KERNELS OF HOMOMORPHISMS BETWEEN UNIFORM QUASI-INJECTIVE MODULES

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Abstract. In this paper, we study the behaviour of endomorphism rings of indecomposable (uniform) quasi-injective modules. A very natural question here is, for a morphism $f : A \to B$, with A, B indecomposable (uniform) quasi-injective right R-modules, and $g : E(A) \to E(B)$ an extension of f where $E(-)$ denotes the injective hull, what is the relation between kernels of f and g, their monogeny classes and their upper parts?

1. Introduction

It is well known by the so-called Krull-Schmidt theorem that if we consider the direct sum of modules $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ such that for all i the rings $\mathrm{End}_R(M_i)$ are local (i.e. they have a unique maximal ideal), then the above direct decomposition of M into a direct sum of indecomposable modules is unique up to an isomorphism and up to a permutation.

Several classes of modules satisfying a weak form of the Krull-Schmidt property have been found recently in the literature. For instance, such a weak form holds for the classes of uniserial modules [4], of cyclically presented modules over a local ring [4], of kernels of homomorphisms between indecomposable injective modules [7], and cyclically finitely presented modules of the projective dimension ≤ 1 [9]. In all these cases, the following holds: there are two equivalence relations $\sim$ and $\equiv$ on the class such that, for any two finite families $\{A_1, \cdots, A_m\}$ and $\{B_1, \cdots, B_n\}$, $\oplus_{i=1}^m A_i \cong \oplus_{j=1}^n B_j$ if and only if $m = n$ and there exist two bijections $\sigma, \tau : \{1, \cdots, m\} \to \{1, \cdots, n\}$ such that $A_i \sim B_{\sigma(i)}$ and $A_i \equiv B_{\tau(i)}$ for every $i = 1, \cdots, n$.

All rings are assumed to be associative and with nonzero identity element; all modules are assumed to be unitary. Let R be a ring, M be a right R-module, and let N be a submodule of the module M. If $N \cap K \neq 0$ for any nonzero submodule K in M, then N is called an essential submodule in M, and we say that M is an essential extension of the module N. If M is an injective module and N is an essential submodule in M, then M is called the injective hull of the module N. The injective hull is unique up to isomorphism and it is denoted by $E(N)$. A submodule X of the module M is said to be closed in M if $X = Y$ for every submodule Y in M that is an essential extension of the module X. A

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module M is said to be uniform if any two nonzero submodules of M have the nonzero intersection, i.e., M does not have proper closed submodules. We refer to [2], [5], [10] and [11] for all the undefined notions in this paper.

For a module N, a module M is said to be injective with respect to N or N-injective if for any submodule K in N, every homomorphism $\alpha : K \to M$ can be extended to a homomorphism $\overline{\alpha} : N \to M$, i.e. $\overline{\alpha}|_K = \alpha$. Note that a module is injective if it is injective with respect to each module. A module is said to be quasi-injective or self-injective provided it is injective with respect to itself. It is well known that a module M is quasi-injective if and only if $f(M) \subseteq M$ for any endomorphism f of the injective hull of the module M (see [8] or [11, 17.11]). It is clear that every injective module is quasi-injective. Every finite cyclic group is a quasi-injective noninjective module over the ring of integers.

Notice that, for a non-zero quasi-injective module M, M is uniform (equivalently, M is indecomposable) iff $E(M)$ is uniform (equivalently, $E(M)$ is indecomposable) iff $\mathrm{End}(E(M))$ is a local ring. So, a natural question to ask is what happens when one considers uniform quasi-injective modules. Hence, the purpose of this article is to study, in an abstract setting, these weak forms of the finite weak Krull-Schmidt theorem for uniform quasi-injective modules, by applying tools and concepts of [6, 7].

2. Some Construction Lemmas and Notations

We start by recalling the following well-known characterizations of quasi-injective modules (see, for example, [8, Theorem 1.1] and [11, 17.9]).

Lemma 2.1. *The following conditions are equivalent for a right R-module M:*

(1) *M is quasi-injective,*
(2) *$\alpha(M) \subseteq M$ for each $\alpha \in \mathrm{End}(E(M))$,*
(3) *M is subbimodule of the bimodule ${}_{\mathrm{End}(E(M))}E(M)_R$,*
(4) *$\mathrm{Tr}(M, E(M)) = M$.*

Lemma 2.2. *The following conditions are equivalent for a non-zero quasi-injective module M:*

(1) *M is uniform,*
(2) *M is indecomposable,*
(3) *$E(M)$ is uniform,*
(4) *$E(M)$ is indecomposable,*
(5) *$\mathrm{End}(M)$ is a local ring,*
(6) *$\mathrm{End}(E(M))$ is a local ring.*

For right R-modules M and N, if $f \in \mathrm{Hom}(M, N)$ and K is a submodule of M, then $f|_K \in \mathrm{Hom}(K, N)$ denotes the restriction of f on K.

