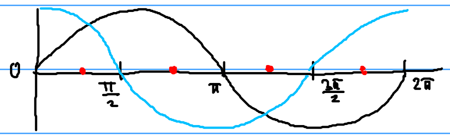
Příklad 4.2 $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,dx$

$f(x) := \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}$



- $\cos^2 x - \sin^2 x = 0$ pro $x = \frac{\pi}{4} + k\cdot\frac{\pi}{2}$, $k \in \mathbb{Z}$

  $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$, $\sin\frac{3\pi}{4} = -\cos\frac{3\pi}{4}$, $\sin\frac{5\pi}{4} = \cos\frac{5\pi}{4}$, $\sin\frac{7\pi}{4} = -\cos\frac{7\pi}{4}$, ...

- $f(x)$ je definovaná a spojitá na intervalech $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + k\frac{\pi}{2}$, $k \in \mathbb{Z}$

  k liché... $k = 2l-1$, $l \in \mathbb{Z}$: interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) + l\pi$

  substituce $t = \operatorname{tg} x$, $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) + l\pi$

  k sudé... $k = 2l$, $l \in \mathbb{Z}$: interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + l\pi$

  substituce $t = \operatorname{tg} x$, $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) + l\pi$ a $x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) + l\pi$

$\cos^2 x = \frac{1}{1+t^2}$, $\sin^2 x = \frac{t^2}{1+t^2}$, $dx = \frac{dt}{1+t^2}$

$$\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,dx \overset{VIII.14}{=} \int \frac{\frac{1}{(1+t^2)^2} + \frac{t^4}{(1+t^2)^2}}{\frac{1}{1+t^2} - \frac{t^2}{1+t^2}} \frac{dt}{1+t^2} = \int \frac{1+t^4}{(1+t^2)^2(1-t^2)}\,dt$$

$$\frac{1+t^4}{(1+t^2)^2(1-t^2)} = \frac{A}{1+t} + \frac{B}{1-t} + \frac{Ct+D}{1+t^2} + \frac{Et+F}{(1+t^2)^2}$$

- vynásobím $1+t$ a dosadím $t=-1$: $A = \frac{1}{4}$
- vynásobím $1-t$ a dosadím $t=1$: $B = \frac{1}{4}$
- vynásobím $(1+t^2)^2$ a dosadím $t=i$: $1 = Ei + F \Rightarrow E = 0,\ F = 1$
- $1+t^4 = A(1-t)(1+t^2)^2 + B(1+t)(1+t^2)^2 + (Ct+D)(1-t^2)(1+t^2) + (Et+F)(1-t^2)$

  $1+t^4 = \frac{1}{4}(1-t)(1+t^2)^2 + \frac{1}{4}(1+t)(1+t^2)^2 + (Ct+D)(1-t^2)(1+t^2) + (1-t^2)$

  zderivuji a dosadím $t=i$:

  $[4t^3]\big|_{t=i} = [(Ct+D)(1-t^2)2t - 2t]\big|_{t=i}$

  $-4i = (Ci+D)4i - 2i$

  $-2i = -4C + 4Di \Rightarrow C = 0,\ D = -\frac{1}{2}$

$$\int \frac{1}{4}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) - \frac{1}{2}\frac{1}{1+t^2} + \frac{1}{(1+t^2)^2}\,dt \overset{(*)}{=} \frac{1}{4}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{2}\frac{t}{1+t^2} + C$$

$$\int \frac{1}{1+t^2}\,dt \overset{PP}{=} \frac{t}{1+t^2} + \int \frac{2t^2}{(1+t^2)^2}\,dt = \frac{t}{1+t^2} + \int \frac{2(1+t^2)}{(1+t^2)^2} - \frac{2}{(1+t^2)^2}\,dt \Rightarrow$$

$$\int \frac{1}{(1+t^2)^2}\,dt = \frac{1}{2}\frac{t}{1+t^2} + \frac{1}{2}\int \frac{1}{1+t^2}\,dt \quad (*)$$

$$\frac{\frac{\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}} = \sin x \cos x$$

Dostáváme

$$\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,dx = \frac{1}{4}\log\left|\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}\right| + \frac{1}{2}\underbrace{\frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x}} + C$$

$$= \frac{1}{4}\left(\log\left|\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}\right| + \sin 2x\right) + C$$

na intervalech $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) + l\pi$, $\left(\frac{\pi}{4}, \frac{\pi}{2}\right) + l\pi$ a $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right) + l\pi$, $l \in \mathbb{Z}$.

Funkce f má ale p.f. na intervalu $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + l\pi$, $l \in \mathbb{Z}$.

$$F(x) := \frac{1}{4}\left(\log\left|\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}\right| + \sin 2x\right), \quad x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) + l\pi$$

spojitě dodefinujeme funkci F v bodě $\frac{\pi}{2} + l\pi$:

$$\lim_{x \to \frac{\pi}{2} + l\pi} \frac{1+\operatorname{tg} x}{1-\operatorname{tg} x} = -1$$

$$\lim_{x \to \frac{\pi}{2} + l\pi} F(x) = \lim_{x \to \frac{\pi}{2} + l\pi} \frac{1}{4}\left(\log\underbrace{\left|\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}\right|}_{\downarrow\, 1} + \underbrace{\sin 2x}_{\downarrow\, 0}\right) = 0$$

def. $F\left(\frac{\pi}{2} + l\pi\right) = 0$

Použijeme větu o limitě derivací (IV.35):

- F je spojitá na $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + l\pi$
- $F'(x) = f(x)$ pro $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) + l\pi \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) + l\pi$
- f je spojitá na $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + l\pi$

věta IV.35 $\Rightarrow$ $F'(x) = f(x)$ pro $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + l\pi$

Závěr: $\displaystyle\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,dx = F(x) + C$, kde

$$F(x) = \begin{cases} \frac{1}{4}\left(\log\left|\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}\right| + \sin 2x\right), & x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) + l\pi,\ l \in \mathbb{Z} \\ 0, & x = \frac{\pi}{2} + l\pi,\ l \in \mathbb{Z} \end{cases}$$

na intervalech $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) + l\pi$ a $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + l\pi$, $l \in \mathbb{Z}$,

tedy na intervalech $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) + k\frac{\pi}{2}$, $k \in \mathbb{Z}$.