_pE$.

**Theorem 4.6.** *Let $A$ be a reduced abelian group and $E = \operatorname{End}(A)$. If, for every $p \in \mathbb{P}$, there exists a central idempotent $e_p \in E$ such that $A_p = e_p(A)$ is homococyclic and $E = \prod_{p\in\mathbb{P}} e_pE$, then $A$ is a C2 group satisfying the Schröder-Bernstein property.*

*Proof.* By Corollary 4.5, the reduced abelian group $A$ is C2. Since $A_p$ satisfies the Schröder-Bernstein property by Lemma 2.3, we obtain that $e_pE \cong \mathrm{End}(A_p)$ satisfies it by [8, Theorem 2.4(a)] for each $p \in \mathbb{P}$. Therefore $E = \prod_{p\in\mathbb{P}} e_pE$ and hence $A$ satisfies the Schröder-Bernstein property by [8, Theorem 2.4(d),(a)]. □

Recall that the class of abelian groups satisfying the Schröder-Bernstein property was not closed under the factor.

**Proposition 4.7.** *Let $M$ be an abelian group and $D$ be its maximal divisible subgroup. The following conditions are equivalent:*

*(1) $M$ satisfies the Schröder-Bernstein property.*
*(2) $M/D$ satisfies the Schröder-Bernstein property.*

*Proof.* (2)⇒(1) Assume $A$ and $B$ are $d$-subisomorphic direct summands of $M$. We denote by $R_A$ and $R_B$ reduced subgroups and $D_A$ and $D_B$ (maximal) divisible subgroups satisfying $A = R_A \oplus D_A$ and $B = R_B \oplus D_B$. Clearly, $D_A$ and $D_B$ are direct summands of $D$ and $R_A \cap D = R_B \cap D = 0$, which implies that $R_A$ and $R_B$ are isomorphic to direct summands of $M/D$. Note that $D_A$ and $D_B$ are $d$-subisomorphic by Lemma 2.2(3) and $R_A$ and $R_B$ are $d$-subisomorphic by Lemma 2.2(4). Hence $D_A \cong D_B$ by [3, Theorem] and $R_A \cong R_B$ by the hypothesis.
(1)⇒(2) This implication follows from [8, Theorem 2.4(b)] since $M \cong D \oplus (M/D)$. □

**Theorem 4.8.** *Let $A$ and $D$ be abelian groups and $E = \mathrm{End}(A)$. If $D$ is divisible and $A$ is reduced C2 such that $E = \prod_{p\in\mathbb{P}} e_pE$, then $A \oplus D$ satisfies the Schröder-Bernstein property.*

*Proof.* By Theorem 4.6, $A$ satisfies the Schröder-Bernstein property and hence the assertion follows from Proposition 4.7. □

**Example 4.9.** *Let $A = \prod_{p\in\mathbb{P}} \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$ for a system of natural numbers $n_p$ and cardinals $\kappa_p$ for each $p \in \mathbb{P}$.*
*(1) $A$ is an abelian reduced group since $\bigcap_{p\in\mathbb{P}} p^{n_p}A = 0$.*
*(2) By applying the idea of* [19, Lemma 2.2 and Proposition 2.4]*, we can easily see that*

$$E = \mathrm{End}(A) = \prod_{p\in\mathbb{P}} e_pE \cong \prod_{p\in\mathbb{P}} \mathrm{End}(\mathbb{Z}_{p^{n_p}}^{(\kappa_p)})$$

*where $e_q = (\delta_{pq})_{p\in\mathbb{P}}$ for the Kronecker's $\delta$ and $e_qE \cong \mathrm{End}(A_q)$, $q \in \mathbb{P}$. Thus $A$ is a C2 group satisfying the Schröder-Bernstein property by Theorem 4.6.*

*(3) By Theorem 4.8, $A \oplus (\mathbb{Q}/\mathbb{Z})^{(\kappa)} \oplus \mathbb{Q}^{(\lambda)}$ also satisfies the Schröder-Bernstein property for every cardinals $\kappa$ and $\lambda$.*

**Proposition 4.10.** *Let $\mathcal{S} \subseteq \mathcal{P}(\mathbb{P})$ be an ideal containing $\{p\}$ for all $p \in \mathbb{P}$ (so $S$ is closed under finite unions and subsets). For each $p \in \mathbb{P}$, let $\kappa$ be a cardinal and $n_p \in \mathbb{N}$. Then*

$$A = \{(x_p)_{p\in\mathbb{P}} \in \prod_{p\in\mathbb{P}} \mathbb{Z}_{p^{n_p}}^{(\kappa_p)} : \ supp((x_p)_{p\in\mathbb{P}}) = p \in \mathbb{P} : x_p \neq 0 \in \mathcal{S}\}$$

*is a C2-group satisfying the Schröder-Bernstein property.*

Proposition 4.10 follows from Theorem 4.6 since $E(A_{\mathcal{S}})$ will naturally equal $\prod_{p\in\mathbb{P}} e_p E$. Note that if $S = \mathcal{P}(\mathbb{P})$, then we have Example 4.9. If $\mathcal{S}$ is just the finite subsets of $\mathbb{P}$, then $A_{\mathcal{S}} = \oplus_{p\in\mathbb{P}} \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$. And of course, there are an uncountably infinite number of intermediate such ideals $\mathbb{S}$ between these two extremes.


## ACKNOWLEDGMENTS

The authors are grateful to the referee for her/his valuable comments. The second author would like to thank the Funds for Science and Technology Development of the Ministry of Education and Training under the Project Number B2023.DNA.14.


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