Lemma 2.3. *Let M be a uniform quasi-injective module, N be a non-zero submodule of M and $f \in \mathrm{End}(M)$. Then the following conditions are equivalent:*

(1) *f is an automorphism,*
(2) *f is injective,*
(3) *$f|_N$ is injective.*

Proof. (1)⇒(2)⇒(3) The implications are trivial.
(3)⇒(1) Let $f|_N$ be injective. Then $\ker f|_N = \ker f \cap N = 0$. Now $\ker f = 0$ and so f is injective because M is uniform. Thus f induces an isomorphism of M onto $f(M)$, say $g : f(M) \to M$, such that $gf = \mathrm{id}_M$. Since M is quasi-injective, there exists an extension $\overline{g} \in \mathrm{End}(M)$ of g. Clearly, $\overline{g}$ is an isomorphism and $\overline{g}f = \mathrm{id}_M$, which implies that f is an isomorphisms as well. □

Similar argument as in the previous proof give us the following elementary but useful result.

Lemma 2.4. *If M and be M' are uniform relatively injective modules and N a non-zero submodule of M, then any monomorphism $N \to M'$ extends to an isomorphism $M \to M'$.*

Example 2.5. Let p be a prime number and n a natural number. Since $E(\mathbb{Z}_{p^m}) = \mathbb{Z}_{p^\infty}$ and $\alpha(\mathbb{Z}_{p^m}) \subseteq \mathbb{Z}_{p^m}$ for each $\alpha \in \mathrm{End}(\mathbb{Z}_{p^\infty})$, the $\mathbb{Z}$-module $\mathbb{Z}_{p^m}$ is uniform quasi-injective but it is not injective.

By Lemma 2.2, $\mathrm{End}(E(M))$ of an indecomposable (a uniform) quasi-injective module M is local.

Proposition 2.6. *Assume M is an indecomposable (a uniform) quasi-injective module. Then*

(1) $\mathrm{End}(M)$ *is a local ring with the maximal ideal*
$$J(\mathrm{End}(M)) = \{f \in \mathrm{End}(M) \mid f \textit{ is non-injective}\}.$$
(2) $\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$.

Proof. (1) By Lemmas 2.2 and 2.3, the ring $\mathrm{End}(E(M))$ is local with the maximal ideal
$$J(\mathrm{End}(E(M))) = \{f \in \mathrm{End}(E(M)) \mid f \text{ is non-injective}\}.$$
(2) Clearly, the map $\rho : \mathrm{End}(E(M)) \to \mathrm{End}(M)$ defined by the rule $\rho(f) = f|_M$, which is well-defined by Lemma 2.1, is a ring homomorphism onto $\mathrm{End}(M)$. Now it is easy to say that $\mathrm{End}(M) \cong \mathrm{End}(E(M))/\ker\rho$ is local as well and
$$J(\mathrm{End}(M)) = \rho(J(\mathrm{End}(E(M)))) = \{f \in \mathrm{End}(M) \mid f \text{ is non-injective}\}$$
by Lemma 2.3. □

For non-zero non-injective homomorphisms $\varphi : M_1 \to M_2$ and $\varphi' : M_1' \to M_2'$, and $f \in \mathrm{Hom}(\ker\varphi, \ker\varphi')$, we fix the following notations that will be used throughout the paper:

$$\begin{array}{rcl} \kappa_1(f) & = & \{f_1 \in \mathrm{Hom}(M_1, M_1') \mid f_1|_{\ker\varphi} = f\} \\ \kappa_2(f) & = & \{f_2 \in \mathrm{Hom}(M_2, M_2') \mid \exists f_1 \in \kappa_1(f) : \varphi' f_1 = f_2\varphi\}. \end{array}$$

Lemma 2.7. *Non-zero non-injective homomorphisms $\varphi : M_1 \to M_2$ and $\varphi' : M_1' \to M_2'$, and $f \in \mathrm{Hom}(\ker\varphi, \ker\varphi')$ satisfies the following properties:*

1. *If M_1' is M_1-injective and M_2' is M_2-injective, then $\kappa_1(f) \neq \emptyset$ and $\kappa_2(f) \neq \emptyset$.*
2. *If $f_1, g_1 \in \kappa_1(f)$, then $f_1 - g_1$ is not injective.*
3. *If M_1' is uniform and $f_2, g_2 \in \kappa_2(f)$, then $f_2 - g_2$ is not injective.*

Proof. (1) Since $\ker\varphi$ is a submodule of M_1 and f can be viewed as a homomorphism to M_1', the existence of $f_1 \in \kappa_1(f)$ follows immediately from the M_1-injectivity of M_1'. If $f_1 \in \kappa_1(f)$, then there exists $\overline{f_1} \in \mathrm{Hom}(\varphi(M_1), \varphi'(M_1'))$ satisfying $\overline{f_1}\varphi = \varphi' f_1$. Thus $\overline{f_1}$ can be extended to $f_2 \in \mathrm{Hom}(M_2, M_2')$ such that $f_2\varphi = \varphi' f_1$ by the M_2-injectivity of M_2'.
(2) This is clear since $\ker\varphi \subseteq \ker(f_1 - g_1)$.
(3) Let $f_1, g_1 \in \kappa_1(f)$ and $f_2, g_2 \in \kappa_2(f)$ such that $f_2\varphi = \varphi' f_1$ and $g_2\varphi = \varphi' g_1$. Assume that $f_2 - g_2$ is injective and denote by $\overline{\varphi} \in \mathrm{Hom}(M_1/\ker\varphi, M_2)$ the injective homomorphism induces by the homomorphism φ. Then there exists a homomorphism g such that the diagram

$$\begin{array}{ccc} M_1/\ker\varphi & \xrightarrow{\overline{\varphi}} & M_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_2 - g_2} \\ M_1' & \xrightarrow{\varphi'} & M_2' \end{array}$$

commutes. Since $\varphi' g = (f_2 - g_2)\overline{\varphi}$ is injective, [5, Lemma 6.26(a)] implies that φ' is injective, a contradiction. □

In the following result, we consider the case when $\varphi = \ker\varphi'$, hence $M_1 = M_1', M_2 = M_2'$ and $\ker\varphi = \ker\varphi'$.

Proposition 2.8. *Let $M_1 = M_1'$, $M_2 = M_2'$ be indecomposable (uniform) quasi-injective modules,* $\ker\varphi = \ker\varphi'$ *and* $f \in \mathrm{End}(\ker\varphi)$. *Then the following conditions are equivalent:*

1. *f is an automorphism,*
2. *all homomorphisms of $\kappa_1(f)$ and $\kappa_2(f)$ are injective,*
3. *there exist homomorphisms $f_1 \in \kappa_1(f)$ and $f_2 \in \kappa_2(f)$ which are injective.*

Proof. (1)⇒(2) The implication is clear.
(2)⇒(3) The implication is an easy consequence of Lemma 2.7.

(3)⇒(1) We follow arguments of the proof of [7, Theorem 2.1]. Note that the existence of the injective map $f_1 \in \kappa_1(f)$ implies that f is injective, so all homomorphisms of $\kappa_1(f)$ are injective. Hence there are injective homomorphisms $f_1 \in \kappa_1(f)$ and $f_2 \in \kappa_2(f)$ such that the diagram with exact rows commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & M_1 & \xrightarrow{\varphi} & M_2 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ 0 & \longrightarrow & \ker\varphi & \longrightarrow & M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

Clearly, it induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & M_1 & \xrightarrow{\varphi} & \varphi(M_1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \overline{f_1}} & & \\ 0 & \longrightarrow & \ker\varphi & \longrightarrow & M_1 & \xrightarrow{\varphi} & \varphi(M_1) & \longrightarrow & 0 \end{array}$$

with exact rows where $\overline{f_1}$ is injective. Since f_1 is an isomorphism by Lemma 2.3, f is an isomorphism by the Snake lemma. □

3. The Endomorphism ring

Theorem 3.1. *Let M_1, M_2 be indecomposable (uniform) quasi-injective modules, and let $\varphi : M_1 \to M_2$ be a non-zero non-injective morphism with $E := \operatorname{End}(\ker\varphi)$. Set*

$$I_1 = \{f \in E \mid f \text{ is non-injective}\}$$

and

$$I_2 = \{f \in E \mid \exists f_2 \in \kappa(f) : f_2 \text{ is non-injective}\}.$$

Then I_1 and I_2 are completely prime maximal ideals of E, and

$$I_1 = \{f \in E \mid \exists f_1 \in \kappa_1(f) \text{ is non-injective}\},$$

$$I_2 = \{f \in E \mid \exists f_1 \in \kappa_1(f), \ker\varphi \subsetneq f_1^{-1}(\ker\varphi)\}.$$

Moreover,

(1) *if $I_1 \subseteq I_2$, then E is local with the maximal ideal I_2,*
(2) *if $I_2 \subseteq I_1$, then E is local with the maximal ideal I_1,*
(3) *if I_1 and I_2 are not comparable, then E is semilocal such that $J(E) = I_1 \cap I_2$ and $E/J(E) \cong E/I_1 \times E/I_2$.*

Proof. We define mappings $\rho_i : E \to \operatorname{End}(M_i)/J(\operatorname{End}(m_i))$ for $i = 1, 2$ by the rule

$$\rho_i(f) = f_i + J(\operatorname{End}(m_i))$$

for $f_i \in \kappa_i(f)$. The correctness of the definition follows from Proposition 2.6 and Lemma 2.7. Moreover, $\ker\rho_i = I_i$ and it is completely prime ideals since $\operatorname{End}(M_i)/J(\operatorname{End}(M_i))$ are division rings. Since I_1 and I_2 are proper ideals, $I_1 \cup I_2$ contains noninvertible elements and all elements of $E \setminus (I_1 \cup I_2)$ are invertible by Proposition 2.8. Thus every proper right ideal of E is contained in $I_1 \cup I_2$. If I_1 and I_2 are comparable, then it holds true either the case (1) or (2).

If they are not comparable, then $J(E) = I_1 \cap I_2$ and I_1 and I_2 are two maximal ideals of E. Now it is easy to see that $E/J(E) \cong E/I_1 \times E/I_2$ by the Chinese reminder theorem. □

Let A and B be two modules. According to [4] and [7], we say that

• A and B have the same monogeny class, denoted by $[A]_m = [B]_m$, if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$;

• A and B have the same upper part, denoted by $[A]_u = [B]_u$, if there exist a homomorphism $\phi : E(A) \to E(B)$ and a homomorphism $\psi : E(B) \to E(A)$ such that $\phi^{-1}(B) = A$ and $\psi^{-1}(A) = B$.

Lemma 3.2. *Let M_1, M_2, M_1' and M_2' be indecomposable quasi-injective modules with M_1, M_1' relative injective and M_2, M_2' relative injective. If $\varphi : M_1 \to M_2$ and $\varphi' : M_1' \to M_2$ are non-injective homomorphisms, then $ker(\varphi) \cong ker(\varphi')$ if and only if either $\varphi = \varphi' = 0$ and $M_1 \cong M_1'$, or there exist isomorphisms $f_1 : M_1 \to M_1'$ and $f_2 : M_2 \to M_2'$ such that $\varphi' f_1 = f_2\varphi$.*

Proof. Suppose that there exists an isomorphism $f : ker(\varphi) \to ker(\varphi')$. Since the indecomposable (uniform) module M_1' is M_1-injective, f extends to a monomorphism $f_1 : M_1 \to M_1'$. Clearly, f_1 is an isomorphism. It is also easy to see that the isomorphism f_1 induces the isomorphism $\bar{f_1} : M_1/ker(\varphi) \to M_1'/ker(\varphi')$. Since the indecomposable (uniform) module M_2' is M_2-injective, there exists a homomorphism $f_2 : M_2 \to M_2'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1/ker(\varphi) & \xrightarrow{\bar{\varphi}} & M_2 \\ & & \downarrow{\scriptstyle \bar{f_1}} & & \downarrow{\scriptstyle f_2} \\ 0 & \longrightarrow & M_1'/ker(\varphi') & \xrightarrow{\bar{\varphi'}} & M_2' \end{array}$$

where $\bar{\varphi}$ and $\bar{\varphi'}$ are monomorphisms induced by φ and φ'. Thus, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & ker(\varphi) & \longrightarrow & M_1 & \xrightarrow{\varphi} & M_2 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ 0 & \longrightarrow & ker(\varphi') & \longrightarrow & M_1' & \xrightarrow{\varphi'} & M_2' \end{array}$$

Now, we have the following two cases:

(i) If $\varphi = 0$, then $ker(\varphi) = M_1$ and $ker(\varphi') = M_1'$. They imply that $M_1 \cong M_1'$ and $\varphi' = 0$.

(ii) If $\varphi \neq 0$, then $M_1/ker(\varphi) \neq 0$. From the isomorphism $\bar{f}_1$, we infer that f_2 is an isomorphism.

The converse follows immediately from Proposition 2.8. □

Proposition 3.3. *Let M_1, M_2, M_1', M_2' be quasi-injective indecomposable modules such that the modules M_1, M_1' are relative injective and the modules M_2, M_2' are relative injective. If $\varphi : M_1 \to M_2, \varphi : M_1' \to M_2'$ are arbitrary homomorphisms, then $ker(\varphi) \cong ker(\varphi')$ if and only if $[ker(\varphi)]_m = [ker(\varphi')]_m$ and $[ker(\varphi)]_u = [ker(\varphi')]_u$.*

Proof. It is enough to prove the reverse implication. The proof follows the arguments of [7, Lemma 2.4].

One can easily check that this observation holds if one of the two homomorphisms φ, φ' is a monomorphism. Thus, we can suppose that both φ and φ' are non-injective.

Assume that $[ker(\varphi)]_m = [ker(\varphi')]_m$ and $[ker(\varphi)]_u = [ker(\varphi')]_u$. Then, there are a monomorphism $f : ker(\varphi) \to ker(\varphi')$ and a homomorphism $k : E(ker(\varphi)) \to E(ker(\varphi'))$ such that $k^{-1}(ker(\varphi')) = ker(\varphi)$. Note that M_1' is M_1-injective, $ker(\varphi)$ is essential in M_1 and $ker(\varphi')$ is essential in M_1'. Therefore, k induces, by the restriction, a homomorphism $h_1 : M_1 \to M_1'$ and $h_1^{-1}(ker(\varphi')) = ker(\varphi)$. If f is an isomorphism, we are done. Thus, we can assume that the monomorphism f is not an isomorphism between $ker(\varphi)$ and $ker(\varphi')$. Inasmuch as the indecomposable module M_1' is M_1-injective, the monomorphism f extends to an isomorphism $f_1 : M_1 \to M_1'$ by Lemma 2.4. Now, the isomorphism f_1 induces the isomorphism $\bar{f}_1 : M_1/ker(\varphi) \to M_1'/ker(\varphi')$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & ker(\varphi) & \longrightarrow & M_1 & \longrightarrow & M_1/ker(\varphi) \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \bar{f}_1} \\ 0 & \longrightarrow & ker(\varphi') & \longrightarrow & M_1' & \longrightarrow & M_1'/ker(\varphi') \end{array}$$

By the Snake lemma, one can check that $ker(\bar{f}_1) \cong coker(f)$. We have that f is not an epimorphism and obtain that $\bar{f}_1$ is not a monomorphism.

By our construction, we have that $h_1(ker(\varphi)) \subseteq ker(\varphi')$, and so h_1 induces, by the restriction, a homomorphism $h : ker(\varphi) \to ker(\varphi')$. Thus, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & ker(\varphi) & \longrightarrow & M_1 & \longrightarrow & M_1/ker(\varphi) \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \bar{h}_1} \\ 0 & \longrightarrow & ker(\varphi') & \longrightarrow & M_1' & \longrightarrow & M_1'/ker(\varphi') \end{array}$$

From $h_1^{-1}(ker(\varphi')) = ker(\varphi)$, we infer that $\bar{h}_1$ is a monomorphism. We have the following two cases:

Case 1. h_1 is an isomorphism. Then, the Snake lemma gives that $ker(\bar{h}_1) \cong coker(h)$, and so h is an epimorphism. On the other hand, h_1 is an extension of h, we obtain that h is a monomorphism. We deduce that h is an isomorphism or $ker(\varphi) \cong ker(\varphi')$.

Case 2. h_1 is not an isomorphism. We have that M_1 is M_1'-injective and M_1' is an indecomposable module and obtain that h_1 is not a monomorphism. It follows that h is not a monomorphism, since $ker(\varphi)$ is essential in M_1. From the sum of the two previous commutative diagrams, we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(\varphi) & \longrightarrow & M_1 & \longrightarrow & M_1/ker(\varphi) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f+h} & & \downarrow{\scriptstyle f_1+h_1} & & \downarrow{\scriptstyle \bar{f}_1+\bar{h}_1} & & \\ 0 & \longrightarrow & ker(\varphi') & \longrightarrow & M_1' & \longrightarrow & M_1'/ker(\varphi') & \longrightarrow & 0 \end{array}$$

Now, we show that $f_1 + h_1$ is a monomorphism. In fact, let x be an element of M_1 with $(f_1 + h_1)(x) = 0$. Then, we have that $f_1(x) = -h_1(x)$. Since M_1 is uniform, $ker(h_1)$ is essential in M_1. Suppose that x is nonzero. Then, there exists an element $r \in R$ such that $xr \neq 0$ and $h_1(xr) = 0$, and so $f_1(xr) = 0$. Inasmuch as f_1 is a monomorphism, we get $xr = 0$, a contradiction. It shows that $f_1 + h_1$ is a monomorphism. By the hypothesis, M_1 is M_1'-injective and M_1' is indecomposable we immediately obtain that $f_1 + h_1$ is an isomorphism. Thus, the restriction $f+h$ of f_1+h_1 to $ker(\varphi)$ is a monomorphism. Similarly, $\bar{f}_1$ non-injective, $\bar{h}_1$ injective and $M_1/ker(\varphi) \cong im(\varphi) \subseteq M_2$ uniform imply that $\bar{f}_1 + \bar{h}_1$ is a monomorphism. From the Snake lemma, $f + h$ is an epimorphism. We deduce that $f + h$ is an isomorphism, and so $ker(\varphi) \cong ker(\varphi')$. ☐

Recall from [6, Section 4.14] that a semilocal category is a preadditive category with a nonzero object such that the endomorphism ring of every nonzero object is a semilocal ring.

Facchini in [6, Section 4.15] remarked that if R is a semilocal ring and $\pi : R \to R/J(R)$ is the canonical projection of R onto R modulo its Jacobson radical, then $\pi : R \to R/J(R)$ is a surjective local morphism, so that $V(\pi) : V(R) \to V(R/J(R))$ is an injective divisor homomorphism by [6, Proposition 3.29]. Moreover, the injective divisor homomorphism $V(\pi)$ is a morphism of monoids with order-units of $(V(R), \langle R_R\rangle)$ into $(V(R/J(R)), \langle R/J(R)\rangle)$.

According to Facchini [6, Page 142-143], if $\mathcal{A}, \mathcal{B}$ are additive categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, we say that F is:

(1) direct-summand reflecting if for every pair A, B of objects of $\mathcal{A}$ with $F(A)$ isomorphic to a direct summand of $F(B)$, A is isomorphic to a direct summand of B. (Here, if A and B are objects of an additive category $\mathcal{C}$, we say that A is isomorphic to a direct summand of B if there exists an object C of $\mathcal{C}$ such that B is a biproduct of A and C.)

(2) weakly direct-summand reflecting if for every pair A, B of objects of $\mathcal{A}$ with $F(A)$ isomorphic to a direct summand of $F(B)$, there exists an object C of A with $F(C) = 0$ and A isomorphic to a direct summand of $B \oplus C$.

Notice that

(a) direct-summand reflecting implies weakly direct-summand reflecting
(b) every additive functor$F : \mathcal{A} \to \mathcal{B}$ induces a monoid homomorphism $V(F) : V(\mathcal{A}) \to V(\mathcal{B})$ between the (possibly large) additive monoids $V(\mathcal{A})$ and $V(\mathcal{B})$. The functor F is isomorphism reflecting if and only if $V(F)$ is an injective mapping, essentially surjective if and only if $V(F)$ is a surjective mapping, and it is direct-summand reflecting if and only if the monoid morphism $V(F)$ is a divisor homomorphism, weakly direct-summand reflecting if and only if the monoid morphism $V(F)$ is an essential morphism, a weak equivalence if and only if $V(F)$ is a monoid isomorphism.

Finally, if $\mathcal{C}$ is a full subcategory of Mod-R, we denote the full subcategory of Mod-R whose objects are all modules that are isomorphic to direct summands of finite direct sums of modules in $Ob(\mathcal{C})$ by $\overline{\mathcal{C}}$. If $\mathcal{C}$ is a semilocal category, then $\overline{\mathcal{C}}$ is also semilocal (see [6, Page 297]).

In a preadditive category $\mathcal{A}$ with a nonzero object, we denote its Jacobson radical by $\mathcal{J}$.

Proposition 3.4. *Let $\varphi_i : M_{i1} \to M_{i2}$ ($i = 1, 2, \dots, n, n \geq 2$) and $\varphi' : M'_1 \to M'_2$ be $n + 1$ non-injective homomorphisms between indecomposable quasi-injective modules $M_{i1}, M_{i2}, M'_1, M'_2$ such that M_{i1}, M'_1 are relative injective and M_{i2}, M'_2 are relative injective. Suppose that $ker(\varphi')$ is isomorphic to a direct summand of $\bigoplus_{i=1}^{n} ker(\varphi_i)$, but $ker(\varphi') \ncong ker(\varphi_i)$ for every $i = 1, 2, \dots, n$. Then there are two distinct indices $i, j = 1, 2, \dots, n$ such that $[ker(\varphi')]_m = [ker(\varphi_i)]_m$ and $[ker(\varphi')]_u = [ker(\varphi_j)]_u$.*

Proof. By Theorem 3.1, $\ker(\varphi_i)$ for $i = 1, \dots, n$ and the modules $\ker(\varphi')$ are all modules whose endomorphism rings are semilocal of type ≤ 2. Set

$$\mathcal{A} := \operatorname{add}(\bigoplus_{i=1}^{n} \ker(\varphi')),$$

i.e. $\mathcal{A}$ contains all direct summands of finite direct sums of modules isomorphic to $\ker(\varphi_i)\}$, so that $\mathcal{A}$ is a semilocal full subcategory of Mod-R. Therefore the canonical monoid morphism $V(\mathcal{A}) \to V(\mathcal{A}/\mathcal{J}(\mathcal{A}))$ is an injective divisor homomorphism, because the canonical projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{J}(\mathcal{A})$ is an isomorphism-reflecting direct-summand reflecting functor by the previous paragraphs. Therefore, the rest follows from [6, Theorem 9.10] and [6, Proposition 9.14] □

Lemma 3.5. *Let $\varphi : M_1 \to M_2$, $\varphi' : M'_1 \to M'_2$ and $\varphi'' : M''_1 \to M''_2$ be non-injective homomorphisms between indecomposable quasi-injective modules such that M_1, M'_1, M''_1 are relative injective and M_2, M'_2, M''_2 are relative injective.*

If we assume $[ker(\varphi)]_m = [ker(\varphi')]_m$ and $[ker(\varphi)]_u = [ker(\varphi'')]_u$, then the following hold:

(1) *$ker(\varphi) \oplus D \cong ker(\varphi') \oplus ker(\varphi'')$ for some module D.*

(2) *The module D in (1) is unique up to isomorphism and is the kernel of a non-injective morphism between indecomposable quasi-injective modules.*

(3) $[D]_m = [ker(\varphi'')]_m$ *and* $[D]_u = [ker(\varphi')]_u$

Proof. (1) By the hypothesis, there exist monomorphisms $f : ker(\varphi) \to ker(\varphi')$ and $g : ker(\varphi') \to ker(\varphi)$, and homomorphisms $k_1 : E(ker(\varphi)) \to E(ker(\varphi''))$ and $k_2 : E(ker(\varphi'')) \to E(ker(\varphi))$ such that $k_1^{-1}(ker(\varphi'')) = ker(\varphi)$ and $k_2^{-1}(ker(\varphi)) = ker(\varphi'')$. We have that M_1, M_1', M_1'' are relative injective and we obtain that $k_1(M_1) \leq M_1''$ and $k_1(M_1'') \leq M_1$.

Call $h_1 := k_1|_{M_1}$ and $l_1 := k_2|_{M_1''}$. Clearly, $h_1 \in \mathrm{Hom}(M_1, to M_1'')$, $l_1 \in \mathrm{Hom}(M_1'', M_1]$ and $h_1^{-1}(ker(\varphi'')) = ker(\varphi)$ and $l_1^{-1}(ker(\varphi)) = ker(\varphi'')$. Let $h : ker(\varphi) \to ker(\varphi'')$ be the restriction of h_1 and $l : ker(\varphi'') \to ker(\varphi)$ be the restriction of l_1 .

We have the following cases:

Case 1. $g \circ f$ is an isomorphism. Then f splits, and so f is an isomorphism, since both $ker(\varphi)$ and $ker(\varphi')$ are uniform. Now $D := ker(\varphi")$ has the required properties.

Case 2. $l \circ h$ is an isomorphism. Then, both l and h are isomorphisms. We deduce that $ker(\varphi) \cong ker(\varphi')$. If $D := ker(\varphi')$, then D has the required properties.

Case 3. Neither $g \circ f$ nor $l \circ h$ are isomorphisms. By the assumption,

$$I_1 = \{\alpha \in \mathrm{End}(ker(\varphi)) \mid \alpha \text{ is non-injective}\}$$

and

$$I_2 = \{\alpha \in \mathrm{End}(ker(\varphi)) \mid \exists \alpha_2 \in \kappa_2(\alpha) : \alpha_2 \text{ is non-injective}\}$$

are completely prime maximal ideals of $\mathrm{End}(ker(\varphi'))$, we get $g \circ f$ is a monomorphism, hence it does not belongs to the ideal I_1. Inasmuch as $g \circ f$ is not an isomorphism we infer that $g \circ f \in I_2$. On the other hand, we have that $I_2 = \{\alpha \in \mathrm{End}(ker(\varphi)) \mid \ker\varphi \subsetneq \alpha_1^{-1}(\ker\varphi)\}$ by Theorem 3.1, so it follows that $l \circ h \notin I_2$. Similarly, we get $l \circ h \in I_1$. From this, we immediately obtain that $g \circ f + l \circ h \notin I_1 \cup I_2$. Thus, $g \circ f + l \circ h$ is an automorphism of $ker(\varphi)$. Then the composite homomorphism of the homomorphisms

$$ker(\varphi) \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} ker(\varphi') \oplus ker(\varphi") \xrightarrow{(g\circ f + l\circ h)^{-1}\circ\begin{pmatrix} g & l \end{pmatrix}} ker(\varphi)$$

is the identity homomorphism, and so $ker(\varphi) \oplus D \cong ker(\varphi') \oplus ker(\varphi")$ for some R-module D.

(2) Assume that $ker(\varphi) \oplus D \cong ker(\varphi') \oplus ker(\varphi'') \cong ker(\varphi) \oplus D'$. Since $\mathrm{End}(ker(\varphi))$ is a semilocal endomorphism ring by Theorem 3.1, we obtain that $D \cong D'$ by [5, Corollary 4.6], hence we have shown that the module D is unique up to isomorphism.

Next, we show that D is the kernel of a non-injective homomorphism between indecomposable quasi-injective modules. In fact, let $M = M_1' \oplus M_1''$. Hence M is quasi-injective. Let $\widehat{N}$ denote the injective hull of N in $\sigma[M]$, i.e. M-injective hull of N ([11, 17.8]). From the M-injectivity of M_1, M_1' and M_1'', we have $\widehat{ker(\varphi)} = M_1$, $\widehat{ker(\varphi')} = M_1'$ and $\widehat{ker(\varphi'')} = M_1''$. The isomorphism $ker(\varphi) \oplus D \cong ker(\varphi') \oplus ker(\varphi'')$ reduces an isomorphism $\widehat{ker(\varphi)} \oplus D \cong \widehat{ker(\varphi') \oplus ker}(\varphi'')$, and so

$$M_1 \oplus \widehat{D} = \widehat{ker(\varphi) \oplus D} \cong \widehat{ker(\varphi') \oplus ker}(\varphi'') = M_1' \oplus M_1''.$$

From $[ker(\varphi)]_m = [ker(\varphi')]_m$, we obtain that M_1 is embeddable into M_1' and M_1' is embeddable into M_1. It follows that $M_1 \cong M_1'$ by [10, Theorem 3.17]. By the direct-sum cancellation of modules with semilocal endomorphism rings again, we infer that $\widehat{D} \cong M_1''$. On the other hand, we have the isomorphism

$$(M_1 \oplus \widehat{D})/(ker(\varphi) \oplus D) \cong (M_1' \oplus M_1'')/(ker(\varphi') \oplus ker(\varphi'')),$$

and so

$$[M_1/ker(\varphi)] \oplus [\widehat{D}/D] \cong [M_1'/ker(\varphi')] \oplus [M_1"/ker(\varphi'')] \cong im(\varphi') \oplus im(\varphi'')$$

is embeddable into $M' := M_2' \oplus M_2''$. Similarly, from the above argument for M_2, M_2', M_2'' we have $M_2 \oplus \widehat{\widehat{D}/D} \cong M_2' \oplus M_2''$ in $\sigma[M']$. Since $[ker(\varphi)]_u = [ker(\varphi'')]_u$ and M_1, M_1'' are relatively injective, there are homomorphisms $\alpha : M_1 \to M_1''$ and $\beta : M_1" \to M_1$ such that $\alpha^{-1}(ker(\varphi'')) = ker(\varphi)$ and $\beta^{-1}(ker(\varphi)) = ker(\varphi'')$. It shows that there exist monomorphisms $M_1/ker(\varphi) \to M_1''/ker(\varphi'')$ and $M_1''/ker(\varphi'') \to M_1/ker(\varphi)$. Thus, there are monomorphisms $M_2 \to M_2''$ and $M_2'' \to M_2$, and so $M_2 \cong M_2''$ by [10, Theorem 3.17]. Then, we infer that $\widehat{\widehat{D}/D} \cong M_2' = \widehat{M_1'/ker(\varphi')}$ in $\sigma[M']$. If $\varphi' = 0$, then $D = \widehat{D} \cong M_1''$. Now, D is the kernel of the zero homomorphism $M_1'' \to M_2''$. If $\varphi' \neq 0$, then D is the kernel of the composite morphism $\widehat{D} \to \widehat{D}/D \to \widehat{\widehat{D}/D}$. Note that $\widehat{D} \cong M_1''$ and $\widehat{\widehat{D}/D} \cong M_2'$. We deduce that it is the kernel of a non-injective morphism between indecomposable quasi-injective modules.

(3) From the proof of (2), we have that D is the kernel of either $M_1'' \to M_2''$ or $M_1'' \to M_2'$.

Case 1. If $D \cong ker(\varphi')$, then $ker(\varphi) \cong ker(\varphi')$ and so D has the required properties. Similarly, it is true for the case $D \cong ker(\varphi'')$.

Case 2. If $D \ncong ker(\varphi')$ and $D \ncong ker(\varphi'')$, then we can apply Proposition 3.4 to the direct summand D of $ker(\varphi') \oplus ker(\varphi'')$, and so we get that either $[D]_m = [ker(\varphi'')]_m$ and $[D]_u = [ker(\varphi')]_u$ or $[D]_m = [ker(\varphi')]_m$ and $[D]_u = [ker(\varphi'')]_u$. Suppose that $[D]_u = [ker(\varphi'')]_u$. From Proposition 3.3, we obtain that $D \cong ker(\varphi)$. Thus, by Proposition 3.4 applied to the direct summands $ker(\varphi')$ and $ker(\varphi'')$ of $ker(\varphi) \oplus D$, we imply that the modules $ker(\varphi'), ker(\varphi''), ker(\varphi)$ and D have the same monogeny part and the same upper part. We deduce that , $ker(\varphi') \cong ker(\varphi'') \cong ker(\varphi) \cong D$, which is a contradiction.

□

Theorem 3.6. (Weak Krull-Schmidt theorem) *Let $\varphi_i : M_{i1} \to M_{i2}$, $i = 1, 2, \dots, n$, and $\varphi_j' : M_{j1}' \to M_{j2}'$, $j = 1, 2, \dots, k$, be non-injective homomorphisms between indecomposable quasi-injective modules $M_{i1}, M_{i2}, M_{j1}, M_{j2}$ such that M_{i1}, M_{j1}' are relative injective and M_{i2}, M_{j2}' are relative injective. Then $\bigoplus_{i=1}^n ker(\varphi_i) \cong \bigoplus_{j=1}^k ker(\varphi_j')$ if and only if $n = k$ and there exist two permutations σ, τ of $\{1, 2, ..., n\}$ such that $[ker(\varphi_i)]_m = [ker(\varphi'_{\sigma(i)})]_m$ and $[ker(\varphi_i)]_u = [ker(\varphi'_{\tau(i)})]_u$ for every $i = 1, 2, \dots, n$.*

Proof. We notice that the kernels $ker(\varphi_i)$ and $ker(\varphi'_j)$ are uniform modules. If $\bigoplus_{i=1}^n ker(\varphi_i) \cong \bigoplus_{j=1}^k ker(\varphi'_j)$, then they have the same Goldie dimension, and so $n = k$. In order to show that the existence of the permutations σ and τ, we use induction on n. The case $n = 1$ being trivial. Assume that $ker(\varphi_i)$ is isomorphic to some $ker(\varphi'_j)$. Cancelling the isomorphic modules $ker(\varphi_i)$ and $ker(\varphi'_j)$ (cancellation of modules holds because they have semilocal endomorphism rings), we can clearly proceed by induction. Then, we can suppose that $ker(\varphi_i) \ncong ker(\varphi'_j)$ for every $i, j = 1, 2, \ldots, n$. Note that $\mathrm{End}(ker(\varphi_i))$ and $\mathrm{End}(ker(\varphi'_j))$ are not local.

Now $ker(\varphi_1)$ is isomorphic to a direct summand of $\bigoplus_{j=1}^n ker(\varphi'_j)$. From Proposition 3.4, we infer that there exist two distinct indices $i, j = 1, 2, \ldots, n$ such that $[ker(\varphi_1)]_m = [ker(\varphi'_i)]_m$ and $[ker(\varphi_1)]_u = [ker(\varphi'_j)]_u$. Without loss of generality we may suppose $i = 1$ and $j = 2$. Now we can proceed as in [1, Theorem 5.3] using Lemma 3.5 instead of [1, Lemma 5.2]. □

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